51.

ON A THEOREM CONCERNING THE COMBINATION OF DETERMINANTS.

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Let ^1A represent the line of terms $^1a_1, {}^1a_2, \ldots {}^1a_m$,

^1B " " " $^1b_1, {}^1b_2, \ldots {}^1b_m$.

Let $^1A \times {}^1B$ represent $\Sigma\,({}^1a_r \times {}^1b_r)$, where of course there are m terms within the symbol of summation.

Again, let ^2A represent the line $^2a_1, {}^2a_2, \ldots {}^2a_m$,

^2B " " $^2b_1, {}^2b_2, \ldots {}^2b_m$,

and let $\begin{vmatrix} {}^1A \\ {}^2A \end{vmatrix} \times \begin{vmatrix} {}^1B \\ {}^2B \end{vmatrix}$ represent $\Sigma \begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix} \times \begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}$,

$\begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix}$ denoting the determinant $({}^1a_r \,.\, {}^2a_s - {}^1a_s \,.\, {}^2a_r)$,

$\begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}$ " " " $({}^1b_r \,.\, {}^2b_s - {}^1b_s \,.\, {}^2b_r)$,

there being of course $\frac{1}{2}m\,(m-1)$ terms comprised within the sign of summation; and so, in general, let

$$\begin{vmatrix} {}^1A \\ {}^2A \\ {}^3A \\ \vdots \\ {}^nA \end{vmatrix} \times \begin{vmatrix} {}^1B \\ {}^2B \\ {}^3B \\ \vdots \\ {}^nB \end{vmatrix},$$ n being less than m,

$\left(\begin{array}{l}\text{and where in general } {}^rA \text{ denotes } {}^ra_1,\ {}^ra_2,\ \dots\ {}^ra_m \\ \qquad\text{and } {}^rB \text{ denotes } {}^rb_1,\ {}^rb_2,\ \dots\ {}^rb_m\end{array}\right)$ represent

$$\Sigma \begin{vmatrix} {}^1a_{h_1}, & {}^1a_{h_2}, & \dots & {}^1a_{h_n} \\ {}^2a_{h_1}, & {}^2a_{h_2}, & \dots & {}^2a_{h_n} \\ \dots & \dots & \dots & \dots \\ {}^na_{h_1}, & {}^na_{h_2}, & \dots & {}^na_{h_n} \end{vmatrix} \times \begin{vmatrix} {}^1b_{h_1}, & {}^1b_{h_2}, & \dots & {}^1b_{h_n} \\ {}^2b_{h_1}, & {}^2b_{h_2}, & \dots & {}^2b_{h_n} \\ \dots & \dots & \dots & \dots \\ {}^nb_{h_1}, & {}^nb_{h_2}, & \dots & {}^nb_{h_n} \end{vmatrix}.$$

Now let r be any integer less than m, and let

$$\mu = \frac{m(m-1)\dots(m-r+1)}{1.2\dots r},$$

and, supposing $\theta_1, \theta_2, \dots \theta_r$ to be r numbers of the set $1, 2, \dots m$, let $G_1, G_2, \dots G_\mu$ denote the μ rectangular matrices of the forms

$$\begin{vmatrix} {}^{\theta_1}A \\ {}^{\theta_2}A \\ \dots\dots \\ {}^{\theta_r}A \end{vmatrix} \text{ respectively,}$$

and let $H_1, H_2, \dots H_\mu$ denote the μ rectangular matrices of the forms

$$\begin{vmatrix} {}^{\theta_1}B \\ {}^{\theta_2}B \\ \dots\dots \\ {}^{\theta_r}B \end{vmatrix} \text{ respectively.}$$

Now form the determinant

$$\begin{vmatrix} G_1 \times H_1, & G_1 \times H_2, & \dots & G_1 \times H_\mu \\ G_2 \times H_1, & G_2 \times H_2, & \dots & G_2 \times H_\mu \\ \dots & \dots & \dots & \dots \\ G_\mu \times H_1, & G_\mu \times H_2, & \dots & G_\mu \times H_\mu \end{vmatrix},$$

then, if we give r the successive values $1, 2, 3 \dots m$ (in which last case the determinant in question reduces to a single term), the values of the determinant above written will be severally in the proportions of

$$K,\ K^m,\ K^{\frac{1}{2}m(m-1)}, \dots K^m, K;$$

that is to say, the logarithms of these several determinants will be as the coefficients of the binomial expansion $(1+x)^m$.

When we make $r = m$, and equate the determinant corresponding to this value of r with that formed by making $r = 1$, the theorem becomes identical with a theorem previously given by M. Cauchy, for the Product of Rectangular Matrices.

It would be tedious to set forth the demonstration of the general theorem in detail. Suffice it here to say that it is a direct corollary from the formula marked (4) in my paper in the *Philosophical Magazine* for April 1851, entitled "On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Functions*," when that formula is particularized by making

$$\begin{Bmatrix} a_{m+1}, & a_{m+2}, & \dots & a_{m+n} \\ b_{m+1}, & b_{m+2}, & \dots & b_{m+n} \end{Bmatrix}$$

represent a determinant all whose terms are zeros except those which lie in one of the diagonals, these latter being all units, which comes, in fact, to defining that

$$\begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 1, \text{ and } \begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 0.$$

The important theorem here referred to is made almost unintelligible by an unfortunate misprint of ${}^q\theta_m$, ${}^1\theta_m$, ${}^2\theta_m$, ${}^\mu\theta_m$, in place of ${}^q\theta_r$, ${}^1\theta_r$, ${}^2\theta_r$, ${}^\mu\theta_r$. I may here take notice of another and still more inexplicable blunder in the same paper, formula (3)†, in the latter part of the equation belonging to which

$$\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2}, & \dots & a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \dots & a_{\theta_{m+s}} \\ a_{\phi_1}, & a_{\phi_2}, & \dots & a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \dots & a_{\phi_{m+s}} \end{Bmatrix}$$

is written in lieu of

$$\begin{Bmatrix} a_1, & a_2, & \dots & a_m, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \dots & a_{\theta_{m+s}} & a_{n+1} & a_{n+2} & \dots & a_{n+m} \\ a_1, & a_2, & \dots & a_m, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \dots & a_{\phi_{m+s}} & a_{n+1} & a_{n+2} & \dots & a_{n+m} \end{Bmatrix}.$$

[* p. 249 above.] [† See pp. 246, 251 above.]