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## ON A FUNDAMENTAL RULE IN THE ALGORITHM OF CONTINUED FRACTIONS.

[*Philosophical Magazine*, VI. (1853), pp. 297—299.]

LET $\frac{1}{a_1 +}\frac{1}{a_2 +}\frac{1}{a_3 +}$ &c. be any continued fraction, and let the successive convergents $\frac{1}{a_1}, \frac{1}{a_1 +}\frac{1}{a_2}$, &c. be called $\frac{N_1}{D_1}, \frac{N_2}{D_2}$, &c., and let $D_i$ be denoted by $(a_1, a_2 \dots a_i)$*, then the following identity obtains which I regard as the fundamental theorem in the theory of continued fractions, but which I have never seen stated in any work where this subject is treated [cf. pp. 530, 618 above].

*Theorem.*

$$(a_1 \dots a_m) \times (a_{m+1} \dots a_{m+n}) + (a_1 \dots a_{m-1}) \times (a_{m+2} \dots a_{m+n})$$
$$= (a_1 \dots a_m, a_{m+1} \dots a_{m+n}).$$

*Corollary* 1.

$$(a_1, a_2 \dots a_m) \times (a_2, a_3 \dots a_{m+1}) - (a_2, a_3 \dots a_m) \times (a_1, a_2 \dots a_{m+1}) = (-)^m 1.$$

This is the well-known theorem

$$D_i N_{i+1} - D_{i+1} N_i = \pm 1,$$

which, however, is only a case of a much more general theorem easily deduced from the fundamental theorem given above. In fact, we may derive immediately from the latter, the equation

$$(a_1, a_2 \dots a_m) \times (a_2, a_3 \dots a_{m+i}) - (a_2, a_3 \dots a_m) \times (a_1, a_2 \dots a_{m+i})$$
$$= (-)^m (a_{m+i}, a_{m+i-1} \dots \text{to } i - 1 \text{ terms}).$$

* It is essential to notice that $(a_1, a_2 \dots a_i) = (a_i, a_{i-1} \dots a_1)$.

Hence

$$D_{m-1}N_m - D_mN_{m-1} = (-)^m 1,$$
$$D_{m-2}N_m - D_mN_{m-2} = (-)^m a_m,$$
$$D_{m-3}N_m - D_mN_{m-3} = (-)^m (a_m a_{m-1} + 1),$$
$$D_{m-4}N_m - D_mN_{m-4} = (-)^m (a_m a_{m-1} a_{m-2} + a_m + a_{m-2}),$$
&c. &c.

*Corollary* 2.

$$(a_1 \dots a_\rho, a_{\rho+1} \dots a_{\rho+f})(a_1 \dots a_\rho, a_{\rho+1} \dots a_{\rho+k})$$
$$- (a_1 \dots a_\rho, a_{\rho+1} \dots a_{\rho+g})(a_1 \dots a_\rho, a_{\rho+1} \dots a_{\rho+h})$$
$$= (-)^\rho \{(a_{\rho+1} \dots a_{\rho+f})(a_{\rho+1} \dots a_{\rho+k}) - (a_{\rho+1} \dots a_{\rho+g})(a_{\rho+1} \dots a_{\rho+h})\}.$$

*Sub-corollary.* If all the several quantities $a_1, a_2, a_3 \dots$ are equal to one another, the quantity $D_fD_k - D_gD_h$ is *constant* in magnitude, but alternating in sign, so long as the differences of the indices $f, g, h, k$ are constant; and as an easy deduction from this sub-corollary, if

$$T_{n+1} = aT_n - bT_{n-1}$$

be the characteristic equation of a recurrent series, and if $f + k = g + h$, $\dfrac{T_fT_k - T_gT_h}{b^{\frac{g+h}{2}}}$ will be constant; and as a particular case of this deduction from the sub-corollary to the second corollary of the fundamental theorem, we have

$$\frac{T^2{}_n - T_{n-1}T_{n+1}}{b^n} = \text{a constant},$$

that is

$$\frac{T^2{}_{n+1} - aT_nT_{n+1} + bT^2{}_n}{b^n} = \text{a constant},$$

which is Euler's theorem. See Terquem's *Nouvelles Annales,* Vol. x. p. 357, and November 1852.

I was led up to a knowledge of the fundamental theorem (be it new or old) by some recent researches connected with my new Rule of Limits, considered with reference to the conditions which must be satisfied when one of the limits found by the rule comes into actual *contact* with a root; a contact which I can demonstrate is *always* possible, as well for the superior as for the inferior limits, and with so much the fewer equations (as distinguished from inequations) of condition between the coefficients of the assumed auxiliary function which the application of the rule of limits requires, as there are fewer pairs of imaginary roots in the function whose roots are to be limited.

I may add that the fundamental theorem is an immediate result of the representation of the terms of the convergents to a continued fraction under the form of determinants. Thus, for example, the determinant

$$\begin{vmatrix} a, & 1 & & & & \\ -1, & b, & 1 & & & \\ & -1, & c, & 1 & & \\ & & -1, & d, & 1 & \\ & & & -1, & e, & 1 \\ & & & & -1, & f \end{vmatrix}$$

is obviously decomposable into

$$\begin{vmatrix} a, & 1 & \\ -1, & b, & 1 \\ & -1, & c \end{vmatrix} \times \begin{vmatrix} d, & 1 & \\ -1, & e, & 1 \\ & -1, & f \end{vmatrix} + \begin{vmatrix} a, & 1 \\ -1, & b \end{vmatrix} \times \begin{vmatrix} e, & 1 \\ -1, & f \end{vmatrix}$$

or into

$$\begin{vmatrix} a, & 1 \\ -1, & b \end{vmatrix} \times \begin{vmatrix} c, & 1 & & \\ -1, & d, & 1 & \\ & -1, & e, & 1 \\ & & -1, & f \end{vmatrix} + a \times \begin{vmatrix} d, & 1 & \\ -1, & e, & 1 \\ & -1, & f \end{vmatrix}$$

or into

$$a \times \begin{vmatrix} b, & 1 & & & \\ -1, & c, & 1 & & \\ & -1, & d, & 1 & \\ & & -1, & e, & 1 \\ & & & -1, & f \end{vmatrix} + \begin{vmatrix} c, & 1 & & \\ -1, & d, & 1 & \\ & -1, & e, & 1 \\ & & -1, & f \end{vmatrix}$$

that is

$$\begin{aligned} (abcdef) &= (abc)(def) + (ab)(ef) \\ &= (ab)(cdef) + a\,(def) \\ &= a\,(bcdef) + (cdef). \end{aligned}$$

Thus the whole of the properties of continued fractions are deduced without algebraical calculation from a theorem which itself springs immediately by inspection from the well-known simple rule for the decomposition of determinants.

If instead of a simple set a triple set of quantities be taken, as

$$\begin{Bmatrix} l_1, & l_2 & \ldots l_{i-1} \\ m_1, & m_2 & \ldots m_i \\ n_1, & n_2 & \ldots n_{i-1} \end{Bmatrix},$$

which, when $i=1$, $i=2$, $i=3$, $i=4$, &c. is to be interpreted to mean

$$m_1; \quad \begin{vmatrix} m_1, & l_1 \\ -n_1, & m_2 \end{vmatrix}; \quad \begin{vmatrix} m_1, & l_1 & \\ -n_1, & m_2, & l_2 \\ & -n_2, & m_3 \end{vmatrix}; \quad \begin{vmatrix} m_1, & l_1 & & \\ -n_1, & m_2, & l_2 & \\ & -n_2, & m_3, & l_3 \\ & & -n_3, & m_4 \end{vmatrix},$$

&c. respectively, the value of the determinant represented by any such set being called $T_i$, we have in general

$$T_i = m_i T_{i-1} + l_i n_i T_{i-2},$$

which, when $m_i$ and $l_i n_i$ are constant, becomes the characteristic equation to an ordinary recurring series. The theorem corresponding to the fundamental theorem for such triple sets will be

$$\begin{Bmatrix} l_1, & l_2 & \ldots & l_{i+i'} \\ m_1, & m_2 & \ldots & m_{i+i'+1} \\ n_1, & n_2 & \ldots & n_{i+i'} \end{Bmatrix} = \begin{Bmatrix} l_1, & l_2 & \ldots & l_{i-1} \\ m_1, & m_2 & \ldots & m_i \\ n_1, & n_2 & \ldots & n_{i-1} \end{Bmatrix} \times \begin{Bmatrix} l_{i+1}, & l_{i+2} & \ldots & l_{i+i'} \\ m_{i+1}, & m_{i+2} & \ldots & m_{i+i'+1} \\ n_{i+1}, & n_{i+2} & \ldots & n_{i+i'} \end{Bmatrix}$$

$$+ l_i n_i \begin{Bmatrix} l_1, & l_2 & \ldots & l_{i-2} \\ m_1, & m_2 & \ldots & m_{i-1} \\ n_1, & n_2 & \ldots & n_{i-2} \end{Bmatrix} \times \begin{Bmatrix} l_{i+2} & \ldots & l_{i+i'} \\ m_{i+2} & \ldots & m_{i+i'+1} \\ n_{i+2} & \ldots & n_{i+i'} \end{Bmatrix}.$$