3.

NOTE ON SIR JOHN WILSON'S THEOREM.

[*Cambridge and Dublin Mathematical Journal*, IX. (1854), pp. 84, 85.]

THE following is probably the best and the briefest mode of deducing Sir John Wilson's Theorem and its cognate Theorems from Fermat's. I can say nothing as to its originality.

p being any prime number, let

$$(x-1)(x-2)(x-3)\ldots\{x-(p-1)\}=x^{p-1}+A_1x^{p-2}+A_2x^{p-3}+\&\text{c.}+A_{p-1}.$$

Let x successively take the values $1, 2, 3, \ldots (p-1)$; then to modulus p, by Fermat's Theorem, we have

$$x^{p-1}+A_{p-1}\equiv 1+A_{p-1},\ \text{ say } A_0,$$

and we derive the $(p-1)$ congruences to modulus p:

$$A_0+A_1+A_2+A_3 \ldots\ldots\ldots\ldots + A_{p-2}\equiv 0,$$
$$A_0+2^{p-2}A_1+2^{p-3}A_2+2^{p-4}A_3\ldots+2A_{p-2}\equiv 0,$$
$$A_0+3^{p-2}A_1+3^{p-3}A_2+3^{p-4}A_3\ldots+3A_{p-2}\equiv 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_0+(p-1)^{p-2}A_1+(p-1)^{p-3}A_2+(p-1)^{p-4}A_3\ldots+(p-1)A_{p-2}\equiv 0.$$

Now the determinant formed by the coefficients of

$$A_0, A_1, A_2, \ldots A_{p-2}$$

is $1.2.3\ldots(p-1)$ multiplied into the product of the differences of $1, 2, 3, \ldots (p-1)$, and is therefore incongruent to zero for the modulus p. Hence, there being $(p-1)$ *independent* homogeneous congruences between $(p-1)$ quantities, each of these quantities must be congruent to zero, that is

$$A_0\equiv 0,\ A_1\equiv 0, \ldots A_{p-2}\equiv 0\ [\text{mod. } p].$$

The congruence $A_0\equiv 0$, that is $1+1.2.3\ldots(p-1)\equiv 0$ [mod. p], is evidently Sir John Wilson's Theorem. We see also (by virtue of the remaining equations) at the same time, that the sums of the binary, ternary, &c., up to the $(p-2)^{\text{ary}}$ combinations of the numbers $1, 2, 3, \ldots (p-1)$, are all severally congruent to zero to the modulus p; that is, are all divisible by that number.