# 54.

## ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

[*Continued from p.* 363 *above.*]

[*Cambridge and Dublin Mathematical Journal,* VIII. (1853), pp. 256—269.]

### SECTION VII. *On Combinants.*

REASONS of convenience have induced me to depart from the plan to which I originally intended to adhere in the development of this theory, and I shall hereafter, from time to time, continue to add sections on such parts of the subject as may chance to be most present to my mind or most urgent upon my attention, without waiting for the exact place which they ought to occupy in a more formal treatise, and without having regard to the separation of the subject into the two several divisions stated at the outset of the first section. The present section will be devoted to a brief and partial exposition of the theory of Combinants*, with a view to the application of this theory to the solution of the problem of throwing the resultant of three general homogeneous quadratic functions under its most simple form, being analogous to that given by Aronhold in the particular case where the three functions are derived from the same cubic, and becoming identical therewith when the coefficients are accommodated to this particular supposition†. I shall confine myself for the present to combinants relating to systems of functions, all of the same degree.

If $\phi_1, \phi_2, \ldots \phi_r$, be homogeneous functions of any number of variables, any invariant or other concomitant of the system which remains unchanged, not only for linear substitutions impressed upon the variables contained within the functions, but also for linear combinations impressed upon the functions themselves, is what I term a Combinant. A Combinant is thus an invariant or other concomitant of a system in its corporate capacity (quâ *system*), being in fact

* Discovered by the Author of this paper in the winter of 1852.

† A similar method will subsequently be applied to the representation of the resultant of two cubic equations as a function of Combinants bearing relations to the quadratic and cubic invariants of a quartic function of $x$ and $y$, precisely analogous to those which the Combinants that enter into the solution above alluded to bear to the Aronholdian invariants of a cubic function.

common to the whole family of forms designated by $\lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_r\phi_r$, where $\lambda_1, \lambda_2, \ldots \lambda_r$, are arbitrary constants. If the coefficients of $\phi_1, \phi_2, \ldots \phi_r$, be supposed to be written out in $r$ lines (the coefficients of corresponding terms occupying the same place in each line), so as to form a rectangular matrix, any combinantive invariant will be a function of the determinants corresponding to the several squares of $r^2$ terms each that can be formed out of such matrix, or, as they may be termed, the *full* determinants belonging to such rectangular matrix. If we call any such combinant $K$, then, over and above the ordinary partial differential equations which belong to it in its character of an invariant, it will be necessary and sufficient, in order to establish its combinantive character, that $K$ shall be subject to satisfy $(r-1)$ pairs of equations of the form

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc}\cdots\right)K = 0,$$

$$\left(a\frac{d}{da'} + b\frac{d}{db'} + c\frac{d}{dc'}\cdots\right)K = 0,$$

where $a, b, c \ldots$ ; $a', b', c' \ldots$, are respectively lines in the matrix above referred to.

So any combinantive concomitant will be a function of the full determinants of the matrix formed by the coefficients of the given system of forms and of the variables, and will be subject to satisfy the additional differential equations just above written.

It will readily be understood furthermore, that an invariant or other concomitant may be combinantive in respect to a certain number of forms of a system, and not in respect of other forms therein ; or more generally, may be combinantive in respect of each, separately considered, of a series of groups into which a given system may be considered to be subdivided, without being so in respect of the several groups taken collectively.

In the fourth section of my memoir [p. 429 below] on a "Theory of the Conjugate Properties of two rational integral Algebraical Functions," recently presented to the Royal Society of London, the case actually arises of an invariant of a system of three functions, which is combinantive in respect only to two of them.

For greater simplicity, let the attention for the present be kept fixed upon combinants which are such in respect of a single group of functions, all of the same degree in the variables. (It will of course have been perceived that when the system is made up of several groups, there would be nothing gained by limiting the groups to be all of the same degree *inter se*; it is sufficient that all of the same group be of the same degree *per se*.)

All such combinants will admit of an obvious and immediate classification. Let us suppose that a combinant is proposed which is in its lowest terms, that is to say, incapable of being expressed as a rational integral algebraical function of combinants of an inferior order. Such a combinant may, notwithstanding this, admit of being decomposed into non-combinantive invariants of inferior dimensions to its own, and in such event will be termed a *complex* combinant; or it may be indecomposable after this method, in which event it will be termed a *simple* combinant. It will presently be shown, that the resultant of a system of three quadratic functions is made up of a complex combinant of twelve dimensions, and of the square of a simple combinant of six dimensions, expressible as a biquadratic function of ten non-combinantive invariants, each of three dimensions in the coefficients. There is an obvious mode of generating complex combinants; according to which they admit of being viewed as invariants of invariants. Supposing $\phi_1, \phi_2, \ldots \phi_r$, to be the functions of the given system, $\lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_r\phi_r$ may conveniently be termed the conjunctive of the system: if now one or more invariants or other concomitants be taken of this conjunctive, there results a derivative function or system of functions of the quantities $\lambda_1, \lambda_2, \ldots \lambda_r$, in which every term affecting any power or combination of powers of the $\lambda$ series is necessarily an invariant or concomitant of the given system. If now an invariant or other concomitant be taken of the new system in respect to $\lambda_1, \lambda_2, \ldots \lambda_r$, (the original variables (supposing them to enter) being treated as constants), this secondarily derived invariant will be itself an Invariant, or at all events a Concomitant in respect of the original system, and being unaffected by linear substitutions impressed upon the $\lambda$ system, is by definition a combinant of such system. A similar method will obviously apply if the original system be made up of various groups; each group will give rise to a conjunctive, and one or more concomitants being taken of this system of conjunctives and treated as in the case first supposed, (the only difference being, that there will on the present supposition be several *unrelated* systems instead of a single system of new variables, that is, several $\lambda$ systems instead of one only) the result, when all the $\lambda$ systems have been *invariantized out* (that is, made to disappear by any process for forming invariants), will be a combinant in respect to each of the groups, severally considered, of the given system of functions.

Here let it be permitted to me to make a momentary digression, in order to be enabled to avoid for the future the inconvenience of using the phrase "invariant or other concomitant," and so to be enabled at one and the same time to simplify the language and to give a more complete unity to the matter of the theory, by showing how every concomitant may in fact be viewed as a simple invariant, so that the calculus of forms may hereafter admit of being cited, as I propose to cite it, under the name of the Theory of Invariants.

Thus, to begin with the case of *simple* contragredience and cogredience, if $\xi, \eta, \zeta \ldots$ are contragredient to $x, y, z \ldots$, any form containing $\xi, \eta, \zeta \ldots$, which is concomitantive to a given form or system of forms $S$, which contains $x, y, z \ldots$, may be regarded as concomitantive to the system $S'$, made up of $S$ and the superadded *absolute* form $\xi x + \eta y + \zeta z + \ldots$, say $\vartheta$; where $\xi, \eta, \zeta \ldots$ are treated no longer as variables, but as *constants*. In like manner every system of variables contragredient to $x, y, z \ldots$, or to any other system of variables in $S$, will give rise to a superadded form analogous to $\vartheta$, the totality of which may be termed $S_1$; and thus the various systems $\xi, \eta, \zeta \ldots$ will no longer exist as variables in the derived form, but purely as constants. Again, if $S$ contain any system of variables $\phi, \psi, \vartheta$, &c., contragredient to $x, y, z$, &c., the system of variables $u, v, w$, &c., cogredient with $x, y, z$, &c., may be considered as constants belonging to the superadded form $\phi u + \psi v + \vartheta w \ldots$; but if $S$ do not contain any system contragredient to $x, y, z$, &c., then $u, v, w$, &c. may be treated as constants belonging to the superadded system of forms $xv - yu$, $yw - zv$, $zu - xw$, &c.; and so in general any concomitant containing any sets of variables in simple relation, whether of cogredience or contragredience, with any of the sets in the given system $S$, may in all cases be treated as an *invariant* of the system $S'$, made up of $S$ and a certain superadded system $S_1$, all the forms contained in which are absolute, by which I mean, that they contain no literal coefficient. The same conclusion may be extended to the case of concomitants containing sets of variables in *compound* relation with the sets in the given system of forms $S$. Thus, suppose $u_1, u_2, \ldots u_n$, to be in compound relation of cogredience with $x^{n-1}, x^{n-2}y, x^{n-3}y^2, \ldots y^{n-1}$; $u_1, u_2, \ldots u_n$, may be regarded as constants belonging to the superadded form

$$u_1 y^{n-1} - (n-1)\, u_2 y^{n-2} x + \tfrac{1}{2}(n-1)(n-2)\, u_3 y^{n-3} x^2 \mp \ldots \pm u_n x^{n-1},$$

say $\Omega$. And thus universally we are enabled to affirm, that a concomitant of whatever nature to a given system of forms, may be reduced to the form of an invariant of a system made up of the given system and a certain other superadded system of absolute forms: without, therefore, abandoning the use of the terms concomitant, cogredience, contragredience, &c., which for many purposes are highly convenient and save much circumlocution, we may regard every concomitant as a disguised invariant, and under the name of the Theory of Invariants comprise the totality of the theory of Concomitance. I have already had occasion to make use of the superadded form $\Omega$ in discussing the theory of the Bezoutiant (a quadratic form concomitant to two functions of the same degree in $x, y$, which plays a most important part in the theory of the relations of their real roots), in the memoir for the Royal Society previously adverted to.

I now return to the question of applying the theory of combinants to the decomposition of the resultant of three general quadratic functions of

$x, y, z$. It will of course be apparent that every resultant of any system of $n$ functions of the same degree of a single set of $n$ variables is a combinantive invariant of the system. This is an immediate and simple corollary to the theorem given by me in this *Journal*, in May, 1851. Accordingly, in proceeding to analyse the composition of the resultant of three quadratic functions, I may, besides impressing linear combinations upon the variables, impress linear combinations upon the functions themselves, in any way most conducive to simplicity and facility of expression and calculation; and whatever relations shall be proved to exist between the resultant and other combinants for such specific representation, must be universal, and hold good for the functions in their most general form.

(1) The system, by means of linear substitutions impressed upon the variables which enter into the functions, may be made to assume the form

$$x^2 + y^2 + z^2,$$
$$ax^2 + by^2 + cz^2,$$
$$lx^2 + my^2 + nz^2 + 2pyz + 2qzx + 2rxy.$$

(2) By means of linear combinations of the functions themselves the system may evidently be made to take the form

$$(c-a)\,x^2 + (c-b)\,y^2,$$
$$(a-b)\,y^2 + (a-c)\,z^2,$$
$$ky^2 + 2pyz + 2qzx + 2rxy;$$

and finally, by taking suitable multipliers of $x, y, z$ in lieu of $x, y, z$, it may be made to become

$$\rho\,(x^2 - y^2),$$
$$\sigma\,(y^2 - z^2),$$
$$y^2 + 2fyz + 2gzx + 2hxy.$$

We have thus reduced the number of constants in the system from eighteen to five; and as it will readily be seen that in any combinant of the system in its reduced form $\rho$ and $\sigma$ can only enter as factors of the simple quantity, $(\rho\sigma)^i$, for all purposes of comparison of the combinants of the system of like dimensions with one another, $\rho$ and $\sigma$ might admit of being treated as being each unity, and accordingly, practically speaking, we have only to deal with three in place of eighteen constants, a marvellous simplification, and which makes it obvious, *à priori*, or at least affords a presumption almost amounting to and capable of being reduced to certainty, that the number of fundamental combinants of the system, of which all the rest must be explicit rational functions, will be exactly four in number; which, for the canonical form hereinbefore written, on making $\rho$ and $\sigma$ each unity, will correspond to

$$1,\quad f^2+g^2+h^2,\quad f^2g^2+g^2h^2+h^2f^2,\quad fgh,$$

and will be of the 3rd, 6th, 12th, and 9th degrees respectively. The reason why the squares of $f, g, h$, instead of the simple terms $f, g, h$, appear in the 2nd and 3rd of these forms is, because, on changing $x$ into $-x$, $y$ into $-y$, or $z$ into $-z$, two of the quantities $f, g, h$ will change their sign, but the forms representing the invariants of even degrees ought to remain absolutely unaltered for such transformations. I shall in the course of the present section set forth the methods for obtaining these four combinants, which, although of the regularly ascending dimensions 3, 6, 9, 12, belong obviously to two different groups, the one of three dimensions forming a class in itself, and the natural order of the three others being that denoted by the sequence 6, 12, and 9, and not that which would be denoted by the sequence 6, 9, 12, the combinant of the ninth degree being properly to be regarded as in some sort an accidentally rational square root of a combinant of 18 dimensions.

Let now
$$\rho(x^2-y^2)=U,$$
$$\sigma(y^2-z^2)=W,$$
$$y^2+2fyz+2gzx+2hxy=V.$$

The resultant will be found by making
$$x=\pm y,$$
$$z=\pm y,$$

when
$$\left.\begin{matrix}x=+y\\ z=+y\end{matrix}\right\},\quad V=(1+2f+2g+2h)\,y^2,$$
$$\left.\begin{matrix}x=+y\\ z=-y\end{matrix}\right\},\quad V=(1-2f-2g+2h)\,y^2,$$
$$\left.\begin{matrix}x=-y\\ z=+y\end{matrix}\right\},\quad V=(1+2f-2g-2h)\,y^2,$$
$$\left.\begin{matrix}x=-y\\ z=-y\end{matrix}\right\},\quad V=(1-2f+2g-2h)\,y^2.$$

Hence the resultant $R$
$$=\rho^4\sigma^4(1+2f+2g+2h)(1-2f-2g+2h)(1+2f-2g-2h)(1-2f+2g-2h)$$
$$=(\rho\sigma)^4\{(1+2h)^2-4(f+g)^2\}\{(1-2h)^2-4(f-g)^2\}$$
$$=(\rho\sigma)^4\{(1+4h^2-4f^2-4g^2)^2-(4h-8fg)^2\}$$
$$=(\rho\sigma)^4[1-8(f^2+g^2+h^2)+16\{(f^4+g^4+h^4)-2(g^2h^2+h^2f^2+f^2g^2)\}+64fgh].$$

Let now
$$K=\lambda U+\mu V+\nu W,$$

$K$ being what I term a linear conjunctive of $U, V, W$. The invariant of $K$, in respect to $x, y, z$, will be the determinant

$$\begin{vmatrix}\rho\lambda, & h\mu, & g\mu\\ h\mu, & \mu-\rho\lambda+\sigma\nu, & f\mu\\ g\mu, & f\mu, & -\sigma\nu\end{vmatrix},$$

that is

$$= (2fgh - g^2)\,\mu^3 + \sigma\,(h^2 - g^2)\,\mu^2\nu - \rho\,(f^2 - g^2)\,\mu^2\lambda - \rho\sigma\mu\lambda\nu + \rho^2\sigma\lambda^2\nu - \rho\sigma^2\lambda\nu^2;$$

or, multiplying by 6, we may write

$$I_{x,y,z}K = 6d\lambda\mu\nu + 3b_3\mu^2\nu + 3b_1\mu^2\lambda + 3a_3\lambda^2\nu + 3c_1\lambda\nu^2 + b_2\mu^3,$$

where

$$d = -\rho\sigma, \qquad b_2 = 12fgh - 6g^2,$$
$$b_1 = -2\rho\,(f^2 - g^2), \quad b_3 = 2\sigma\,(h^2 - g^2),$$
$$a_3 = \rho^2\sigma, \qquad c_1 = -2\rho\sigma^2,$$

the notation being accommodated to that employed by Mr Salmon in *The Higher Plane Curves*, $\lambda$, $\mu$, $\nu$ in $IK$ being correspondent to $x$, $y$, $z$ in Mr Salmon's form. If now we employ Mr Salmon's expression for the $S$ (the biquadratic Aronholdian of $IK$), observing that

$$a_2 = 0, \quad c_2 = 0, \quad a_1 = 0, \quad c_3 = 0,$$

we have the complex combinant

$$S_{\lambda,\mu,\nu}I_{x,y,z}K = d^4 - 2d^2(b_1c_1 + a_3b_3) + da_3b_2c_1 - a_3c_1b_1b_3 + b_1^2c_1^2 + a_3^2b_3^2$$

$$= \rho^4\sigma^4\begin{pmatrix}1 - 8\,(f^2 + h^2 - 2g^2) + 4\,(12fgh - 6g^2)\\ -16\,(f^2 - g^2)(h^2 - g^2) + 16\,(f^2 - g^2)^2 + (h^2 - g^2)^2\end{pmatrix}$$

$$= \rho^4\sigma^4\,\{1 - 8\,(f^2 + g^2 + h^2) + 16\,(f^4 + g^4 + h^4 - h^2g^2 - g^2f^2 - f^2h^2) + 48fgh\}.$$

Hence, calling the resultant $R$, we have

$$-3R + 4S_{\lambda,\mu,\nu}I_{x,y,z}K = 1 - 8\,(f^2 + g^2 + h^2) + 16\,(f^4 + g^4 + h^4)$$
$$+ 32\,(f^2g^2 + g^2h^2 + h^2f^2) = \{1 - 4\,(f^2 + g^2 + h^2)\}^2 = P^2.$$

Let $\Omega$ be taken the polar reciprocal to the conjunctive

$$-\lambda U + \mu V + \nu W;$$

and for greater simplicity, as we know, *à priori*, from the fundamental definition of a combinant, which (save as to a factor) must remain unaltered by any linear modification impressed upon the functions to which it appertains, that $\rho$ and $\sigma$ can enter factorially only in any combinant, let $\rho$ and $\sigma$ be each taken equal to unity in performing the intermediary operations.

Then

$$\Omega = \begin{vmatrix} -\lambda, & h\mu, & g\mu, & \xi \\ h\mu, & \lambda + \mu + \nu, & f\mu, & \eta \\ g\mu, & f\mu, & -\nu, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}$$

$$= \left\{\begin{array}{l} \xi^2(\nu^2 + \nu\mu + \nu\lambda + f^2\mu^2) \\ +\,\eta^2(-\lambda\nu + g^2\mu^2) \\ +\,\zeta^2(\lambda^2 + \lambda\mu + \lambda\nu + h^2\mu^2) \\ -\,2\eta\zeta\,(f\lambda\mu + hg\mu^2) \\ +\,2\zeta\xi\,\{g\,(\mu\lambda + \mu\nu) + (g - fh)\,\mu^2\} \\ -\,2\xi\eta\,(h\mu\nu + fg\mu^2) \end{array}\right\}.$$

Upon $\Omega$, which is a quadratic function in respect of each of the two unrelated systems $\xi, \eta, \zeta$; $\lambda, \mu, \nu$, and also in respect of the coefficients in $(U, V, W)$, we may operate with the commutantive symbol

$$\left.\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu}, & \frac{d}{d\nu} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu}, & \frac{d}{d\nu} \end{matrix}\right\},$$

which, for facility of reference, I shall term $8E$.

Considering the first line as stationary, we shall obtain, for the value of $8E(\Omega)$, 216 commutantives, which may be expressed under the following forms:

$$\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d^2}{d\lambda^2},\right. & \frac{d^2}{d\mu^2}, & \left.\frac{d^2}{d\nu^2}\right] \end{matrix}\right|,$$

$$-\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d^2}{d\lambda^2},\right. & \frac{d}{d\mu}\frac{d}{d\nu}, & \left.\frac{d}{d\mu}\frac{d}{d\nu}\right] \end{matrix}\right|,$$

$$-\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\nu},\right. & \frac{d^2}{d\mu^2}, & \left.\frac{d}{d\lambda}\frac{d}{d\nu}\right] \end{matrix}\right|,$$

$$-\left|\begin{matrix} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\mu},\right. & \frac{d}{d\lambda}\frac{d}{d\mu}, & \left.\frac{d^2}{d\nu^2}\right] \end{matrix}\right|,$$

$$2\left|\begin{matrix} \frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta} \\ \frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\mu},\ \frac{d}{d\mu}\frac{d}{d\nu},\ \frac{d}{d\nu}\frac{d}{d\lambda}\right] \end{matrix}\right|.$$

In this expression the first lines may be considered stationary, the second lines are subject to the usual process of commutation, which makes three of the six permutations positive and three negative; and the third or bracketed lines are subject to the simple process which makes all the permutations of the same sign. In the three middle groups two of the terms in the final line are always identical; it will therefore be more convenient to introduce the multiplier 2, and then to consider each such line to represent the three distinct permutations, taken singly.

Let now

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2},\ \frac{d^2}{d\eta^2},\ \frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega),$$

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2},\ \frac{d}{d\eta}\frac{d}{d\zeta},\ \frac{d}{d\eta}\frac{d}{d\zeta}\right\}\Omega = (\Omega)',$$

$$\frac{1}{8}\left\{\frac{d}{d\xi}\frac{d}{d\zeta},\ \frac{d^2}{d\eta^2},\ \frac{d}{d\xi}\frac{d}{d\zeta}\right\}\Omega = (\Omega)'',$$

$$\frac{1}{8}\left\{\frac{d}{d\eta}\frac{d}{d\xi},\ \frac{d}{d\eta}\frac{d}{d\xi},\ \frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega)''',$$

$$\left\{\frac{d}{d\xi}\frac{d}{d\eta},\ \frac{d}{d\eta}\frac{d}{d\zeta},\ \frac{d}{d\zeta}\frac{d}{d\xi}\right\}\Omega = (\Omega)_1.$$

And let

$$\left[\frac{d^2}{d\lambda^2},\ \frac{d^2}{d\mu^2},\ \frac{d^2}{d\nu^2}\right] = L,$$

$$\left[\frac{d^2}{d\lambda^2},\ \frac{d}{d\mu}\frac{d}{d\nu},\ \frac{d}{d\mu}\frac{d}{d\nu}\right] = L',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\nu},\ \frac{d^2}{d\mu^2},\ \frac{d}{d\lambda}\frac{d}{d\nu}\right] = L'',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\mu},\ \frac{d}{d\lambda}\frac{d}{d\mu},\ \frac{d^2}{d\nu^2}\right] = L''',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\mu},\ \frac{d}{d\mu}\frac{d}{d\nu},\ \frac{d}{d\nu}\frac{d}{d\lambda}\right] = L_1.$$

Then, attending to the convention just previously explained, we shall have

$$E(\Omega) = (L - 2L' - 2L'' - 2L''' + 2L_1)$$
$$\times \{(\Omega) - 2(\Omega)' - 2(\Omega)'' - 2(\Omega)''' + 2(\Omega)_1\},$$

a symbolical product, any term in which such as $L'\Omega''$ will mean

$$\left\{\begin{matrix}\left[\dfrac{d^2}{d\lambda^2}, & \dfrac{d}{d\mu}\dfrac{d}{d\nu}, & \dfrac{d}{d\mu}\dfrac{d}{d\nu}\right] \\ \dfrac{d}{d\xi}\dfrac{d}{d\zeta}, & \dfrac{d^2}{d\eta^2}, & \dfrac{d}{d\xi}\dfrac{d}{d\zeta}\end{matrix}\right\}\tfrac{1}{8}\Omega,$$

and a similar interpretation must be extended to each of the 25 partial products; we have then

$$L(\Omega)=8g^2,\quad -2L'(\Omega)=0,\quad -2L'''(\Omega)=0,$$
$$-2L''(\Omega)=-4g^2,\quad 2L_1(\Omega)=-2,$$
$$-2L(\Omega)'=0,\quad -2L(\Omega)'''=0,$$
$$4L'(\Omega)'=0,\quad 4L''(\Omega)'''=0,$$
$$4L''(\Omega)'=0,\quad 4L'''(\Omega)'''=0,$$
$$4L'''(\Omega)'=8f^2,\quad 4L'(\Omega)'''=8h^2,$$
$$-2L(\Omega)''=0,\quad 4L'(\Omega)''=0,\quad 4L''(\Omega)''=0,\quad 4L'''(\Omega)''=0,$$
$$-4L_1(\Omega)'=0,\quad -4L_1(\Omega)'''=0,$$
$$-4L_1(\Omega)''=4g^2;$$

and, finally, the five terms comprised in

$$2L(\Omega)_1,\ \ldots,\ 4L_1(\Omega)_1,$$

each $=0$. All the above equations can be easily verified by direct inspection, it being observed that $8(\Omega)$ represents

$$\nu^2+\lambda\nu+\mu\nu+f^2\mu^2,\quad -\lambda\nu+g^2\mu^2,\quad \lambda^2+\lambda\mu+\lambda\nu+h^2\mu^2,$$

that $8(\Omega)'$ represents

$$\nu^2+\mu\nu+\lambda\nu+f^2\mu^2,\quad -f\lambda\mu-hg\mu^2,\quad -f\lambda\mu-hg\mu^2,$$

that $8(\Omega)''$ represents

$$-\lambda\nu+g^2\mu^2,\quad g(\mu\lambda+\mu\nu)+(g-fh)\mu^2,\quad g(\mu\lambda+\mu\nu)+(g-fh)\mu^2,$$

that $8(\Omega)'''$ represents

$$\lambda^2+\mu\lambda+\nu\lambda+h^2\mu^2,\quad -h\mu\nu-fg\mu^2,\quad -h\mu\nu-fg\mu^2,$$

and that $(\Omega)_1$ represents

$$-f\lambda\mu-hg\mu^2,\quad g(\mu\lambda+\mu\nu)+(g-fh)\mu^2,\quad -h\mu\nu-fg\mu^2.$$

We have thus

$$E(\Omega)=8g^2-4g^2-2+8f^2+8h^2+4g^2$$
$$=2\{4f^2+4g^2+4h^2-1\}.$$

Hence

$$3R=4S_{\lambda,\mu,\nu}I_{x,y,z}K-\tfrac{1}{4}\{E\Omega\}^2. \qquad \text{(A)}$$

If we restore to $U$, $V$, $W$ their general values, and make

$$U = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$
$$V = a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy,$$
$$W = a''x^2 + b''y^2 + c''z^2 + 2f''yz + 2g''zx + 2h''xy,$$

and construct the cubic function

$$\begin{aligned}\vartheta = &(ax + a'y + a''z)(bx + b'y + b''z)(cx + c'y + c''z)\\ &-(ax + a'y + a''z)(fx + f'y + f''z)^2 - (bx + b'y + b''z)(gx + g'y + g''z)^2\\ &-(cx + c'y + c''z)(hx + h'y + h''z)^2\\ &+2(fx + f'y + f''z)(gx + g'y + g''z)(hx + h'y + h''z),\end{aligned}$$

that is

$$\begin{aligned}&\Sigma(abc - af^2 - bg^2 - ch^2 + 2fgh)x^3\\ &+\Sigma\{a'bc + ab'c + abc' - (a'f^2 + 2aff') - (b'g^2 + 2bgg') - (c'h^2 + 2chh')\\ &\qquad + 2f'gh + 2fg'h + 2fgh'\}x^2y\\ &+\{a'b''c + a'bc'' + a''b'c + a''bc' + ab'c'' + ab''c' - 2a'ff'' - 2af'f'' - 2a''ff'\\ &\qquad - 2b'gg'' - 2bg'g'' - 2b''gg' - 2c'hh'' - 2ch'h'' - 2c''hh'\\ &\qquad + 2f''g'h + 2f'g''h + 2fg'h'' + 2f''gh' + 2f''gh' + 2fg''h'\}xyz,\end{aligned}$$

$S_{\lambda,\mu,\nu} I_{x,y,z} K$ in the preceding equation becomes simply the Aronholdian $S$ to $\vartheta$, which may be calculated by Mr Salmon's formula previously quoted.

$\Omega$ may be taken equal to the determinant

$$\begin{vmatrix} ax + a'y + a''z, & hx + h'y + h''z, & gx + g'y + g''z, & \xi \\ hx + h'y + h''z, & bx + b'y + b''z, & fx + f'y + f''z, & \eta \\ gx + g'y + g''z, & fx + f'y + f''z, & cx + c'y + c''z, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}.$$

And the cubic commutant of this, obtained by affecting it with the commutantive operator,

$$\left.\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \end{matrix}\right\}$$

will give $48E(\Omega)$ if each of the four lines of the operator undergoes permutation, or $8E(\Omega)$, if one of the four lines is kept stationary. Thus it falls within the limits of practical possibility to calculate explicity, by the formula (A), the value of the resultant. I give to the $S$ of $\vartheta$ the appellation of the Hebrew letter שׁ (*shin*), and to the commutant of $\Omega$ the appellation of the Hebrew letter ט (*teth*). These letters are chosen with design; for I shall presently show that when the three given quadratic functions are the differential derivatives of the same cubic function $\psi$, the ט becomes the Aronholdian $T$ to the cubic function, or, as we may write it, $T\psi$, and the שׁ becomes the Aronholdian $S$ of the Hessian thereto, that is $SH\psi$.

Thus for the first time the true inward constitution of the resultant of three quadratics is brought to light. The methods anteriorly given by me, and the one subsequently added by M. Hesse for finding this resultant, adverted to in Section II., lead, it is true, to the construction of the form, but throw no light upon the essential mode of its composition.