The Bohr Model of the Hydrogen Atom Revisited

Stanisław Olszewski

Institute of Physical Chemistry, Polish Academy of Sciences,

Kasprzaka 44/52, 01-224 Warsaw, Poland

E-mail: olsz@ichf.edu.pl

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Abstract

The parameters of the original Bohr model of the hydrogen atom are applied in an extension presented for that model.

In the first step the quanta of the magnetic field for each of the energy levels of the hydrogen atom are taken into account. A consequence of the presence of these quanta are the quanta of the magnetic flux being for any hydrogen energy state equal to a simple multiple of the flux quantum known from the theory of superconductors.

In the second step a model of the electron and proton spin is developed indepensently of the Dirac's formalism. The model is an effect

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of modification of the Heisenberg uncertainty principle for energy and time which yields a minimal distance between the particles in an ensemble of each fermions kind. This minimal distance has been next assumed as equal to the radius of circulation of a spinning particle respectively for the electron and proton.case, and the speed of the particle is then close to rhat of the speed of light. This model gives proper values for the angular mechanical momentum and the magnetic moment for each of considered particle kinds.

The third application of the developed model concerns the time of energy emission connected with the electron transitions in the hydrogen atom. If the energy transmitted between two neighbouring quantum levels is considered as a dissipated Joule-Lenz energy, the transmission time between the levels can be calculated in a definite non-probabilistic way. The product of that time and the amount of transmitted energy approaches the Planck constant h for any considered pair of the quantum states.

Keywords:

1 Introduction

A look on the Bohr model.

A little more than one hundred years ago the Bohr model of the hydrogen atom has been developed. Its spectacular success was due rather equally to the model simplicity as well as a tight agreement of the obtained results with

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the experimental spectroscopical data.

The main idea of the model was very simple. If the hydrogen atom is composed of a positively charged nucleus and a negative electron particle, a dominant force in the atom is the electrostatic force between the nucleus and electron. Since the nucleus is heavy and practically can be immobile, the light electron is moving about it along the circles called the orbits. The orbits are assumed to be planar similarly to the planetary orbits surrounding the Sun.

The centrifugal force acting on the electron along its motion is compensated by the attractive electron-nucleus electrostatic force, so we have - in the first step - a classical relation

$$m\frac{v^2}{r} = \frac{e^2}{r^2}. \tag{1}$$

Here m is the mass of the electron and $-e$ and e are respectively the electron and nuclear charges, r is the radius of the orbit, v is the electron orbital velocity. A compensation of e by $-e$ makes the atom equal to an electrically neutral system.

The second step of the Bohr formalism makes a reference to the quantum theory: the mechanical angula r momentum possessed by the electron is quantized, so

$$L = L_n = mv_n r_n = n\hbar \tag{2}$$

where n is here a positive integer number.

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A system of equations (1) and (2) can be easily solved with respect to the unknown parameters r and v:

$$r = r_n = \frac{n^2 \hbar^2}{me^2} \tag{3}$$

and

$$v = v_n = \frac{e^2}{n\hbar}. \tag{4}$$

The formulae (3) and (4) yield the the time period of the electron circulation about the atomic nucleus:

$$T = T_n = \frac{2\pi r_n}{v_n} = \frac{2\pi n^2 \hbar^2}{me^2} \frac{n\hbar}{e^2} = \frac{2\pi n^3 \hbar^3}{me^4}. \tag{5}$$

Because of (2) only a discrete spectra of r_n, v_n and T_n are possible. This property can be prolongated to the expression of the electron energy

$$E_{tot} = E_{kin} + E_{pot} \tag{6}$$

where

$$E_{kin} = \frac{mv^2}{2} = \frac{mv_n^2}{2} = \frac{1}{2}\frac{me^4}{n^2\hbar^2} \tag{7}$$

and

$$E_{pot} = -\frac{e^2}{r} = \frac{e^2}{r_n} = \frac{me^4}{n^2\hbar^2}. \tag{8}$$

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Evidently

$$E_{\text{tot}} = \frac{1}{2} \frac{me^4}{n^2 \text{n}^2} - \frac{me^4}{n^2 \text{n}^2} = -E_{\text{kin}} \tag{9}$$

is also a quantized parameter. A negative energy spectrum of E_{tot} indicates the bound states of the electron in the hydrogen atom. Finally from (7) and (8) we obtain the virial relation

$$2E_{\text{kin}} + E_{\text{pot}} = 0. \tag{10}$$

Having

$$E = E_{\text{tot}} \tag{11}$$

we can calculate the differences of (11) and (9) for different pairs of numbers n, say $n = n^t$ and $n = n^{tt}$. This gives

$$\Delta E = E_{n^f} - E_{n^{ff}} = \frac{1}{2} \frac{me^4}{n^2 \text{n}^2} \left(\frac{1}{n^{t2}} - \frac{1}{n^{tt2}} \right). \tag{12}$$

Because of the Planck quantum theory there exists the relation

$$\Delta E = h\nu \tag{13}$$

where

$$h = 2\pi \text{n} \tag{13a}$$

and ν is the frequency of the electromagnetic wave.

An enormous success of the Bohr theory was due to the fact that ν cal-

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culated for different pairs of n' and n'' in (12) could fit the experimental v measured for the hydrogen atom with a very high accuracy, see e.g. [Sommerfeld].

But the success of the Bohr model for the one-electron atom could not be extended to other systems than hydrogen atom, or hydrogen-like ions. A sharp difficulty came out already with the spectrum of the two-electron atom of helium, the obstacles became still more acute for the atomic systems having more than two electrons. In effect the modern quantum mechanics had to be developed and applied and the Bohr theory remained suitable solely for the one-electron systems.

But even for such systems the limitations of the model seemed to be rather evident; moreover not all of these difficulties could be removed by the quantum mechanics. The main problem seemed to be a lack of insight into the time parameter which certainly accompanies any physical effect, therefore any quantum effect produced in the hydrogen atomic system, for example the phenomenon of the quantum transitions.

The next point may concern a shortage of the treatment of the magnetic effects in the Bohr model; see Sec. 2. Also any approach to the spin effect is absent in that model. But simulataneously the model remains a well-founded tool for further investigations.

The aim of the paper was to demonstrate how at least a part of the shortcomings exhibited by the Bohr model could be removed with the aid of the model framework.

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2   Magnetic field and the magnetic flux in the hydrogen atom

A kind of asymmetry can be noted when the electromagnetic properties of the Bohr atomic model are considered. The electric field

$$|\mathbb{E}| = |\mathbb{E}_n| = \frac{e}{r_n^2} = \frac{m^2 e^5}{n^4 \mathrm{n}^4} \tag{14}$$

responsible for the electron-nucleus interaction is taken into account, whereas the magnetic field is not considered. But the presence of the magnetic field strength B in the atom can be easily detected on condition we note that a circular motion of any charged particle should produce such a field. The field $B = B_n$ in the central point of the nth orbit area is connected with the circular frequency Ω_n of the motion by the formula (see e.g. [42])

$$\Omega_n = \frac{eB_n}{mc}. \tag{15}$$

On the other hand, Ω_n, r_n and T_n are coupled together by the equation

$$\frac{2\pi r_n}{T_n} = \Omega_n r_n = \frac{eB_n}{mc} \frac{\mathrm{n}^2}{me^2} n^2 = v_n = \frac{e^2}{\mathrm{n}n}. \tag{16}$$

Expressions (15) and (16) give

$$B_n = \frac{e^3}{\mathrm{n}n} m^2 c \frac{1}{\mathrm{n}^2 n^2} = \frac{e^3 m^2 c}{\mathrm{n}^3 n^3}. \tag{17}$$

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A check of B_n is obtained with the aid of the orbital magnetic moment

$$M_{\text{orb}} = \frac{e}{2mc}n\hbar \tag{18}$$

which should give the electron energy (46) on the level n:

$$E_n = -M_{\text{orb}}B_n = -\frac{e}{2mc}n\hbar\frac{e^3m^2c}{n^3\hbar^3} = -\frac{e^4m}{2n^2\hbar^2}. \tag{19}$$

Let us note that n entering the Bohr model is a result of quantization of the orbital angular momentum

$$L_n = n\hbar. \tag{20}$$

3 Quanta of the magnetic flux absorbed/emitted in course of the electron transition

The quanta of B_n given by the formula (17) are rather complicated for further examinations. But instead of B_n a quantum change of the magnetic flux Φ due to the change of the electron quantum level n may become of interest. Having B_n from (17) and the area S_n enclosed by the orbit n expressed by the relation

$$S_n = \pi r_n^2 = \pi \left(\frac{\hbar^2}{me^2}n^2\right)^2, \tag{21}$$

the magnetic flux is

$$\Phi_n = S_n B_n = \pi \frac{n^4}{m^2 e^4} \, n^4 \frac{e^3 m^2 c}{n^3 n^3} = \pi \frac{cnn}{e} = \frac{1}{2} \frac{ch}{e} n. \qquad (22)$$

Hence each orbit *n* provides us with a flux quantum (22) proportional to *n*. The quanta

$$\frac{ch}{2e} \qquad (23)$$

are the well-known objects in the physics of superconductors [11, 5, 47].

It can be noted that the energy quanta

$$n\Omega_n = n\frac{eB_n}{mc} = n \frac{e}{mc} \frac{e^3 m^2 c}{n^3 n^3} = \frac{e^4}{n^2 c^2} mc^2 \frac{1}{n^3} = \alpha^2 \frac{mc^2}{n^3}, \qquad (24)$$

due to transitions between the neighbouring oscillator levels in the magnetic field, occur coincident with the quanta of the transition energy in the Bohr atom calculated also for the neighbouring levels. We obtain from (46):

$$\Delta E = E_{n+1} - E_n = -\frac{me^4}{2n^2} \left[\frac{1}{(n+1)^2} - \frac{1}{n^2} \right] = \frac{me^4}{2n^2} \frac{(n+1)^2 - n^2}{(n+1)^2 n^2}$$
$$\cong \frac{me^4}{2n^2} \frac{2n+1}{n^4} \approx \frac{me^4}{2n^2} \frac{2}{n^3} = mc^2 \alpha^2 \frac{1}{n^3}. \qquad (25)$$

For large *n* the energies of (25) approach evidently the result of (24).

A characteristic point is that the ratio of M_{orb} in (18) to Φ in (22) is a

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constant number for all quantum levels n:

$$\frac{M_{\text{orb}}}{\Phi_n} = \frac{ehn}{4\pi mc} : \frac{chn}{2e} = \frac{1}{2\pi} \frac{e^2}{mc^2}. \qquad (26)$$

A similar constant behaviour concerns also the ratio of the orbital angular momentum L_n to Φ_n:

$$\frac{L_n}{\Phi_n} = n\text{n} : \frac{nch}{2e} = \frac{1}{\pi} \frac{e}{c}. \qquad (27)$$

Let us note that an excitation energy of the electron spin or proton spin is very high. For the electron spin this energy will amount to

$$\text{n}\Omega_s = \text{n}\frac{eB_s}{mc} = \text{n}\frac{e}{mc} \frac{m^2c^3}{en} = mc^2, \qquad (28)$$

on the basis of a formula similar to (48) and $B = B_s$ from ( ). A similar result for the proton case will become

$$\text{n}\Omega_{sp} = \text{n}\frac{eB_{sp}}{m_pc} = \text{n}\frac{e}{m_pc} \frac{m_p^2c^3}{en} = m_pc^2. \qquad (28a)$$

4 Energy of the electron spin-orbit interaction in the Bohr model

This is the interaction energy between the orbital magnetic moment M_{orb} in (18) induced at the position of the proton nucleus and the spin magnetic

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moment on the orbit equal to the Bohr magneton

$$\mu_B = \frac{en}{2mc}.$$ (29)

The distance between two moments is represented by the length of the radius of the correspoding Bohr orbit; see (40). The influence of the proton magnetic moment on the moments (18) and (29) has been neglected. The interaction energy between the moments becomes

$$\frac{M_{orb}\mu_B}{r_n^3} = \overline{e}\ \overline{2me}\ n^2\ n^2\frac{1}{r_n^3} = \frac{1}{4}\frac{e^2n^2}{m^2c^2}n\ \frac{me^2}{n^2n^2}\ ^3 = \frac{1}{4}\frac{e^8}{n^4c^4}mc^2\frac{1}{n^5} = \frac{1}{4}\alpha^4 mc^2\frac{1}{n^5}.$$ (30)

Evidently, because of

$$\alpha \approx (137)^{-1},$$ (31)

the result in (30) is a number much smaller than a change of E given in (46) and (19) due to a small change of integer n; see (25).

5   A proposal of the magnetic flux operator and its application

$$\frac{\Phi}{L_B} = \frac{B\pi r^2}{m\Omega r^2} = \frac{\pi c}{e}.$$ (32)

Our proposal is that the operator for Φ should be proportional to that applied for L_B. The angular momentum along a circle performed with a

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constant speed is dependent only on a single variable ϕ which is the angle about the axis parallel to B. This implies that an operator for L_B is equal to (see e.g. [41, 40, 6]):

$$\overset{''}{L}_B = \frac{\text{n}}{i}\frac{\partial}{\partial\phi}. \tag{33}$$

The eigenfunctions of the operator in (33) are evidently

$$(2\pi)^{-1/2}\, e^{\pm i n_B \phi} \tag{34}$$

giving – for the integer n_B – the eigenvalues

$$(n_B |\overset{''}{L}_B| n_B) = \pm n_B \text{n}. \tag{35}$$

In effect the average values of the operator for the magnetic flux become equal to

$$(n_B |\overset{'''}{\phi}| n_B) = \pm\frac{\pi c}{e}n_B\text{n} = \pm n_B\,\frac{ch}{2e}. \tag{36}$$

A characteristic point is that the result in (36) has no reference to the size of the magnetic field B defining Ω in ( ).

A wave-mechanical treatment of the hydrogen atom implies (see e.g. [6])

$$n_B = n - 1,\ n - 2,\ \ldots,\ 1,\ 0, \tag{37}$$

where n is the quantum number defining the electron energy in the atom.

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But calculations done in the earlier Sections show that

$$n_B = n \qquad (38)$$

as the only correct choice for n_B; the minus sign in (35) and (36) holds in case of the change of the direction of B to an opposite one. In fact a special behaviour of the lowest inner shell (K-shell) of the atom is a quantum-mechanical effect absent in a semiclassical treatment of the atomic problem. It should be noted that the factor of $2e$ in the denominator of the last step in (36) is also a result of a quantum-mechanical calculation of the angular momentum in (35). But the same value for $L_B = L_n$ in (20) has been applied in the foundations of the Bohr orbital model, see the end of Sec 12.

Since for E equal to the electric field intensity along the orbit there holds the formula

$$\frac{\partial \Phi}{c\,\partial t} = \frac{\partial(BS)}{c\,\partial t} = \frac{1}{} E\, dt = 0, \qquad (39)$$

so the electron energy parameter E_n, as well as the angular momentum and magnetic flux on any level n, are conserved; see e.g. [18].

To this purpose an extension of the well-known Heisenberg principle of uncertainty for energy and time given by the relation [43, 41]

$$\Delta E \Delta t > n \qquad (40)$$

is taken into account. In fact the formula (40) has been objected on many

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occasions [44, 15, 42] and numerous textbooks on quantum mechanics neglect (40) at all; see e.g. [11, 5]. An extension of (40) is [47, 27, 28]:

$$2mc^2 \Delta E(\Delta t)^2 > n^2. \tag{41}$$

Here, as well as in (40), ΔE is the interval of energy and Δt the interval of time taking part in the quantum process.

If we assume that ΔE is the excitation energy of a free electron particle from the level $E = 0$ to the level $E = \Delta E$, we have

$$\Delta E = \frac{p_x^2}{2m} \tag{42}$$

where p_x is the electron momentum. On the other hand we have

$$p_x = mv_x = m\frac{dx}{dt} \cong m\frac{\Delta x}{\Delta t} \tag{43}$$

obtained in effect of one of the Hamilton equations

$$\frac{dx}{dt} = \frac{\partial(\Delta E)}{\partial p_x} = \frac{p_x}{m}. \tag{44}$$

A transformation of (43) gives

$$\Delta t = \frac{m}{p_x}\Delta x. \tag{45}$$

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The expression (45) substituted together with (42) into (41) gives [29]

$$2mc^2 \frac{p_x^2}{2m} \frac{m}{p_x^2} (\Delta x)^2 = m^2 c^2 (\Delta x)^2 > n^2. \qquad (46)$$

This implies that

$$\Delta x > \frac{n}{mc}, \qquad (47)$$

so the smallest admissible quantum of distance is

$$\Delta x_0 = \frac{n}{mc}. \qquad (48)$$

The smallest acceptable interval of time can be obtained from (48) by requirement that the distance Δx_0 is travelled with a maximal speed close to c, so

$$\Delta t_0 = \frac{\Delta x_0}{c} = \frac{n}{mc^2}. \qquad (49)$$

The formulae similar to (48) and (49) have been derived a time ago on both the experimental and theoretical footing [40, 18, 6, 17]. Instead of Δx_0 in (15) the Compton length

$$\Delta x_0^{Compton} = \frac{h}{mc} \qquad (48a)$$

has been there proposed. It is larger than (48) by a factor of 2π. Preliminary calculations on the hydrogen atom done with the aid of (41) are presented in   [35].

6   Elementary quantum of distance (48) and the electron spin

We assume that the trajectory of the spinning electron particle is represented by a circle having the radius equal to $\triangle x_0$ in (48) (see [25]):

$$r_c = \frac{n}{mc} = \frac{v_c}{\Omega_c}. \tag{50}$$

This distance is coupled in (50) with the circulation frequency

$$\Omega_c = \frac{eB_c}{mc} \tag{51}$$

[cf. (23)] and the electron speed represented approximately by

$$v_c \cong c. \tag{52}$$

A general warning on the treatment of spin is that it should not be seeked as a result of the circulation effect of a particle about its own axis (see e.g. [15]), and this view is shared also in the present approach. But instead of the motion about an axis which crosses the particle body, a charged particle may perform its spontaneous circulation in the magnetic field about an axis located outside the particle mass. The sense of such behaviour is – as we shall see – that in effect of the particle interaction with the magnetic field created by the particle motion, the particle energy becomes much lowered

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below the zero value of energy which can be assumed to be associated with the particle at rest.

In defining the position of the axis of the particle circulation in the magnetic field, the uncertainty relation for energy and time can be of use [20, 21, 22]. Beyond of time t and energy E, the principle contains also a reference to the particle mass m and the speed of light c:

$$2m_e c^2 \, \Delta E (\Delta t)^2 > \text{n.} \tag{53}$$

Evidently the rule (53) derived for electrons in [20, 21, 22, 26] does apply to the particles which obey the Fermi statistics. But, for example, instead of electrons of the mass $m = m_e$ considered in [20, 21, 22, 26], we can have also the gas of the proton particles of the mass $m = m_p$ distributed in the field of a negative background which makes the gas electrically neutral. A reasoning of [20, 21, 22, 26] repeated in the case of an ensemble of the proton particles gives the result

$$2m_p c^2 \, \Delta E (\Delta t)^2 > \text{n.} \tag{54}$$

This makes (54) different from (53) solely by a replacement of m_e in (53) by m_p in (54). Certainly ΔE and Δt in (54) refer to the proton particle.

A consequence of the principle in (53) and (54) is a rule that two Fermi particles of the same kind cannot approach together to an arbitrarily small distance but they should be separated at least by the interval which – in view

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of (53) – is equal to [26]

$$\Delta x_e = \frac{n}{m_e c} \qquad (55)$$

for electrons, but becomes equal to

$$\Delta x_p = \frac{n}{m_p c} \qquad (56)$$

for the protons case; see e.g. [44] for the proton mass, spin angular momentum and spin magnetic moment.

The minimal distances (55) and (56) between particles represent respectively the Compton length of the electron and proton particle, on condition the rationalized Planck constant n is replaced by the original Planck constant h. The kind of the formulae given in (55) and (56) has been derived before in [37, 7, 8]; see also [10].

In Sec. 12 we apply (55) and (56) to define the positions of the axes of a spontaneous particle circulation giving, respectively, the electron and the proton spin. Before these motions take place we assume that the particle energy of the electron (E_e) and proton (E_p) is at zero:

$$E_e = E_p = 0. \qquad (57)$$

7   Spinning process of the electron and proton

A general law of physics is that any particle tends to assume a possibly lowest level of energy. In case of a charged particle this can be attained in effect of the particle circulation about some axis along which the particle motion induces the presence of the magnetic field. This situation implies that the kinetic energy of the orbital motion is associated with a particle. The axis of the motion can be located outside the extension area of the particle mass. As a distance of the axis from the particle location (r_e for the electron and r_p for proton) let us assume that (55) and (56) hold respectively in the electron and proton case.

The magnetic field B causes the velocity v along a circle normal to B, and the balance of the forces requires that

$$eB = m\frac{v^2}{r} \tag{58}$$

where $m = m_e$ or m_p, $v = v_e$ or v_p, and $r = r_e$ or r_p. In effect the force in (58) represents an equilibrium between the force of the field and the mechanical force due to the acceleration of a particle toward the track center (see e.g. [42]). We postulate that

$$r_e = \Delta x_e \tag{59}$$

in the case of the electron particle, and

$$r_p = \Delta x_p \qquad (60)$$

in the proton case.

The mechanical angular momenta of electron and proton become respectively

$$L_e = m_e r_e v_e = m_e r_e^2 \omega_e, \qquad (61)$$

$$L_p = m_p r_p v_p = m_p r_p^2 \omega_p. \qquad (62)$$

For the sake of simplicity the same size of charge $|e|$ for the electron and proton is assumed.

The ω_e and ω_p in (61) and (62) are the electron and proton circulation frequencies equal to

$$\omega_e = \frac{|e| B_{ce}}{m_e c}, \qquad (63)$$

$$\omega_p = \frac{|e| B_{cp}}{m_p c}. \qquad (64)$$

The B_{ce} and B_{cp} are the strengths of the magnetic field suitable for the electron and proton case. For both kinds of particles we assume that the strength of B is so large that electron or proton gyrate in the magnetic field with a speed close to c. This requirement for the particle velocity is dictated

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by examination of the particle acceleration expressed in terms of the electric field E and magnetic field B [14]. In this case

$$\frac{dv}{dt} = \frac{e}{m}\sqrt{1 - \frac{v^2}{c^2}}\left[E + \frac{1}{c} v \times B - \frac{1}{c^2} v(vE) \right].$$ (65)

Evidently the acceleration (65) vanishes when the particle velocity becomes a constant $v = c$. Thus we have

$$v_e = r_e \omega_e = r_e \frac{|e|B_{ce}}{m_e c},$$ (66)

$$v_p = r_p \omega_p = r_p \frac{|e|B_{cp}}{m_p c},$$ (67)

and

$$v_e = v_p = c.$$ (68)

With the aid of (55) and (56) we obtain from (66), (67) and (68):

$$B = B_{ce} \cong \frac{m_e c^2}{r_e |e|} = \frac{m_e^2 c^3}{n|e|}$$ (69)

and

$$B = B_{cp} \cong \frac{m_p c^2}{r_p |e|} = \frac{m_p^2 c^3}{n|e|},$$ (70)

on condition the absolute values of B are taken into account. The orbital radii r_e and r_p [see (59) and (60)] substituted together with the velocities of (68)

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into the formulae (61) and (62) for the angular momentum give respectively

$$L_e \cong m_e \frac{n}{m_e c} c = n \qquad (71)$$

for the electron and

$$L_p \cong m_p \frac{n}{m_p c} c = n \qquad (72)$$

for the proton particle. In effect we have

$$L_e = L_p \cong n. \qquad (73)$$

Evidently the formulae obtained in (71)–(73) do not depend on the particle mass. But a mass dependent parameter becomes the magnetic moment M of a particle. For the electron case we obtain:

$$M_e = \frac{þ}{2m_e c} L_e = \frac{þ h}{2m_e c} = M_B \qquad (74)$$

(which is the Bohr magneton) and for proton

$$M_p = \frac{þ}{2m_p c} L_p = \frac{þ h}{2m_p c}, \qquad (75)$$

called also the theoretical nuclear magneton applied in considering the nuclear particles [44]. The ratio between (75) and (74) is defined by

$$M_p / M_e = m_e / m_p \qquad (76)$$

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which is not very far from the ratio obtained from the experimental data for the magnetic moment of electron and proton [44]. In many cases the experiments performed on the nuclear magnetic momenta M_n give the ratio M_n/M_e not much different from m_e/m_n where m_n is the nuclear mass.

The energy of a spinning particle in the magnetic field is respectively represented by

$$E_e = -B_{ce}M_e = \frac{1}{2}m_e c^2 \qquad (77)$$

for an electron, and by

$$E_p = -B_{cp}M_p = \frac{1}{2}m_p c^2 \qquad (78)$$

for a proton. Therefore the gain of energy in the magnetic field due to formation of the particle spin is large. This gain of energy is expensed to provide the kinetic energy to a spinning particle having its velocity close to c.

8   Magnetic flux of a spinning particle, conservation of energy and quantization of the spin motion

A parameter concerning spin which has its established experimental counterpart is the magnetic flux. Let us choose for an elementary planar area of

that flux the circle

$$S_e = \pi r_e^2 = \pi(\Delta x_e)^2 = \pi \left(\frac{n}{m_e c}\right)^2 \tag{79}$$

for electrons, and the circle

$$S_p = \pi r_p^2 = \pi(\Delta x_p)^2 = \pi \left(\frac{n}{m_p c}\right)^2 \tag{80}$$

for protons. From (79), (80), as well as for the magnetic field strength taken respectively from (69) and (71), we obtain

$$\Phi_{ce} = B_{ce} S_e = \frac{m_e^2 c^3}{n|e|} \pi \frac{n^2}{m_e^2 c^2} = \frac{1}{2} \frac{hc}{|e|} \tag{81}$$

and

$$\Phi_{cp} = B_{cp} S_p = \frac{m_p^2 c^3}{n|e|} \pi \frac{n^2}{m_p^2 c^2} = \frac{1}{2} \frac{hc}{|e|} \tag{82}$$

respectively in the electron and proton case. An evident result is that

$$\Phi_{ce} = \Phi_{cp} = \frac{1}{2} \frac{hc}{|e|}. \tag{83}$$

Therefore the flux extended over the elementary areas in (79) and (80) does not depend on the particle kind represented by the particle mass. Moreover, the flux calculated in (81) and (82) is equal to a constant quantum term observed experimentally since a long time in superconductors [11].

The time derivative of the flux term is zero, so we have the fundamental

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relation of electrodynamics

$$\frac{1}{L} \oint \boldsymbol{E}\, d\boldsymbol{l} = -\frac{d}{dt} \int_S \boldsymbol{B}\, d\boldsymbol{S} = -\frac{d}{dt}\left(\frac{hc}{2|e|}\right) = 0. \tag{84}$$

Physically this means that a linear integral over $\boldsymbol{E}$ representing the electric field along a circular path of the electron is equal to zero, therefore the energy of the circular motion in the magnetic field of B is conserved.

Having the magnetic flux

$$\pi r_c^2 B_c = \frac{hc}{2|e|} = \Phi_c \tag{85}$$

the spin motion can be quantized according to a rule of the old quantum theory [42, 33]. It originates from a general rule given by Sommerfeld that momentum $\boldsymbol{p}$ integrated over a closed path $d\boldsymbol{r}$ of the particle motion should be a multiple of the Planck constant h:

$$\oint \boldsymbol{p}\, d\boldsymbol{r} = nh; \tag{86}$$

here n is usually considered as an integer number. But according to [42] equation (86) can be transformed into

$$\frac{\pi r_c^2 |e| B_c}{c} = nh. \tag{87}$$

By taking into account the first equation in (85) we obtain for (87) the

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relation

$$\frac{h}{2} = nh \tag{88}$$

from which the spin quantum number becomes:

$$n = 1/2. \tag{89}$$

This is a well-known result confirmed experimentally by the measurements on the gyromagnetic ratio in ferromagnets [2] performed a time before the spin discovery [45].

9 Drift velocity of a spinning electron in the electric field of the proton nucleus

Till the present time no other field than B_c spontaneously created by a spinning particle has been considered. Now let us assume that the spinning electron meets the electrostatic field of the proton nucleus. A minimal distance which can appear between the electron moving particle and the proton being at rest is defined in (55) because (56) is too small to have a decisive influence. In this case

$$eE_{ce} = \frac{e^2}{(\Delta x_{min})^2} = \frac{e^2}{\left(\frac{n}{m_e c}\right)^2} = \frac{(em_e c)^2}{n^2}, \tag{90}$$

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where E_{ce} is the absolute value of the electric field acting on the electron. Another force acting on the electron is eB_{ce} where B_{ce} is the magnetic field intensity of the electron spin; see (69). Assuming that E_{ce} is normal to B_{ce}, the driving electron velocity obtained as a result of the joined action of both fields is [18]

$$v_d = c\frac{|E_{ce} \times B_{ce}|}{B_{ce}^2} = c\frac{E_{ce}B_{ce}}{B_{ce}^2} = c\frac{E_{ce}}{B_{ce}}. \tag{91}$$

But it is easy to check from (69) and (90) that

$$\frac{E_{ce}}{B_{ce}} = \frac{e^2}{nc} = \alpha \approx \frac{1}{137} \tag{92}$$

is the fine-atomic-structure constant [15, 40], so

$$v_d = c\frac{e^2}{nc} = \frac{e^2}{n}. \tag{93}$$

The result in (93) is precisely the electron velocity on the lowest orbit of the Bohr atom [3]. Therefore a combined action of the spin magnetic force of the electron and electrostatic force acting between electron and the proton nucleus, gives the speed of electron equal to that possessed on the lowest quantum state in the hydrogen atom. The spin action of the proton on the electron spin moment present on the orbit has been neglected.

In effect the velocity along the lowest orbit of the Bohr's hydrogen atom can be considered as a consequence of a drift motion being a result of su-perposition of many spinning rotations along very small orbits having their

radii equal to (59) and travelled with a speed equal to c. The time necessary to travel along the Bohr orbit having the well-known radius

$$a_B = \frac{n^2}{m_e e^2} \qquad (94)$$

is

$$T_1 = \frac{2\pi a_B}{v_d} = \frac{2\pi n^2}{m_e e^2} \cdot \frac{n}{e^2} = \frac{2\pi n^3}{m_e e^4}, \qquad (95)$$

whereas the travel time along the spin orbit calculated from (59) and (55) is equal to

$$T_2 = \frac{2\pi r_e}{c} = 2\pi \frac{n}{m_e c^2}. \qquad (96)$$

In consequence the number of spinning circular motions which take place in course of the electron drift along the first Bohr orbit is equal to

$$\frac{T_1}{T_2} = \frac{2\pi n^3}{m_e e^4} \cdot \frac{m_e c^2}{2\pi n} = \frac{n^2 c^2}{e^4} = \frac{1}{\alpha^2} \simeq 137.04^2 \approx 18780. \qquad (97)$$

This is a number independent of the mass m_e. A diagram presenting schematically the motion of a spinning electron along the lowest Bohr orbit in the hydrogen atom is given in Fig. 1. The circular frequency of a spinning electron is

$$\frac{2\pi}{T_2} = \frac{m_e c^2}{n} = 0.78 \times 10^{21} \text{ sec}^{-1}. \qquad (98)$$

The mass m_e has to be replaced by m_p in case of a spinning proton frequency
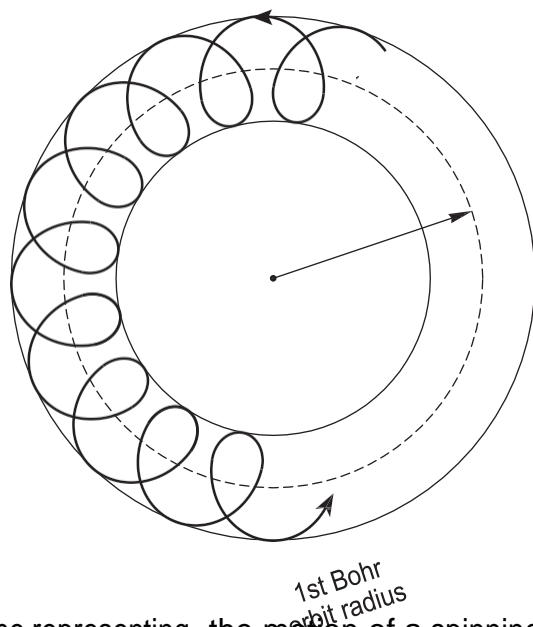
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Figure 1: A scheme representing the motion of a spinning electron along the shortest (lowest) circular Bohr orbit of the hydrogen atom. The orbit circle is represented by a dashed line, the separation distance between two circles enclosing the motion is twice the radius r_e given by the formulae (55) and (59). For the number of the spin oscillations along the orbit see formula (97).

10 Quanta of energy change in the hydrogen atom and the intervals of time necessary to supply them

The rate of change of the kinetic energy E_{kin} of an electron in course of the time interval dt is [14]

$$\frac{dE_{kin}}{dt} = e\bar{E}\,v; \qquad (99)$$

e is the electron charge, $\bar{E}$-intensity of the electric field, v-the electron velocity. The formula (99) can be transformed into

$$dE_{kin} = e\bar{E}\,v\,dt = e\bar{E}\,d\bar{r} \qquad (100)$$

where $d\bar{r}$ is an element of the electron path. For a closed path of the electron present on a quantum level in the atom we obtain

$$E_{kin} = e \oint \bar{E}\,d\bar{r} = -E_{tot} = \text{const} \qquad (101)$$

where E_{tot} is the total electron energy on the level. Relation (101) is satisfied because of the virial theorem

$$2E_{kin} + E_{pot} = 0 \qquad (102)$$

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which gives

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = -E_{\text{kin}}. \tag{103}$$

E_{pot} is the potential energy of the elctron.

But a situation of a constant E_{kin} in (101) can be changed due to a transition process of the electron between two quantum levels. In this case the formalism of electrodynamics implies [14]

$$\Delta E_{\text{kin}} = \Delta e \int \boldsymbol{E}\, d\boldsymbol{l} \longrightarrow -\frac{e}{c}\frac{\partial}{\partial t}\int \boldsymbol{B}\, d\boldsymbol{f} \cong -\frac{e}{c}\frac{\Delta \int \boldsymbol{B}\, d\boldsymbol{f}}{\Delta t}. \tag{104}$$

The integral on the right-hand side of (104) concerns the magnetic field B extended over the area $d\boldsymbol{f}$ circumvented by the electron in course of its travel along a closed path.

In the hydrogen atom any term of the magnetic flux is quantized according to the formula [30]

$$\int \boldsymbol{B}\, d\boldsymbol{f} = \frac{n}{2}\frac{hc}{e} \tag{105}$$

where n is a quantum number of the level. Therefore

$$\Delta \int \boldsymbol{B}\, d\boldsymbol{f} = \frac{\Delta nhc}{2e}. \tag{106}$$

In calculating (105) we noted the quantum properties of B in the hydrogen atom [30] which give

$$B = B_n = \frac{e^3 m^2 c}{n^3 n^3}, \tag{107}$$

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so B_n are constant numbers for any n. Moreover, it is assumed that B_n is extended over a circular area

$$d\boldsymbol{f} = S_n = \pi r_n^2 \qquad (108)$$

circumvented by the electron on level n; the radius of the area is [43]

$$r_n = \frac{n^2 \mathrm{n}^2}{me^2}. \qquad (109)$$

In effect we obtain

$$B_n S_n = \frac{e^3 m^2 c}{\mathrm{n}^3 n^3} \pi \frac{n^4 \mathrm{n}^4}{m^2 e^4} = n \frac{hc}{2e} \qquad (110)$$

which is a result identical with (105). The quanta of (105) and (110) are well known because of their presence in superconductors [12].

The relaton (104) combined with the virial theorem in (102) and the result in (106) implies

$$-\Delta E_{\text{kin}} = \Delta E_{\text{tot}} = \frac{e}{c} \Delta n \frac{hc}{2e} \frac{1}{\Delta t} \qquad (111)$$

or

$$\Delta E_{\text{tot}} \Delta t = \frac{1}{2} h \qquad (111a)$$

on condition we put

$$\Delta n = 1. \qquad (112)$$

If we note that both kinds of the electron orbital motion – a clockwise

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and counterclockwise – are equally possible, we may consider (111a) as a half of the result expected for both kinds of the circular motion. The both kinds of the motion taken together give evidently

$$\Delta E_{\text{tot}} \Delta t = h. \qquad (113)$$

The product in (113) is identical with that quoted in ( ).

The aim of the present paper is to avoid a statistical way of calculations and try to approach the time of transition for an individual electron particle. The first step on this way is to point out that formula (53) does not represent the only manner on which the energy of a quantum system can be changed. For, beyond of the radiation transition of energy, we can have also a dissipation process of that energy similar to production of the heat in course of a transport process of electrons in a metal. In such a process the energy change is supplied by the electric current. In fact the stationary states in many quantum systems have a time-dependent periodic nature in general. A well-known example is the electron state in the hydrogen atom which semiclassically is considered to be a particle incessibly gyrating about the atomic nucleus; see e.g. [43]. This implies a permanent electric current connected with such electron motion in any of its quantum states.

In consequence, when the quantum state is changed, there exists also a change of energy supplied by the current. This change can be regulated by the Joule-Lenz law for the production rate of the current dissipation energy

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[9, 18]

$$Q = \frac{\Delta E}{\Delta t} = R i^2. \tag{114}$$

Here R is the current resistance, i is the current intensity.

In the present paper we demonstrate that in quantum systems ΔE and Δt entering (114) are coupled together in a definite way. This property makes it possible to obtain the time interval Δt associated with the energy change ΔE.

$$i = \frac{e}{T_n} = \frac{me^5}{2\pi n^3 \mathrm{n}^3} \tag{115}$$

for each quantum level n.

The electron dissipation energy is assumed to be equal to the electron transition energy between two neighbouring quantum levels $n + 1$ and n:

$$\Delta E_n = -\frac{me^4}{2\mathrm{n}^2} \left[\frac{1}{(n+1)^2} - \frac{1}{n^2} \right]. \tag{116}$$

This energy is equivalent in both quantum-mechanical and semiclassical (Bohr's) formalisms.

But in the framework of quantum mechanics the time of electron transitions between quantum levels of the atomic hydrogen, combined with the problem of intensity of transitions, is approached on the basis of a complicated probabilistic background; see e.g. [4].

In the present paper, however, we can limit the problem to a single electron transition by calculating first the electric resistance associated with the

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orbit change. Next, having such resistance, it is easy to obtain the time rate with which the electron energy due to the orbit change has been left, on condition we assume that the whole of the energy change is produced in the form of the Joule–Lenz heat.

In effect we obtain the time interval needed to produce the Joule-Lenz heat connected with a given electron transition. Though this time is not necessarily the time of radiation, nevertheless such calculation can give an estimate of an upper limit of the time associated with the electron transition.

A simplification which we assume in calculations is that

$$n \quad 1 \tag{117}$$

which gives from (116)

$$\Delta E_n = \frac{me^4}{n^2} \frac{1}{n^3}. \tag{118}$$

The electric resistance connected with transition between $n + 1$ and n let be

$$R = \frac{V_n}{i_n} \tag{119}$$

where the potential change is

$$V = \frac{\Delta E_n}{e}. \tag{120}$$

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This formula, together with (115) and (119), provides us with the resistance

$$R = \frac{\Delta E_n}{e^2} T_n = \frac{me^4}{n^2 n^3} \frac{2\pi n^3 n^3}{me^4} = \frac{2\pi n}{e^2} = \frac{h}{e^2}. \tag{121}$$

Expression (121) is a resistance result which is well-known from the integer quantum Hall effect: it has been observed for numerous planar free-electron structures; see e.g. [17]. It seems, however, that the presence of the resistance amount in (121) has been neglected in former calculations on small quantum systems, including the hydrogen atom.

A characteristic point is that R in (121) became fully independent of the quantum number n, as well as the size of the mass m.

11 The time necessary for the Joule-Lenz heat expense and product of $\triangle E$ and $\triangle t$

According to classical electrodynamics the time rate of the expense of the Joule-Lenz heat is equal to:

$$Q = \frac{\Delta E_n}{\Delta t_n} = I_n^2 R = \left(\frac{e}{T_n}\right)^2 \frac{h}{e^2} = \left(\frac{2\pi n^3 n^3}{me^4}\right)^{-2} h = h \frac{m^2 e^8}{(2\pi)^2 n^6 n^6}. \tag{122}$$

Evidently from (122) the time interval $\triangle t$ of the heat expense becomes

$$\Delta t_n = \frac{\Delta E_n}{Q} = \frac{me^4}{n^2 n^3} \frac{(2\pi)^2 n^6 n^6}{hm^2 e^8} = \frac{2\pi n^3 n^3}{me^4} = T_n. \tag{123}$$

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This time interval of the expense is – in the present approximation – equal to the time period T_n of the electron circulation in the hydrogen atom; see ( ).

It is of interest to multiply the energy interval ΔE_n in (118) by Δt_n in (123). We obtain

$$\Delta E_n \Delta t_n = \frac{me^4}{n^2 n^3} \frac{2\pi n^3 n^3}{me^4} = 2\pi n = h. \tag{124}$$

In the next Sections we show that results similar to (124) can be obtained for the free-electron particle in a one-dimensional potential box as well as the harmonic oscillator.

A typical application of Δt_n is an expense rate of energy. With the aid of (123) and (124) we have

$$(\Delta t_n)^{-+} = T_n^{-1} = \frac{\Delta E_n}{h}. \tag{125}$$

Evidently in virtue of (125) the emission time rate of the energy ΔE_n is equal to

$$\frac{\Delta E_n}{\Delta t_n} = \frac{(\Delta E_n)^2}{h}. \tag{126}$$

This rate can be compared with quantum-mechanical results [4]. Since (126) is proportional to $(\Delta E_n)^2$ in Table 1 we present the factors entering $(\Delta E_n)^2$ dependent on n, i.e. the expressions

$$(\Delta E_n)^2 \sim \frac{1}{(n+1)^2} - \frac{1}{n^2} {}^2 \tag{127}$$

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calculated according to (117), for the levels beginning with $n = 1$ up to $n = 5$.

The quantum-mechanical results for the total transition probabilities between the quantum states

$$(n + 1)p \longrightarrow ns, \tag{128}$$

where $n = 1, 2, 3, 4$, and 5, and probabilities are expressed in $10^8 \sec^{-1}$ units, are [4]:

$$2p \longrightarrow 1s \quad 6.25 \tag{129}$$

$$3p \longrightarrow 2s \quad 0.22 \tag{130}$$

$$4p \longrightarrow 3s \quad 0.030 \tag{131}$$

$$5p \longrightarrow 4s \quad 0.0075 \tag{132}$$

$$6p \longrightarrow 5s \quad 0.0021 \tag{133}$$

Table 1: Expression (127) entering $(\Delta E_n)^2$ in (126) calculated for different quantum levels n.

n	expression (127)
1	$(3/4)^2$
2	$(5/16)^2$
3	$(7/144)^2$
4	$(9/400)^2$
5	$(11/900)^2$

In Table 2 the ratios of (127), calculated for different pairs of n and $n+1$,

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Table 2: Ratios of expressions (127) entering Table 1 calculated for different pairs of the level indices n and $n+1$ compared with the ratios of the transition intensity data given by the formulae (129)–(133).

n	$n+1$	Indices ratio	Ratios of expressions (127)	Ratios of probabilities from the eqs. (19)–(23)
1	2	$1:2$	$(3/4)^2 : (5/36)^2$ $= 28.7$	$(19):(20) =$ $6.25 : 0.22 = 28.4$
2	3	$2:3$	$(5/36)^2 : (7/144)^2$ $= 8.16$	$(20):(21) =$ $0.22 : 0.30 = 7.33$
3	4	$3:4$	$(7/144)^2 : (9/400)^2$ $= 4.74$	$(21):(22) =$ $0.30 : 0.0075 = 4.00$
4	5	$4:5$	$(9/400)^2 : (11/900)^2$ $= 3.40$	$(22):(23) =$ $0.0075 : 0.0021 = 3.57$

are compared with the ratios of the quantum-mechanical data obtained for the same pairs of n and $n+1$ entering (129)–(133). A parallelism of the quantum-mechanical results with those calculated from the present semiclassical theory is evident.

12 Quanta of the dissipated energy and intevals of time necessary to produce them

In the first step, the aim of the formalism developed in the present paper is to demonstrate that Δt for an electron transition can be a defined quantity similar to ΔE and v. To this purpose the ΔE is referred to the dissipation

energy ΔQ which accompanies the electron transition by the equation:

$$\Delta Q = \Delta E \tag{134}$$

Evidently ΔQ occuring in the transition seems to be of a typically emissive character. In short we assume that the emission of ΔE should not be necessarily of a radiation character dictated by ( ), but can materialize also in the form of ΔQ. Consequencies of such assumption are illustrated on three examples concerning respectively the hydrogen atom, a particle enclosed in the potential box and the harmonic oscillator. In Sec. 11 the results of the present Section are applied in calculating the electron acceleration produced in course of the quantum transitions performed in the mentioned three systems.

The energy differences

$$\Delta E = E_{n+1} - E_n \tag{135}$$

between two neighbouring quantum states having the indices

$$n + 1 \quad \text{and} \quad n \tag{136}$$

(n 1) are mainly considered. We have

$$\Delta E = -\frac{me^4}{2n^2}\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] \approx \frac{me^4}{n^2 n^3} \tag{137}$$

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for the hydrogen atom [43],

$$\Delta E = \frac{h^2}{8mL^2}[(n+1)^2 - n^2] \approx \frac{h^2 n}{4mL^2} \qquad (138)$$

for a free particle having mass m enclosed in a one-dimensional potential box of length L [6], and

$$\Delta E = n\omega \left(n + \frac{3}{2} - n - \frac{1}{2} \right) = n\omega \qquad (139)$$

for the harmonic oscillator having the frequency ω.

The time periods of the electron particle circulation on the orbits are defined by a physical character of each of the above systems. They are

$$T_n = \frac{2\pi n^3 n^3}{me^4} \qquad (140)$$

for the case of electron in the hydrogen atom occupying the state n [43],

$$T_n = \frac{2L}{v_n} = \frac{2L}{\frac{hn}{2mL}} = \frac{4mL^2}{hn} \qquad (141)$$

for the particle of mass m being in state n in the potential box because of the relation

$$E_n = \frac{mv_n^2}{2} = \frac{h^2 n^2}{8mL^2},$$

and

$$T_n = T = \frac{2\pi}{\omega} \qquad (142)$$

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for all states n of the harmonic oscillator.

A characteristic property of expressions (137)–(142) is that

$$\Delta E = \frac{h}{T_n} = \frac{me^4 h}{2\pi n^3 \mathrm{n}^3} = \frac{me^4}{n^3 \mathrm{n}^2} \qquad (143a)$$

holds for the atomic orbit n of the hydrogen [see (65)],

$$\Delta E = \frac{h}{T_n} = \frac{h^2 n}{4mL^2} \qquad (144a)$$

for a free particle (free electron) in state n in the box [see (66)], and

$$\Delta E = \frac{h}{T_n} = \frac{h}{T} = \frac{h}{2\pi}\omega = \mathrm{n}\omega \qquad (145a)$$

for the harmonic oscillator [see (139) and (142)]. A common feature of (143a)–(145a) is that

$$\Delta E = \frac{h}{T_n}. \qquad (146)$$

If we note that the resistance R connected with any of transitions examined above can be defined by the ratio

$$R = \frac{V}{i} = \frac{\Delta E}{ei} \qquad (147)$$

where V is a potential and i is a current intensity given by the electron particle of charge e, so

$$i = \frac{e}{T_n}, \qquad (148)$$

we obtain

$$R = \frac{\Delta E}{e^2} T_n = \frac{\Delta E}{e^2} \frac{h}{\Delta E} = \frac{h}{e^2}. \tag{149}$$

In the last step in (149) the result of (146) is taken into account.

The R in (149) is equal to the well-known quantum of resistance examined experimentally in the planar free-electron structures [17]; some theoretical results connected with R are presented in [23, 24, 31].

Let the dissipation heat ΔQ of the quantum emission process satisfy the Joule–Lenz equation [16, 18]

$$\frac{\Delta Q}{\Delta t} = R i^2. \tag{150}$$

By putting

$$\Delta Q = \Delta E \tag{151}$$

[see (134)] we obtain from (148)–(151) the following relation

$$\frac{\Delta E}{\Delta t} = \frac{h}{e^2} \left(\frac{e}{T_n}\right)^2 = \frac{h}{T_n^2} \tag{152}$$

which gives

$$\frac{\Delta E \, T_n^2}{\Delta t} = h \tag{153}$$

But because of (146) the formula (153) can be transformed into

$$\frac{T_n^2}{\Delta t} = \frac{h}{\Delta E} = T_n \tag{154}$$

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which implies that

$$\Delta t = T_n. \tag{155}$$

In effect of (155) the relation (153) can be presented in a more familiar form:

$$\Delta E \Delta t = h; \tag{156}$$

cf. here ( ).

A comparison of the time rate of energy emission calculated according to the method presented above with the quantum-mechanical method is done in [32].

13 A check of the theory: acceleration of electrons obtained in effect of their quantum transitions

The physics of the test is much similar to that entering the Tolman experiment [18]. The point is that the change of the electron energy in course of its transition between the quantum levels is accompanied by a change Δv of the velocity which the electron has along its path. For example in the hydrogen atom the emission of energy from some level $n+1$ to level n is associated with an increase of the tangential velocity along the orbit. Let us assume that the expense of energy connected with acceleration is approximately equal to the

emission energy ΔE. In this case

$$\Delta E \approx m \frac{\Delta v}{\Delta t} l \qquad (157)$$

where the time of emission Δt is also a time of the acceleration Δv. Consequently the length l of the electron path covered in course of acceleration should approach that obtained during the electron circulation $\Delta t = T_n$; see (155). In effect instead of (157) we should have a transformed relation

$$\frac{\Delta E}{m l_n} = \frac{|\Delta v|}{T_n} \qquad (157a)$$

where l_n is the path travelled in course of T_n. The calculations of (157a) effectuated for the quantum systems examined in Sec. 13 are presented below. They seem to confirm the validity of (157a). The absolute value of Δv is taken in (157a) in order to make this formula applicable to both positive and negative cases of acceleration; see [18].

Beginning with the hydrogen atom we have ΔE in the formula (137), the velocity in state n is [43]

$$v_n = \frac{e^2}{n\hbar}, \qquad (158)$$

$$|\Delta v| = \frac{e^2}{n\hbar} - \frac{e^2}{(n+1)\hbar} \approx \frac{e^2}{n^2\hbar}, \qquad (158a)$$

$$\Delta t = T_n \qquad (158b)$$

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is given in (140), and

$$I_n = 2\pi r_n = 2\pi \frac{n^2 \mathrm{n}^2}{me^2};$$
(159)

r_n is the nth orbit radius [43]. A substitution of parameters (137) and (159) into the left-hand side of (157a) gives the following expression

$$\frac{\Delta E}{m I_n} = \frac{\Delta E}{m 2\pi r_n} = \frac{me^4}{\mathrm{n}^2 n^3} \frac{1}{m} \frac{me^2}{2\pi n^2 \mathrm{n}^2}.$$
(160)

On the other side, a substitution of the absolute value of Δv given in (158a) and the results of (158b) and (140) give for the right-hand side of (157a) the formula

$$|\Delta v| \frac{1}{T_n} = \frac{e^2}{n^2 \mathrm{n}} \frac{me^4}{2\pi n^3 \mathrm{n}^3}.$$
(161)

Evidently both expressions (160) and (161) are equal:

$$(160) = (161).$$

A similar operation can be repeated for the electron in the potential box. Here [see (141) and equation below of it]

$$v_n = \left(\frac{2E_n}{m}\right)^{1/2} = \left(\frac{2n^2 h^2}{8mL^2} \frac{1}{m}\right)^{1/2} = \frac{nh}{2mL}$$
(162)

so

$$\Delta v_n = \frac{n+1-n}{2mL} h = \frac{h}{2mL}.$$
(163)

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The time period in state n is that given in (141) and

$$I_n = 2L \qquad (164)$$

for any state n. Therefore the left-hand side of (157a) becomes

$$\frac{\Delta E}{ml} = \frac{nh^2}{4mL^2} \frac{1}{2L} \frac{1}{m} = \frac{nh^2}{8m^2L^3} \qquad (165)$$

and the right-hand side is

$$\frac{\Delta v_n}{\Delta t} = \frac{\Delta v_n}{T_n} = \frac{h}{2ml} \frac{nh}{4mL^2} = \frac{nh^2}{8m^2L^3}. \qquad (166)$$

In effect we obtain

$$(165) = (166)$$

which is a similar property to that calculated above in the case of the hydrogen atom.

The case of the harmonic oscillator is rather different than that of the electron in the hydrogen atom or the potential box because the velocity v_n ceases to be a constant number within the oscillation time period T: it changes gradually from zero value at the turning ponts of the oscillator to a maximal absolute value at the central point of the oscillator motion which is represented by the Hamiltonian:

$$H = \frac{p_x^2}{2m} + k\frac{x^2}{2} \qquad (167)$$

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where k is the force constant. In consequence the formalism described in (157) and (157a) is applied solely to the velocity acceleration at a single point

$$x = 0. \tag{168}$$

The electron velocity v_n in state n and point (168) is obtained from the formula

$$\frac{mv_n^2}{2} \cong \hbar n\omega = E_n \tag{169}$$

where E_n is the oscillator energy in state n. This gives

$$v_n = \left(\frac{2nn\omega}{m}\right)^{1/2}. \tag{170}$$

By considering solely the positive sign in (170) the increment of velocity due to the change of the quantum state becomes

$$\Delta v_n \cong \left(\frac{2n\omega}{m}\right)^{1/2} \left[(n+1)^{1/2} - n^{1/2}\right] = \left(\frac{2n\omega}{m}\right)^{1/2} \frac{1}{(n+1)^{1/2} + n^{1/2}}$$
$$\cong \left(\frac{2n\omega}{m}\right)^{1/2} \frac{1}{2n^{1/2}}. \tag{171}$$

The length l_n is

$$l_n = 4a_n \tag{172}$$

where a_n – the amplitude of the oscillator – is coupled to the oscillator energy in state n by the formula

$$E_n = \frac{1}{2}m\omega^2 a_n^2 \tag{173}$$

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so

$$a_n = \left(\frac{2E_n}{m\omega^2}\right)^{1/2} \cong \left(\frac{2nn\omega}{m\omega^2}\right)^{1/2}. \tag{174}$$

In effect the left-hand side of equation (157a) becomes

$$\frac{\Delta E}{ml} = \frac{n\omega}{m4a_n} = \frac{n\omega}{4m}\left(\frac{m\omega^2}{2nn\omega}\right)^{1/2} \tag{175}$$

and the right-hand side of (157a) is

$$\frac{\Delta v_n}{\Delta t} = \frac{\Delta v_n}{T} = \left(\frac{2n\omega}{m}\right)^{1/2} \frac{1}{2n^{1/2}} \cdot \frac{\omega}{2\pi}. \tag{176}$$

In consequence we obtain an approximate equality of both sides of (157a) represented by the relation

$$4 \times (175) = 2\pi \times (176)$$

but not precisely the relation

$$(175) = (176).$$

14 Energy emission and its time in the case of

$\Delta n \gg 1$

In preceding Sections the case of the neighbouring quantum states $n+1$ and n has been mainly considered. An opposite situation is the case of

$$\Delta n \gg 1. \tag{177}$$

For example for the hydrogen atom the situation (177) implies

$$\Delta E = \frac{me^4}{2n^2 n^2}; \tag{178}$$

see (65) where $n+1$ is replaced by an almost infinite number $n + \Delta n$. The formula (178) is $n/2$ times larger than (65) and for $n = 1$ we have

$$\Delta E = \Delta E_1 = \frac{me^4}{2n^2}. \tag{179}$$

Assuming that the end state of the energy emission has the index $n = 1$ the time interval of the emission becomes [see (68)]

$$\Delta t = T_1 = \frac{2\pi n^3}{me^4}. \tag{180}$$

The length of the electron path covered within the time period T_1 is

$$l_1 = 2\pi r_1 = 2\pi \frac{n^2}{me^2}; \tag{181}$$

see (84).

Our aim is to check the validity of the formula (157a) for the case of the emission from the state $n \quad 1$. In this case the absolute value of the electron velocity change between $n \quad 1$ and $n = 1$ is

$$\Delta v_1 = \frac{e^2}{n} \left(\frac{1}{1} - \frac{1}{n+1} \right) \cong \frac{e^2}{n}. \tag{182}$$

We substitute on the left-hand side of (157a) the quantities ΔE_1 and l_1 respectively from (179) and (181) so

$$\frac{\Delta E_1}{ml_1} = \frac{\Delta E_1}{m2\pi r_1} = \frac{me^4}{2n^2} \frac{1}{m} \frac{me^2}{2\pi n^2}. \tag{183}$$

On the right-hand side of (157a) a substitution of Δv_1 calculated from the formula (182) should be done together with the time expression for Δt given in (180). We obtain the acceleration expression

$$\frac{\Delta v_1}{\Delta t} = \frac{\Delta v_1}{T_1} = \frac{e^2}{n} \frac{me^4}{2\pi n^3}. \tag{184}$$

We find that the left-side of (157a) presented in (183) differs from the right-

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hand side presented in (184) solely by a factor of 2:

$$2 \times (183) = (184).$$

It is easy to demonstrate that a particle in the potential box and the harmonic oscillator submitted to the check given by the equation (157a) do not satisfy this equation. For an electron in the potential box the energy ΔE coming from a transition between the state n 1 and state $n = 1$ is approximately proportional to n^2 [the left side of (157a)], whereas the velocity change entering the right-hand side of (157a) is solely proportional to n. A similar discrepancy between the both sides of (157a) occurs for the harmonic oscillator. Here the transition energy between n 1 and $n = 1$ is approximately proportional to n but the velocity change in the central point of the oscillator is approximately proportional only to the square root of n.

It should be noted that for the transition Δn 1 in the hydrogen atom the product of ΔE [see (179)] and Δt [see (180)] gives

$$\Delta E\, \Delta t = \Delta E_1\, T_1 = \frac{me^4}{2n^2}\,\frac{2\pi n^3}{me^4} = \frac{h}{2}. \qquad (185)$$

This result is different from a similar product calculated in the case on $\Delta n = 1$ [see (81)] solely by the factor of $1/2$.

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15   Poynting's vector in the hydrogen atom and the emission time

We define [16, 18, 9]

$$S_p = \frac{c}{4\pi} \left[\vec{E} \times \vec{B} \right] \tag{186}$$

as Poynting's vector. The time rate of the loss of energy is [9]

$$\frac{dQ}{dt} = -\frac{1}{8\pi} \frac{d}{dt} \int (E^2 + B^2)\, d\tau + \frac{c}{4\pi} \int_S \left[\vec{E} \times \vec{B} \right] dS. \tag{187}$$

A well-known formal asymmetry of the Bohr model of the hydrogen atom is the presence of the electric field strength

$$|E_n| = \frac{e}{r_n^2} \tag{188}$$

in the orbit plane for any quantum state n, but this presence is combined with the absence of a similar magnetic field strength B_n. In fact B_n are also present in the hydrogen atom if we note that the electron is circulating along the orbit n having the radius r_n [see (84)] with the velocity v_n [see (83)]. In
effect we obtain the relation

$$\frac{2\pi r_n}{T_n} = \omega_n r_n = v_n \tag{189}$$

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[T_n given in (68)] which supplies the frequency

$$\omega_n = \frac{2\pi}{T_n} = \frac{v_n}{r_n} = \frac{e^2}{nn}\frac{me^2}{n^2n^2}. \tag{189a}$$

With the electron circulating with frequency (189a) is associated the field strength B_n according to the well-known formula [42]:

$$\omega_n = \frac{eB_n}{cm}. \tag{190}$$

In effect

$$B_n = \frac{e^3m^2c}{n^3n^3} \tag{191}$$

which is the size of a vector normal to the orbit plane.

A substitution of E_n and B_n into the expression (186) gives for the absolute value of the Poynting's vector the expression

$$|\vec{S}_p| = \frac{c}{4\pi}\left|\vec{E}_n \times \vec{B}_n\right| = \frac{c}{4\pi}\frac{e}{r_n^2}\frac{e^3m^2c}{n^3n^3}. \tag{192}$$

Since E_n and B_n are numbers constant in time, the time derivative of the expressions E^2 and B^2 composed of these vectors being on the right-hand side of (187) should vanish. There remains solely the surface integral of the Poynting vector value given in (192). This is a toroidal surface which encloses the electron orbit as the torus axis. Approximately the torus surface becomes

equal to a surface of a thin cylinder having its axis length of the size

$$2\pi r_n. \tag{193}$$

On the other side, the cross-section of the toroidal cylinder is dictated by the radius (see e.g. [18])

$$r_e = \frac{e^2}{mc^2} \tag{194}$$

of the electron particle which moves along the orbit. In effect the toroidal surface is approximated by the product of (193) and the circumference of the cross-section of the torus cylinder which is

$$2\pi r_e. \tag{195}$$

The value of the Poynting vector for a thin electron orbit can be assumed as a constant number given in (192), therefore a non-vanishing term on the right of (187) becomes equal to

$$|S_p| 2\pi r_n 2\pi r_e = \frac{c}{4\pi} \frac{e}{r_n^2} \frac{e^3 m^2 c}{n^3 n^3} 2\pi r_n 2\pi \frac{e^2}{mc^2} = \pi \frac{e^8 m^2}{n^5 n^5}. \tag{196}$$

This is a product of (192), (193) and (195). In effect the equation (187) has the form

$$\frac{dQ}{dt} = \frac{\Delta Q}{\Delta t} = |S_p| 2\pi r_n 2\pi r_e = \pi \frac{e^8 m^2}{n^5 n^5}. \tag{197}$$

Since the emitted energy in course of the electron transition between levels

$n + 1$ and n is [see (65)]

$$\Delta Q \approx \Delta E \approx \frac{me^4}{n^2 n^3},$$

the emission time for that energy is

$$\Delta t = \Delta Q \ \frac{\pi e^8 m^2}{n^5 n^5}^{-+} = \frac{me^4}{n^2 n^3} \frac{n^5 n^5}{\pi e^8 m^2} = \frac{n^3 n^2}{\pi me^4} \tag{198}$$

where the term taken in brackets is that calculated in (197).

The result of (198) should be compared with that given by the quantum-mechanical formula (81). This gives

$$\Delta t = \frac{h}{\Delta E} = \frac{h}{\Delta Q} = \frac{h}{me^4} n^2 n^3 = \frac{2\pi n^3 n^3}{me^4} \tag{199}$$

which is a number larger by the factor of

$$2\pi^2 n \tag{200}$$

than that of (198).

This is an expected situation because the emission rate described by the Poynting vector is not restricted to a single transition from level $n+1$ to level n [a limitation which exists in calculations leading to (199)] but concerns emission from n to any level below n.

It seems of interest to demonstrate that B_n in the hydrogen atom can be

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obtained with the aid of the Biot–Savart law [16, 18]. For a constant current intensity $\tilde{j}_n$ along the nth orbit, we have from the Biot–Savart law

$$B_n = \tilde{j}_n \int \frac{\mathbf{r}_n \times d\mathbf{r}_n}{c r_n^3}.$$ (201)

The current $\tilde{j}_n$ along the orbit is defined by the formula

$$\tilde{j}_n = e v_n Q S_e = e v_n \frac{1}{V_e} S_e$$ (202)

where

$$Q^{-1} \sim V_e$$ (203)

is the volume occupied by the electron particle and

$$S_e = \pi r_e^2$$ (204)

is the cross-section area of both of the volume V_e and the electron orbit.

Since the integral of $\mathbf{r}_n \times d\mathbf{r}_n$ leads to result proportional to r_n^2, and r_n is a constant applied in (84), we obtain from (201) and (202):

$$B_n = e v_n \frac{1}{c r_n} \frac{1}{r_e} = e \frac{e^2}{nn} \frac{me^2}{cn^2 n^2} \frac{mc^2}{e^2} = \frac{e^3 m^2 c}{n^3 n^3}$$ (205)

which is a formula identical to that given in (191).

*   * Deficiency of the time intervals for electron transitions in the hydrogen atom and their removal.

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A deficiency of the quantum theory. also that applied in the Bohr atom, was that any calculation of ΔE in (13) could not be combined with a calculation of the time interval deltat necessary to perform the electron transition between the levels n' and n" entering the formula (12). This lack was dictated by the very foundations of the quantum theory which was mainly of a probabilistic nature in its approach to the intensity of transitions from n' to n" or vice versa. This statistical background of the transition time between quantum levels was characteristic not only for quantum mechanics but also existed in the old quantum theory devloped by Planck [ ] and Einstein [ ].

16 Absorption time of energy ΔE compared with the emission time

Contrary to the emission effect, the process of the electron promotion from level n of a lower energy E_n to level $n + 1$ having its energy E_{n+1} higher than E_n can be never spontaneous because of the conservation of energy: the energy amount

$$\Delta E = E_{n+1} - E_n \tag{206}$$

should be first supplied to the electron on level n in order to begin a rise of that electron to level $n + 1$. Consequently some interaction time between, for example, a photon carrying ΔE and the electron on n can be expected. In result of that interaction the E_n is increased by ΔE and the electron way

from level n to $n+1$ may begin. A calculation of the mentioned interaction time seems to be outside of possibility of the present method, but the time necessary for an electron to travel from n to $n+1$ is accessible on the basis of the Ohm's law [ ]. The point is that the electron velocity v_n on the orbit n is slowed down to similar velocity v_{n+1} on the orbit $n+1$, so the velocity difference

$$\Delta v = v_n - v_{n+1} = \frac{e^2}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) \approx \frac{e^2}{n n^2} \tag{207}$$

should be cancelled. The problem of the velocity change (207) is similar to that entering the Tolman experiment [ ] where the original current i in a solenoid having resistance R loses its velocity Δv within the time interval Δt according to the formula

$$i\Delta t = \frac{m}{e}\frac{L}{R}\Delta v. \tag{208}$$

Here L is the length of the path travelled by the current.

If we assume that L is the length of the electron orbit n, so

$$L = 2\pi r_n = 2\pi \frac{n^2 n^2}{me^2} \tag{209}$$

and the current intensity on the level n is

$$i = \frac{e}{T_n} \tag{210}$$

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the formula (208) gives:

$$\Delta t \frac{e}{T_n} = \frac{m}{e} \frac{2\pi n^2 \mathrm{n}^2}{me^2} \frac{1}{R} \frac{e^2}{\mathrm{n}n^2} = \frac{2\pi \mathrm{n}}{eh} e^2 = e \qquad (211)$$

where for R has been substituted the resistance

$$R = \frac{h}{e^2} \qquad (212)$$

characteristic for the energy dissipation; see Sec. . In effect we obtain from (211):

$$\Delta t = T_n \qquad (213)$$

which is the transition time known from the emission process; see ( ). In effect an application of the Ohm's law gives the interval Δt necessary to satisfy the second law of the Newtonian dynamics; see Sec.

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17 Transition time between the electron levels in the atom compared with the relaxation time characteristic for the electron transport process

Let us note that the effective electric field E_{eff} which gives the Ohm relation with the electric current j on the orbit n, viz.

$$j = \lambda E_{\text{eff}} \tag{214}$$

where λ is a constant, should satisfy the relation [Matveev]

$$E_{\text{eff}}\, dl = |E_{\text{eff}}|\, 2\pi r_n = Ri = E \tag{215}$$

where E is the electromotive force. The rate of loss of the dissipated energy ΔQ is

$$\frac{\Delta Q}{\Delta t} = Ri^2 = \frac{1}{2}\frac{e^2 l}{m(v)}|E_{\text{eff}}|^2 = \frac{1}{2}\frac{e^2 l}{m(v)}\frac{R^2 i^2}{(2\pi r_n)^2} \tag{216}$$

where l is the mean free path of electron between its two collisions, say similar to that with the metal atoms, and (v) is the average velocity of the random electron motion. From (216) we obtain

$$\frac{1}{2}\frac{l}{(v)} = \frac{1}{R}\frac{m}{e^2}(2\pi r_n)^2 \tag{217}$$

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where the ratio

$$\frac{l}{\langle v \rangle} = \tau \tag{218}$$

is called the relaxation time of the electron transport.

A substitution of the resistance R from (212) characteristic for the dissipation process of energy into (217) gives

$$\tau = \frac{2m}{n} \left(2\pi \frac{n^2 \mathrm{n}^2}{me^2} \right)^2 = \frac{2m}{n} 4\pi^2 \frac{n^4 \mathrm{n}^4}{m^2 e^4} = 8\pi^2 \frac{n^4 \mathrm{n}^3}{me^4}. \tag{219}$$

This τ is longer than the transition time

$$\Delta t = T_n = \frac{2\pi n^3 \mathrm{n}^3}{me^4} \tag{220}$$

chracteristic for an individual emission process between states $n+1$ and n of the hydrogen atom by the factor

$$2\pi n. \tag{221}$$

But it should be noted that τ in (219) is due mainly to the idea of many electron collisions expected to be met in course of the electron random path

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Calculation of the Einstein probability coefficient for an electron transition

The reaction of the atom to the radiation field should be understood as a reaction existent in a given stationary state [Kramers-Heisenberg]. Since transitions between two stationary states are very short in their duration, the detailed nature of transitions becomes of only slight importance in the description of the optical phenomena. In general, when irradiated by a monochromatic light, an atom is expected to emit spherical waves distributed into surrounding space.

Let the oscillating dipole have a moment $P(t)$ which is a complex vector dependent on the frequency v and the time-independent electric field vector E. For the total intensity of light scattered per unit of time one can apply the formula

$$S = \frac{(2\pi v)^4}{3c^3}(P\,P^*) \tag{S1}$$

where P^* is the complex conjugate vector to P. From the point of view of the quantum radiation theory [Bohr, Kramers, Slater] the irradiated atom acts as a source of spherical waves where frequencies v_q are associated with energy jumps

$$hv_q \tag{S2}$$

characteristic for Bohr frequency condition. This kind of radiation acts as a

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classical dipole whose moment is represented by the real part of the expression

$$A_q e^{2\pi i v_q t}.\tag{S3}$$

The amplitude vectors A_q are linked to the Einstein probability coefficients a_q by means of the relation

$$a_q h v_q = \frac{(2\pi v_q)^4}{3c^3}(A_q A_q^*) = \frac{(2\pi v_q)^4}{3c^3}|A_q|^2.\tag{S4}$$

Here the vector A_q refers to an individual dipole frequency v_q.

But the energy relation [see (S1) and Kuhn]

$$\frac{dE}{dt} = \frac{(2\pi v_q)^4}{3c^3}|A_q|^2\tag{S5}$$

represents the energy dispersed per unit time with the frequency v_q. When v_q concerns the emissive transition between two neighbouring quantum levels in the hydrogen atom ($\Delta n = 1$) we obtain

$$\frac{dE}{dt} \cong \frac{\Delta E}{\Delta t} = \frac{(\Delta E)^2}{h}\tag{S6}$$

because the formula

$$\frac{1}{\Delta t} = \frac{\Delta E}{h}\tag{S7}$$

is satisfied in this case; see ( ). A substitution of the result of (S5)–(S7) into

(S4) gives the relation

$$a_q h v_q = a_q \Delta E = \frac{(\Delta E)^2}{h} \tag{S8}$$

so

$$a_q = \frac{\Delta E}{h}. \tag{S9}$$

A characteristic point is a change of a_q obtained in (S9) for transitions $\Delta n > 1$. If the interval ΔE refers invariably to $\Delta n = 1$, we have for $\Delta n > 1$ the relation

$$\Delta n \Delta E \Delta t \approx h \tag{S10}$$

so

$$\frac{1}{\Delta t} \approx \frac{\Delta n \Delta E}{h}. \tag{S11}$$

The intensity of the energy emission for $\Delta n > 1$ is

$$\frac{\Delta n \Delta E}{\Delta t} = \frac{(\Delta n\ \Delta E)^2}{h} \tag{S12}$$

and the energy emitted in this case is given by the equation

$$h v_q = \Delta n \Delta E. \tag{S13}$$

In effect the formula (S8) is replaced by the relation

$$a_q \Delta n \Delta E = \frac{(\Delta n\ \Delta E)^2}{h} \tag{S14}$$

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from which

$$a_q = \frac{\Delta n \Delta E}{h}. \qquad (S15)$$

This means that a_q behaves in a way similar to that presented in (S9): its change is dictated by the change of energy interval ΔE valid in the case of $\Delta n = 1$ to the interval $\Delta n \Delta E$ which is approximately valid for transitions having $\Delta n > 1$.

18 Summary

The Bohr model of the hydrogen atom has been revisited with the aim to develop it.

In the first step the quanta B_n of the magnetic induction have been taken into account. In fact these quanta can lead to the same spectrum of electron energy as the original Bohr model. First of advantages of the B_n calculation was to show that the quanta of the magnetic flux identical to those which are well-known in superconductors are present also on any energy level of the hydrogen atom. The next advantage of B_n was their use in an approach to calculate the the emission time of energy from the levels of the atom, see Sec. .

Another extension of the original Bohr model concerned the electron and proton spin. Instead of the Dirac theory, a modification of the Heisenberg uncertainty principle for energy and time has been applied for the fermions

case. The modification was based on a reference to the theory of relativity

by assuming that any transition velocity in a system should not exceed the speed of light. In effect a minimal distance between the fermion particles in a fermion ensemble has been derived. The idea of spin came from the assumption that a spontaneous magnetization can be obtained by any fermion particle due to its circulation along a circle having its radius equal to a minimal distance acceptable for a given fermions kind. As a consequence correct expressions for the spim angular momentum and magnetic moment could be obtained equally for the electron and proton particles. Simultaneously the paradox of the equal angular momentum porsessed by the electron and proton could be explained - a problem which seemed to trouble the physicists since a long time [ ].

The last step of the development of the Bohr model concerned the calculation of the time intervals spent by electrons in their transitions between individual quantum levels. Here a probabilistic-and-statistical way of approach to the transition time, applied both in the old and modern quantum theories, has been abandoned. Instead if it a well-known electrodynamical formula for the dissipation energy considered as the energy ΔE entering the electron transition has been applied. For a small distance between the quantum levels in the atom, viz. represented by the pair of states having indices n and $n + 1$, this treatment leads to the transition time interval Δt between $n + 1$ and n represented by the formula

$$\Delta E \Delta t = h. \tag{222}$$

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For the transitions which are between n and $n + \Delta n$, where $\Delta n > 1$ does not exceed much unity, we obtain

$$\Delta n \, \Delta E \, \Delta t \cong \hbar, \tag{223}$$

on conditions ΔE is the energy difference between the neighbouring levels. A characteristic point is that for both relations (222) and (223) we obtain

$$\Delta t \cong T_n \tag{224}$$

where T_n is the Bohr time period of circulation along the level n.

Appendix: Decay time of a high-frequency oscillator found to be close to the elementary interval of time

In the Kramers–Heisenberg search on dispersion properties of radiation by atoms there was Appendix introduced the decay time τ of a classically oscillating electron with frequency v:

$$\tau_v = \frac{3c^3 m}{8\pi^2 e^2 v^2} \tag{A1}$$

Let us assume that v is so large that its reciprocal value attains the size

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of the decay time τ_v given in (A1), so

$$\tau_v = \frac{1}{v}. \tag{A2}$$

The relation (A2) substituted to (A1) gives the following equation for τ_v:

$$\tau_v = \frac{3c^3 m \tau_v^2}{8\pi^2 e^2} \tag{A3}$$

from which

$$\frac{1}{\tau_v} = \frac{3c^3 m}{8\pi^2 e^2}. \tag{A4}$$

Our aim is to show that the expression (A4) does not differ much from the reciprocal value of the elementaty time derived in ( ) which is

$$\frac{1}{\Delta t_0} = \frac{mc^2}{n}. \tag{A5}$$

By multiplying the expression $1/\tau_v$ in (A4) by α and putting it equal to (A5) we obtain the equation

$$\frac{\alpha}{\tau_v} = 3\frac{\alpha c^3 m}{8\pi^2 e^2} = \frac{1}{\Delta t_0} = \frac{mc^2}{n} \tag{A6}$$

from which

$$\alpha = \frac{8\pi^2}{3} \frac{e^2}{cn} \approx \frac{8\pi^2}{3} \frac{1}{137} = 0.24. \tag{A7}$$

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References

[1] J. Avery, *Creation and Annihilation Operators*, McGraw-Hill, New York, **(1976)**.

[2] E. Beck, *Annalen der Physik* 60, 109, **(1919)**.

[3] N. Bohr. *The Theory of Spectra and the Atomic Constitution*, Cambridge University Press, Cambridge (UK), **(1922)**.

[4] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra*, Cambridge University Press, Cambridge **(1970)**.

[5] M. Cyrot and D. Pavuna, *Introduction to Superconductivity and High T_c Materials*, World Scientific, Singapore **(1992)**.

[6] H. Eyring, J. Walter and G. E. Kimball, *Quantum Chemistry*, Wiley, New York, **(1957)**.

[7] H. E. Flint, *Proceedings of the Royal Society A*, London, Vol. 117, 630, **(1928)**.

[8] H. E. Flint and O. W. Richardson. *Proceedings of the Royal Society A*, London, Vol. 117, 637, **(1928)**.

[9] D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., Prentice Hall, New Jersey **(1999)**.

[10] M. Jammer, *The Conceptual Development of Quantum Mechanics*, McGraw-Hill, New York, **(1966)**.

[11] C. Kittel, *Quantum Theory of Solids*, 2nd ed., Wiley, New York **(1987)**.

[12] C. Kittel, *Quantum Theory of Solids* 2nd ed., Wiley, New York **(1989)**.

[13] A. M. Kobos, *Postępy Fizyki* Vol. 64, 86, **(2013)**.

[14] L. D. Landau and E. M. Lifshitz, *Mechanics. Electrodynamics*, Izd. Nauka, Moscow **(1969)** (in Russian).

[15] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, (in Russian) Izd. Nauka, Moscow **(1972)**.

[16] H. Lass, *Vector and Tensor Analysis*, McGraw-Hill, New York **(1950)**.

[17] A. H. MacDonald (ed.), *Quantum Hall Effect. A Perspective*, Kluwer, Milano **(1989)**.

[18] A. N. Matveev, *Electrodynamics and the Theory of Relativity*, (in Russian) Izd. Wyzszaja Szkola, Moscow **(1964)**.

[19] R. A. Millikan and I. Bowen, *Phys. Rev.* Vol. 23, 1, 1924; ibid. Vol. 24, 209 and 223, **(1924)**.

[20] S. Olszewski, *J. of Modern Physics* Vol. 2, 1305, **(2011)**.

[21] S. Olszewski, *J. of Modern Physics* Vol. 3, 217, **(2012)**.

[22] S. Olszewski, *Quantum Matter* Vol. 1, 127, **(2012)**.

[23] S. Olszewski, *Quantum Matter*, Vol. 2, 102, **(2013)**.

[24] S. Olszewski, *J. of Modern Physics*, Vol. 4, 14, **(2013)**.

[25] S. Olszewski, *Reviews in Theoretical Science* Vol. 1, 344 **(2013)**.

[26] S. Olszewski, *J. of Modern Physics* Vol. 5, 1264, **(2014)**.

[27] S. Olszewski, *Journal of Modern Physics* Vol. 5, 1502 **(2014)**.

[28] S. Olszewski, *Journal of Modern Physics* Vol. 5, 2022 **(2014)**.

[29] S. Olszewski, *Journal of Modern Physics* Vol. 5, 2030 **(2014)**.

[30] S. Olszewski, *Quanta of the magnetic flux and magnetic induction in the hydrogen atom*, Quantum Matter (in press).

[31] S. Olszewski, *Quantum Matter*, Vol. 3, 155 **(2014)**.

[32] S. Olszewski, *Quantum Matter* (in press).

[33] L. Onsager, *Philosophical Magazine* Vol. 43, 1006, **(1952)**.

[34] M. Planck, *Acht Vorlesungen ueber Theoretische Physik*, S. Hirzel Verlag, Leipzig **(1910)**.

[35] R. E. Prange and S. M. Girvin (eds.), *The Quantum Hall Effect* 2nd ed., Springer, New York **(1990)**.

72

[36] M. E. Rose, *Relativistic Electron Theory* Wiley, New York, **(1961)**.

[37] A. E. Ruark, *Proceedings of the National Academy of Sciences of the United States of America*, Vol. 14, 322, **(1928)**.

[38] A. Rubinowicz, *in Handbuch der Physik* (eds. H. Geiger and K. Scheel), Vol. 24, Part 1, Springer, Berlin, **(1933)**.

[39] S. das Sarma and A. Pinczuk (eds.), *Perspectives in Quantum Hall Effects*, Wiley, New York **(1997)**.

[40] L. I. Schiff, *Quantum Mechanics*, 3rd ed., McGraw-Hill, New York, **(1968)**.

[41] J. C. Slater, *Quantum Theory of the Atomic Structure*, 1, McGraw-Hill, New York **(1960)**.

[42] J. C. Slater, *Quantum Theory of Molecules and Solids*, 3, McGraw-Hill, New York **(1967)**.

[43] A. Sommerfeld, *Atombau und Spekrallinien*, Vol. 1, 5th ed., Vieweg, Braunschweig **(1931)**.

[44] S. Tolansky, *Hyperfine Structure in Line Spectra and Nuclear Spin*, 2nd ed., Methuen, London **(1948)**.

[45] G. E. Uhlenbeck and S. A. Goudsmit, *Naturwissenschaften* Vol. 13, 953, **(1925)**.

[46] H. E. White, *Introduction to Atomic Spectra*, McGraw-Hill, New York, **(1934)**.

[47] J. M. Ziman, *Principles of the Theory of Solids*, 2nd ed., University Press, Cambridge **(1972)**.