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Note on displacement in shakedown of elastic-plastic-creeping structures

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THIS COMMUNICATION is a complement to an earlier paper which deals with an upper bound to maximum deflections of elastic-plastic structures at shakedown and is the extension of the results to creeping deformations. The residual deformations are estimated on the basis of total dissipation work during the shakedown process.

Komunikat jest kontynuacją wcześniejszej pracy podającej górne oszacowanie ugięć sprężysto-plastycznych konstrukcji w stanie przystosowania i przedstawia rozszerzenie tych wyników w przypadku uwzględnienia pełzania materiału. Resztkowe ugięcia są oszacowane na podstawie analizy całkowitej energii dysypowanej w procesie przystosowania.

Сообщение является продолжением более ранней работы приводящей верхнюю оценку остаточных упруго-пластических прогибов конструкций в состоянии приспособления и представляет расширение этих результатов в случае учета ползучести материала. Остаточные прогибы оценены на основе анализа полной энергии диссипованной в процессеприспособления.

1. Introduction

SOME STRUCTURES are subjected to cycles of temperature variation as well as to cycles of repeated loading. The shakedown performance becomes important because, by restricting the cyclic loading to the shakedown limit, the designer is assured that after initial plastic deformation further deformation is in the elastic range, the possibilities of incremental collapse or reversed plasticity are thus avoided.

When the operating conditions exceed approximately 0.3 T_m, where T_m is the melting temperature, the effects of creep must be taken into account. Nowadays a great effort is being made in order to know better the behaviour of elastic-plastic-creeping structures under cyclic actions. Except for a few simple idealized cases, the analytical solutions of such problems are in general very difficult to solve, even numerical solutions present considerable problems. Therefore various types of bounding techniques on displacements and on total dissipation work have been studied and developed. These techniques have been reviewed in [2, 4, 5].

This communication is complementary to an earlier paper [1] which dealt with an upper bound to maximum deflections of elastic-plastic structures at shakedown and its purpose is to extend the results to account also for the creeping deformations. The method which has been obtained leads to an estimate of the residual deformation of the structure on the basis of the total dissipation work during the shakedown process. Illustrations of the applications are left to a further paper.

2. Simplified material model

A useful first assumption which has been generally accepted is the separation of the total strain ε_{ij}, into the sum of the elastic strain ε^e_{ij}, the creep strain ε^c_{ij}, and the plastic strain ε^P_{ij}. The elastic strain ε^e_{ij} is linked with stress by Hooke's law:

(2.1) $$\varepsilon^e_{ij} = A_{ijkl}\sigma_{kl},$$

where A_{ijkl} is a symmetric, positive definite elasticity tensor.

The elastic domain is specified by the yield function $f(\sigma_{ij}) = 0$ which also plays the role of the plastic potential function. The plastic strain rate is therefore

(2.2) $$\dot{\varepsilon}^P_{ij} = \begin{cases} 0 & \text{for} \quad f(\sigma_{ij}) < 0, \\ \dot{\lambda}\dfrac{\partial f}{\partial \sigma_{ij}} & \text{for} \quad f(\sigma_{ij}) = 0, \end{cases}$$

where $\dot{\lambda}$ is a non-negative scalar multiplier.

The creep strain rate $\dot{\varepsilon}^c_{ij}$ is expressed by the law attributed to Norton

(2.3) $$\dot{\varepsilon}^c_{ij} = \Phi^n(\sigma_{ij})\frac{\partial \Phi(\sigma_{ij})}{\partial \sigma_{ij}},$$

where Φ is a convex homogeneous function of degree one, n is the creep constant. It may be noted that the bound proved below does not explicitly involve the form of Φ.

The creep rate energy dissipation per unit volume is defined by

(2.4) $$D_c(\sigma_{ij}) = \sigma_{ij}\dot{\varepsilon}^c_{ij} = \Phi^{n+1}(\sigma_{ij}).$$

It is assumed that the material is isotropic, incompressible and the strain rates are independent of hydrostatic pressure.

The actual displacements u_i, strain ε_{ij} and stresses σ_{ij} at any instant t can be decomposed in the following way:

(2.5) $$u_i = u^E_i + u^R_i,$$

(2.6) $$\sigma_{ij} = \sigma^E_{ij} + \sigma^R_{ij},$$

(2.7) $$\varepsilon_{ij} = \varepsilon^E_{ij} + \varepsilon^R_{ij} + \varepsilon^c_{ij} + \varepsilon^P_{ij},$$

in which the displacements u^E_i, strains ε^E_{ij} and stresses σ^E_{ij} specify the response of the body in a linearly elastic regime, while u^R_i, σ^R_{ij}, denote the residual displacements, strains and stresses, respectively, caused by the incompatible creep strains ε^c_{ij} and plastic strains ε^P_{ij}. At any instant t the following equilibrium requirements apply within the body of volume V and on the surface part S_p subjected to quasistatic variable repeated load $P_i(x, t)$:

(2.8) $$\sigma^E_{ij,j} + X_i = 0 \quad \text{in} \quad V, \qquad \sigma^E_{ij}n_j = P_i \quad \text{on} \quad S_p,$$

(2.9) $$\sigma^R_{ij,j} = 0 \quad \text{in} \quad V, \qquad \sigma^R_{ij}n_j = 0 \quad \text{on} \quad S_p,$$

where X_i denotes the body forces.

The following equations are satisfied:

(2.10) $$\varepsilon^E_{ij} = \frac{1}{2}(u^E_{i,j} + u^E_{j,i}) \quad \text{in} \quad V,$$

$$\varepsilon_{ij}^{R}+\varepsilon_{ij}^{c}+\varepsilon_{ij}^{P}=\frac{1}{2}(u_{i,j}^{R}+u_{j,i}^{R}) \quad \text{in} \quad V, \tag{2.11}$$

$$u_i^E=u_i^R=0 \quad \text{on} \quad S-S_p. \tag{2.12}$$

3. Displacement bounding

The method which is of interest here falls into an approximate technique of calculation of the displacement at a specified point x_0 of the body for the shakedown. Making use of the well-known reciprocal relation of the elasticity extended to the case of non-compatible strains which are identified here with the creeping and plastic strains [1], we derive the following expression:

$$\int_{S_p} \hat{P}_i u_i dS_p=\int_V \hat{\sigma}_{ij}\varepsilon_{ij}^{E}dV+\int_V \hat{\sigma}_{ij}(\varepsilon_{ij}^{c}+\varepsilon_{ij}^{P})dV, \tag{3.1}$$

which provides a relation for displacement at the considered point, when $\hat{P}_i$ becomes a concentrated point force, where $\hat{\sigma}_{ij}(x, x_0)$ stands for the elastic stress field in a fictitious purely elastic structure having the same geometry and loaded by the concentrated force $\hat{P}_i$ at point x_0. Such a problem is in general very difficult to solve; for this reason bounding technique will be developed.

The actual displacement u_i can be estimated by the extremal magnitudes u_i^E and u_i^R as follows:

$$\max u_i \leqslant \max u_i^E+\max u_i^R, \tag{3.2}$$

$$\min u_i \geqslant \min u_i^E+\min u_i^R. \tag{3.3}$$

Bounds to u_i^E are straightforward or mostly existing analytical solutions. To evaluate u_i^R which is represented by the second integral on the right hand side of Eq. (3.1), we invoke:

If the yield condition contains the origin of the coordinate system, then there exist two constants $\varkappa$ and μ such that

$$\varkappa||\dot{\varepsilon}_{ij}^{P}|| \leqslant \sigma_{ij}\dot{\varepsilon}_{ij}^{P} \leqslant \mu||\dot{\varepsilon}_{ij}^{P}||, \tag{3.4}$$

where

$$\varkappa=\inf\frac{\sigma_{ij}\dot{\varepsilon}_{ij}^{P}}{||\dot{\varepsilon}_{ij}^{P}||}, \qquad \mu=\sup\frac{\sigma_{ij}\dot{\varepsilon}_{ij}^{P}}{||\dot{\varepsilon}_{ij}^{P}||} \tag{3.5}$$

and

$$||\dot{\varepsilon}_{ij}^{P}|| \equiv (\dot{\varepsilon}_{ij}^{P}\dot{\varepsilon}_{ij}^{P})^{\frac{1}{2}}, \tag{3.6}$$

in which the optimization is to be performed over the whole hypersurface $f(\sigma_{ij})=0$.

The assumption concerning the Norton law Eq. (2.3) permits the dissipation of creep energy to be bounded in a similar way:

$$\gamma||\dot{\varepsilon}_{ij}^{c}|| \leqslant \sigma_{ij}\dot{\varepsilon}_{ij}^{c} \leqslant \delta||\dot{\varepsilon}_{ij}^{c}||, \tag{3.7}$$

where

$$\gamma=\inf\frac{\sigma_{ij}\dot{\varepsilon}_{is}^{c}}{||\dot{\varepsilon}_{ij}^{c}||}, \qquad \delta=\sup\frac{\sigma_{ij}\dot{\varepsilon}_{ij}^{c}}{||\dot{\varepsilon}_{ij}^{c}||}, \tag{3.8}$$

$$(3.9)\qquad ||\dot{\varepsilon}^c_{ij}|| \equiv (\dot{\varepsilon}^c_{ij}\dot{\varepsilon}^c_{ij})^{\frac{1}{2}},$$

in which the optimization is to be performed over the whole hypersurface $\Phi(\sigma_{ij}) = 0$.

Also, if at $t = 0$ there is no plastic strain and no creep strain, $\varepsilon^P_{ij} = \varepsilon^c_{ij} = 0$, then

$$(3.10)\qquad ||\varepsilon^P_{ij}|| \equiv (\varepsilon^P_{ij}\varepsilon^P_{ij})^{\frac{1}{2}} = \int_0^t \frac{d}{dt}(\varepsilon^P_{ij}\varepsilon^P_{ij})^{\frac{1}{2}}dt \leqslant \int_0^t ||\dot{\varepsilon}^P_{ij}||dt$$

and, in a similar way,

$$(3.11)\qquad ||\varepsilon^c_{ij}|| \leqslant \int_0^t ||\dot{\varepsilon}^c_{ij}||dt.$$

Thus the above inequalities permit the residual displacements to be bounded as follows:

$$(3.12)\qquad \int_{S_p} \hat{P}_i u^R_i dS_T = \int_V \hat{\sigma}_{ij}(\varepsilon^c_{ij} + \varepsilon^P_{ij})dV \leqslant \int_V ||\hat{\sigma}_{ij}|| \cdot (||\varepsilon^c_{ij}|| + ||\varepsilon^P_{ij}||)dV$$

$$\leqslant \int_V ||\hat{\sigma}_{ij}|| \cdot \int_0^t (||\dot{\varepsilon}^c_{ij}|| + ||\dot{\varepsilon}^P_{ij}||)dtdU \leqslant \int_V ||\hat{\sigma}_{ij}|| \cdot \left(\frac{1}{\gamma}\int_0^t \sigma_{ij}\dot{\varepsilon}^c_{ij}dt + \frac{1}{\varkappa}\int_0^t \sigma_{ij}\dot{\varepsilon}^P_{ij}dt\right)dV$$

$$\leqslant \max_V\left(\frac{||\hat{\sigma}_{ij}||}{\gamma}\right)\int_V\int_0^t \sigma_{ij}\dot{\varepsilon}^c_{ij}dtdV + \max_V\left(\frac{||\hat{\sigma}_{ij}||}{\varkappa}\right)\int_V\int_0^t \sigma_{ij}\dot{\varepsilon}^P_{ij}dtdV,$$

where $||\hat{\sigma}_{ij}||$ denotes $(\hat{\sigma}_{ij}, \hat{\sigma}_{ij})^{1/2}$.

We see that the integrals on the right-hand side of the inequality (3.12) are the total creep energy and the plastic energy, respectively, dissipated during the shakedown process. By substituting the bound of the total energy dissipated at shakedown, given in [3] or [5], the inequality (3.12) provides a bound on the residual displacement at the considered point x_0 when $\hat{P}_i$ becomes a concentrated point force. All the terms appearing on the right-hand side of the inequality (3.12) can be calculated provided the solution of the shakedown problem is known.

The bound obtained may provide a useful addition to shakedown analysis and seems that it could have application where a safe bound is sufficient.

For this general bound, a more extensive investigation for various creep law and load conditions will appear in a separate paper.

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