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NOTE ON THE "ENUMERATION OF THE CONTACTS OF LINES AND SURFACES OF THE SECOND ORDER."

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IN the month of February, 1851, I gave in this *Magazine* an *à priori* and exhaustive process, founded upon the method of determinants, for determining every different kind of simple or collective contact capable of happening between lines and surfaces, and in general between all loci (whether intraspatial or extraspatial) of the second order [Vol. I., p. 219]. The question was shown to resolve itself into that of determining the number of singular relations capable of existing between two quadratic homogeneous functions of any given degree. My object in the paper referred to was actually to calculate the geometrical and analytical characters of these contents and singularities for intraspatial loci, that is loci representable by homogeneous quadratics of two, three, and four variables; but I incidentally appended [Vol. I., p. 239] a statement of the number of such for loci of five, six, seven, and eight variables, without, however, dwelling upon the means of representing the general law. This statement is, however, affected with certain inaccuracies of computation which will be presently pointed out.

It will be at once apparent, from an inspection of the principle of my method, that it remains equally applicable (*mutatis mutandis*) to the more general question of determining the relative singularities (in character and amount) of two functions, each linear in respect of two systems of variables $x_1, x_2 \ldots x_n$; $x_1', x_2' \ldots x_n'$; which species of functions degenerate into quadratic forms, when the two systems of variables become identical so as to coalesce into a single system. Some researches of Mr Cayley into the autometamorphic substitutions of quadratic forms (meaning thereby the linear substitutions which leave the form unaltered) required him to consider the nature of the singular relations capable of existing between two linear substitutions, which is precisely the question, differently stated, of the singular relations connecting two lineo-linear functions above adverted to; accordingly, I am indebted to Mr Cayley for making an observation on the effect of my rule for finding such singularities, which leads to a most elegant formulization of the number of singularities in question, and which I proceed to introduce to the notice of my readers.

If U and V be two quadratic functions, each of n variables, and if we call D the discriminant of $U+\lambda V=D(\lambda)$, $D(\lambda)$ will be a function of λ of the nth degree. Now, first, I have observed that if any of these n roots be repeated any number of times, there will be a corresponding degree of singularity about one of the points of intersection of the loci represented by $U=0$, $V=0$; so that if the n roots of $D(\lambda)$ be made up of r_1 roots a_1, r_2 roots a_2, r_3 roots a_3, &c., there will be an *inclusive singularity* r_1 at one point, r_2 at another, r_3 at a third, and so on—by *inclusive singularity* meaning a number one unit greater than the index of singularity properly so termed; the inclusive-singularity at an *ordinary* intersection being called 1, at a point of simple singularity 2, of double singularity 3, and in general at a point of the $(r-1)$th degree of singularity r.

Hence the total-inclusive singularity (which is an unit greater than the total-singularity, properly so called) may be broken up into as many partial heaps of inclusive-singularity as there are modes of decomposing n into integers. We may now confine our attention exclusively to the different modes in which a given amount of inclusive-singularity at a single point admits of subdivision into distinct species of singularity, for which I have given in my paper referred to the following rule: The minor systems of determinants corresponding to the matrix of $U+\lambda V$ are to be considered in succession; and if a be any root of the complete determinant of the matrix occurring r times, every hypothesis is to be exhausted as regards the number of times in which $(\lambda-a)$ may be conceived to enter as a factor into each of the system of 1st minors, into each minor of the system of 2nd minors, into each minor of the system of 3rd minors, and so on; the number of such hypotheses being limited by the condition that, if *quoad* the root a, $(\lambda-a)^{k_1}$, $(\lambda-a)^{k_2}$, $(\lambda-a)^{k_3}$ be the greatest common factors respectively to three consecutive systems of minor determinants, k_1 must be not less than $2k_2-k_3$. Here steps in the beautiful observation of Mr Cayley, that the question of assigning the different species of singularities respondent to the factor a supposed to occur r times, is, by virtue of the above condition, tantamount precisely to that of assigning the total number of decreasing* series of positive integers, commencing with a given number r, subject to the condition that the second differences shall be all positive; which (he adds), calling the successive second differences δ, δ', δ'', &c., is tantamount to finding

* Such a series must, from its very nature, be *always* decreasing or increasing in the same direction.

the number of ways that the equation $r = \delta + 2\delta' + 3\delta'' +$ &c., admits of being solved by positive integers, which is obviously the same as the number of modes in which r admits of being decomposed into positive integer parts. Thus the idea of partition, which arises naturally in the first part of the process (that, namely, of the decomposition of the collective inclusive-singularity in every possible way into modes of distributive inclusive-singularity), reappears quite unexpectedly (it may almost be said miraculously), and as the result of an analytical transformation in the second part of the same.

It should be observed that the case of complete coincidence between U and V, which, supposing them to be functions of n variables, corresponds to the supposition of the same factor occurring respectively n times, $(n-1)$ times, $(n-2)$ times, &c., 2 times and 1 time in the complete determinant, the 1st minor system, the 2nd minor system, &c., the $(n-2)$th minor system and the $(n-1)$th minor system respectively, is here taken as the highest case of singularity; this and the case of non-singularity, which also adds a unit to the index of singularity, properly so called, will together make a difference of two units in the numbers given by me in the paper referred to, which numbers will accordingly be 3, 6, 14, &c., in lieu of 1, 2, 12*, &c. We are now enabled to give the following simple statement of the law for determining the total number of singularities which can exist between two quadratic forms of n variables (or if we like so to say, more generally between two linear substitution-systems of the nth order), namely the number of the singularities (including absolute unrelatedness and entire coincidence within the purview of the term) is the index of double decomposition into parts of the number n. To raise up in the mind a clear conception of the idea of double decomposition, we may proceed as follows: First. Suppose a state of things in which a body is supposed to be determined completely, provided that the number of molecules which it contains, and the different number of atoms in each molecule are given, the index of simple decomposition, that is of ordinary partitionment of the number of n, will be the number of different bodies which are capable of being formed out of n atoms. Now imagine that, for the complete determination of a body, another step in the hierarchy of aggregation is to be taken into account, and that we must know for this purpose not only the number of molecules in the body and the number of atoms in each molecule, but also the number of monads in each atom; the number of bodies (differing by definition) capable of being formed out of n monads will then represent what I mean by the index of double decomposition of (or if we like so to say), to the modulus, n. And it is obvious that this idea admits of indefinite extension, and that we may speak of the index of decomposition of any order of multiplicity (single, double, treble, &c.) of, or to the modulus, n.

* These numbers refer to quadratic homogeneous functions, containing respectively 2, 3, 4, &c. variables. For the case of functions containing but one variable there is no distinction between coincidence and unrelatedness, and the number of modes of relation is a single unit.

For single decomposition it is well known and immediately obvious, that the indices to the successive moduli given by the rational numbers in regular progression will be the coefficients of x, x^2, x^3, &c. in the continued product

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} \text{ \&c. } ad\ inf.;$$

calling these n_1, n_2, n_3, &c., it is of course obvious, as Mr Cayley has observed, that the indices of double decomposition to the same successive moduli will be the coefficients of the same arguments x, x^2, x^3, &c., in the continued product

$$(1-x)^{-n_1}(1-x^2)^{-n_2}(1-x^3)^{-n_3} \text{ \&c. } ad\ inf.;$$

and by aid of this formula he has calculated (with extreme facility) the indices in question up to the modulus 11, and found that they form the series 1, 3, 6, 14, 27, 58, 111, 223, 424, 817, 1527, which accordingly is the series representing the number of singularities capable of existing between quadratic loci commencing with 1 and ending with 11 variables.

The values of n_1, n_2, n_3, ... n_{11}, &c. themselves are given in Euler's introduction, and are respectively

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, &c.,

which numbers will accordingly represent to their respectively corresponding moduli the number of *classes* of singularity, whether these classes be defined with reference to the different modes of distribution of the total collective singularity about different points, or with reference to the degree of the lowest system of minor determinants of the matrix to the determinant to $U+\lambda V$ having one or more factors in common, which latter is the mode of forming the classes adopted by me in the "Enumeration."

Let me be permitted to express the satisfaction which I have felt in finding this theory, which appeared to be doomed to hopeless oblivion, thus unexpectedly, after three years of interment, coming back to life, and at once filling a desired place in analytical researches pursued with apparently a totally different aim.