

10.

NOTE ON BURMAN'S LAW FOR THE INVERSION OF THE INDEPENDENT VARIABLE.

[*Philosophical Magazine*, VIII. (1854), pp. 535—540.]

THIS Note refers to the development of the n th differential coefficient of u in respect to x in terms of the n th and lower differential coefficients of x in respect to u .

The late Mr Gregory, in his very valuable book of examples on the Calculus, in alluding to this development, speaks of it as "extremely complicated, and involving so much preliminary matter for its demonstration," that he contents himself "with referring to a memoir by Mr Murphy on the subject in the *Philosophical Transactions*, 1837, p. 210." The development there given is of course essentially no other than that included in Burman's general formula. I recently have had occasion (as a preliminary step to the investigation of the laws of inverse transformation between two systems of t variables each, instead of between two single variables only, an investigation in which I have already made such progress that I expect shortly to be in possession of the general formula for the purpose) to reconsider what I shall term Burman's law, and have been somewhat surprised to find that, so far from affording a complicated expression, it does, when properly stated, give rise to an expression of the very simplest form that could be conceived or desired, and one that admits of an easy and elementary proof.

To fix the ideas, let us take the case of $\frac{d^r u}{dx^r}$, where $x = \phi u$. For greater brevity write $\frac{d^r x}{du^r}$ as x_r . The most cursory consideration will suffice to show, irrespective of all calculation, that we should have the following form of expansion, namely,

$$\begin{aligned} \frac{d^7 u}{dx^7} = & -x_7 \div x_1^8 \\ & + \{(2, 6) x_2 x_6 + (3, 5) x_3 x_5 + (4, 4) x_4 x_4\} \div x_1^9 \\ & - \{(2, 2, 5) x_2 x_2 x_5 + (2, 3, 4) x_2 x_3 x_4 + (3, 3, 3) x_3 x_3 x_3\} \div x_1^{10} \\ & + \{(2, 2, 2, 4) x_2 x_2 x_2 x_4 + (2, 2, 3, 3) x_2 x_2 x_3 x_3\} \div x_1^{11} \\ & - \{(2, 2, 2, 2, 3) x_2 x_2 x_2 x_2 x_3\} \div x_1^{12} \\ & + (2, 2, 2, 2, 2, 2) \div x_1^{13}. \end{aligned}$$

In the first group of a single term, 7 is taken in one part, in the second group of 3 terms, 8 is taken in every possible way of partition in two parts, in the third group of 3 terms, 9 is taken in every possible way of partition in three parts, and so on, until finally 12, that is, the double of the number next inferior to the given index 7, is taken in the sole possible way in which it can be taken of six parts; I ought to add, that in the groups of indices, *unity* is always understood to be inadmissible.

The groups of indices in the parentheses indicate numerical coefficients to be determined, and the whole and sole real difficulty (if any) of the question consists in determining the value of these numerical symbols. Now the law which furnishes these values would be seen on the most perfunctory examination to be the very simplest law that could possibly be stated, namely, any such symbol as (r, s, t, \dots) is to be understood to denote the number of distinct ways in which a number of things equal to the sum of the indices $r, s, t, \&c.$ admit of being thrown into combination groups of $r, s, t, \&c.!$

Thus, for example,

$$\begin{aligned} (2, 6) &= \frac{8!}{2!6!} = 28, & (3, 5) &= \frac{8!}{3!5!} = 56, & (4, 4) &= \frac{1}{2} \frac{8!}{(4!)^2} = 35, \\ (2, 2, 5) &= \frac{1}{2} \frac{9!}{(2!)^2 5!}, & (2, 3, 4) &= \frac{9!}{2!3!4!}, & (3, 3, 3) &= \frac{1}{6} \frac{9!}{(3!)^3}, \\ (2, 2, 2, 4) &= \frac{1}{3!} \frac{10!}{(2!)^3 4!}, & (2, 2, 3, 3) &= \frac{1}{(2!)^2 (2!)^2 (3!)^2}, \end{aligned}$$

and so on. The general law is obvious; and to prove its applicability in general, we have only to show that if it be true for the case of $\frac{d^r u}{dx^r}$, it is true for $\frac{d^{r+1} u}{dx^{r+1}}$. The proof is as follows. Let in general $[l, m, n, \&c.]$ indicate the value of

$$\frac{1 \cdot 2 \cdot 3 \dots (l + m + n + \&c.)}{1 \cdot 2 \dots l \times 1 \cdot 2 \dots m \times 1 \cdot 2 \dots n \times \&c.},$$

without reference to $l, m, n, \&c.$ being equal or unequal *inter se*.

Lemma 1. It is very easily seen that

$$[l, m, n, \&c.] = [l - 1, m, n, \&c.] + [l, m - 1, n, \&c.] + [l, m, n - 1, \&c.] + \&c.$$

If now we use the notation $[\rho^r, \sigma^s, \tau^t, \dots]$ as an abbreviated form of the notation $[\rho, \rho, \rho \dots$ to r terms, $\sigma, \sigma, \sigma \dots$ to s terms, $\tau, \tau \dots$ to t terms, $\&c.]$, it is obvious that the equation last written becomes

$$[\rho^r, \sigma^s, \tau^t, \dots] = r[\rho - 1, \rho^{r-1}, \sigma^s, \tau^t, \dots] + s[\rho^r, \sigma - 1, \sigma^{s-1}, \tau^t, \dots] + t[\rho^r, \sigma^s, \tau - 1, \tau^{t-1}, \dots] + \dots$$

Lemma 2. Let $C(\rho^r, \sigma^s, \tau^t, \dots)$ denote the number of ways in which $r\rho + s\sigma + t\tau + \dots$ can be taken in combinations of $\rho, \rho \dots$ to r places, $\sigma, \sigma \dots$ to s places, $\&c.$, then upon the supposition that $\rho, \sigma, \tau, \&c.$, which are to be understood as arranged in an ascending order of magnitude, are all unequal, we shall have

$$C(\rho^r, \sigma^s, \tau^t, \dots) = [\rho^r, \sigma^s, \tau^t, \dots] / r! s! t! \dots,$$

which by Lemma 1

$$\begin{aligned} &= \frac{[\rho - 1, \rho^{r-1}, \sigma^s, \tau^t, \dots]}{(r - 1)! s! t! \dots} + \frac{[\rho^r, \sigma - 1, \sigma^{s-1}, \tau^t, \dots]}{r! (s - 1)! t! \dots} + \frac{[\rho^r, \sigma^s, \tau - 1, \tau^{t-1}, \dots]}{r! s! (t - 1)! \dots} + \dots \\ &= C(\rho - 1, \rho^{r-1}, \sigma^s, \tau^t, \dots) + \{1 + rF(\sigma - \rho)\} C(\rho^r, \sigma - 1, \sigma^{s-1}, \tau^t, \dots) \\ &\quad + \{1 + sF(\tau - \sigma)\} C(\rho^r, \sigma^s, \tau - 1, \tau^{t-1}, \dots) + \dots \end{aligned}$$

$F(\sigma - \rho)$, $F(\tau - \sigma)$, $\&c.$ meaning quantities which are respectively zero when $\sigma - 1 > \rho$, $\tau - 1 > \sigma$, $\&c.$, and respectively units when $(\sigma - 1) = \rho$, $\tau - 1 = \sigma$, $\&c.$; for it will be obvious that if $\sigma - 1 = \rho$, the quantity

$$[\rho^r, \sigma - 1, \sigma^{s-1}, \tau^t, \dots]$$

becomes

$$[\rho^{r+1}, \sigma^{s-1}, \tau^t, \dots],$$

and consequently when divided by $r! (s - 1)! t! \dots$ does not give

$$C(\rho^{r+1}, \sigma^{s-1}, \tau^t, \dots),$$

but

$$(r + 1) C(\rho^{r+1}, \sigma^{s-1}, \tau^t, \dots),$$

and so similarly for the cases of $\tau - 1 = \sigma$, $\&c.$

Now let us suppose that we are considering any group ($\rho, \rho \dots$ to r places, $\sigma, \sigma \dots$ to s places, $\&c.$), or more briefly $(\rho^r, \sigma^s, \tau^t, \dots)$, the numerical coefficient of the term $x_\rho^r x_\sigma^s x_\tau^t \dots$ in the inverse development of $\frac{d^\mu x}{du^\mu}$.

And first, suppose that ρ is not 2.

The coefficient in question will evidently be made up exclusively of the following parts, each, however, affected with the factor $(-)^{N-1}$, derived from

the expansion of $\frac{d^{\mu-1}x}{du^{\mu-1}}$, for which the law to be established is supposed to hold, namely,

$$\left. \begin{aligned} & C(\rho-1, \rho^{r-1}, \sigma^s, \tau^t, \dots) \\ & + \{1+rF(\sigma-\rho)\} C(\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \dots) \\ & + \{1+sF(\tau-\sigma)\} C(\rho^r, \sigma^s, \tau-1, \tau^{t-1}, \dots) \\ & + \&c. \end{aligned} \right\} \quad (\mathfrak{S}),$$

each part being affected with the factor $(-1)^{N-1}$, derived from the differentiations performed upon

$$x_{\rho-1} x_{\rho}^{r-1} x_{\sigma}^s x_{\tau}^t \dots \div x_1^N,$$

$$x_{\rho}^r x_{\sigma-1} x_{\sigma}^{s-1} x_{\tau}^t \dots \div x_1^N,$$

$$x_{\rho}^r x_{\sigma}^s x_{\tau-1} x_{\tau}^{t-1} \dots \div x_1^N,$$

&c.

Secondly, suppose ρ , the lowest index, is 2, then the term

$$x_{\rho-1} x_{\rho}^r x_{\sigma}^s x_{\tau}^t \dots$$

must be rejected, because $x_{\rho-1}$ becomes x_1 , which is excluded from appearing in any numerator. But then, *per contra*, in this case there will be a portion of the coefficient derivable from the differentiation of the denominator of the term

$$(-)^{N-2} \cdot \frac{(2^{r-1}, \sigma^s, \tau^t \dots) x_2^{r-1} x_{\sigma}^s x_{\tau}^t \dots}{x_1^{N-1}},$$

where

$$(N-1) = 1 + (r-1)2 + s\sigma + t\tau + \&c.$$

This portion will be

$$(-)^{N-1} (N-1) C(2^{r-1}, \sigma^s, \tau^t, \dots),$$

or, which is the same thing,

$$C(1, 2^{r-1}, \sigma^s, \tau^t, \dots),$$

and therefore the portion of the coefficient corresponding to $x_{\rho-1} x_{\rho}^r x_{\sigma}^s x_{\tau}^t \dots$, &c. is supplied from another source, and the expression (\mathfrak{S}) remains good for all values of ρ, σ, τ , &c., and consequently, by virtue of the second lemma, is equal to $C(\rho^r, \sigma^s, \tau^t, \dots)$; and thus we see that if the law assumed is true for $\frac{d^r u}{dx^r}$ it remains true for $\frac{d^{r+1} u}{dx^{r+1}}$, as was to be shown. And as it is evidently true for $r=1$, it is true generally.



POSTSCRIPT.

The formula expressing Burman's law may be exhibited as follows: x_r will still be understood to denote $\frac{d^r x}{d u^r}$, and $C\{p, q, \dots, m\}$ will, as before, denote the number of distinct modes of combining $p + q + \dots + m$ things in sets of p, q, \dots, m at a time; so that, for example, $C\{2, 2, 4, 4\}$ will denote

$$\frac{1 \times 2 \times 3 \dots \times 16}{(1 \cdot 2)^2 \cdot (1 \cdot 2 \cdot 3 \cdot 4)^2} \cdot \frac{1}{1 \cdot 2} \cdot \frac{1}{1 \cdot 2 \cdot 3}$$

Let now $n - 1$ be broken up *without restriction in every possible way* into parts, and let $r, s, t \dots l$ denote one such system of parts so that

$$r + s + t + \dots + l = n - 1,$$

$r, s, \&c.$ being all *actual* positive integers. Then is $\frac{d^n u}{d x^n}$ equal to

$$\Sigma C\{(1+r), (1+s), (1+t) \dots (1+l)\} \cdot \frac{1}{x_1^n} \cdot \left\{ \frac{-x_{1+r}}{x_1} \cdot \frac{-x_{1+s}}{x_1} \cdot \frac{-x_{1+t}}{x_1} \dots \frac{-x_{1+l}}{x_1} \right\},$$

than which nothing more clear and simple can be desired or imagined. And so more generally, if we make, as before, $r + s + t + \dots + l = n - g$, and give g in succession every different value from 1 to n , we shall have $\frac{d^n \Sigma}{d x^n}$ equal to

$$\Sigma \Sigma \left[\{[(1+r), (1+s), \dots (1+l)], (g-1)\} \frac{d^g \Sigma}{d u^g} \cdot \frac{1}{x_1^n} \left(\frac{-x_{1+r}}{x_1} \cdot \frac{-x_{1+s}}{x_1} \dots \frac{-x_{1+l}}{x_1} \right) \right],$$

where $\{[(1+r), (1+s), \dots (1+l)], (g-1)\}$ means the number of ways in which $(1+r) + (1+s) + \dots + (1+l) + (g-1)$ elements can be partitioned off into groups of one kind containing respectively $(1+r), (1+s), \dots (1+l)$ of the elements, and into a group of another kind containing the remainder $(g-1)$ of the elements. This distinction of the groups into two kinds has no effect upon the result except when $g-1$ is equal to any of the numbers $(1+r), (1+s), \dots (1+l)$. If we write, according to the notation above employed, $(1+r), (1+s), \dots (1+l)$ under the form $(\alpha^a, \beta^b, \dots \gamma^c)$, then

$$\{[(1+r), (1+s), \dots (1+l), (g-1)]\}$$

will represent

$$\frac{(\alpha\alpha + b\beta + \dots + c\gamma + g - 1)!}{a! (\alpha!)^a b! (\beta!)^b \dots c! (\gamma!)^c (g-1)!}$$

This more general theorem may of course be demonstrated by a similar method to that employed in the text for the case of $\mathfrak{S} = u$, for which all the terms in the expansion vanish except those in which $g = 1$.

I have, since this paper was sent to the press, obtained a new solution of the far more difficult and interesting question of the change from one *system* of independent variables to another system*. I say a new solution, because one has already been *virtually* effected, but under a form leaving much to be desired, by the great Jacobi in his Memoir *De Resolutione Aequationum per series infinitas*, Crelle, Vol. vi. 1830. In my solution, a remarkable species of quantities, to which I give the name of Arborescent Functions, make their appearance in analysis for the first time.

[* p. 65 below.]