50.

ON THE EXPRESSIONS FOR THE QUOTIENTS WHICH APPEAR IN THE APPLICATION OF STURM'S METHOD TO THE DISCOVERY OF THE REAL ROOTS OF AN EQUATION.

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MANY years ago I published expressions for the residues which appear in the application of the process of common measure to fx and $f'x$, and which constitute Sturm's auxiliary functions. These expressions are complete functions of the factors of fx and of differences of the roots of fx, and are therefore in effect functions of the factors exclusively, since the difference between any two roots may be expressed as the difference between two corresponding factors. Having found that in the practical applications of Sturm's theorem the quotients may be employed with advantage to replace the use of the residues, I have been led to consider their constitution; and having succeeded in expressing these quotients (which are of course linear functions of x) under a similar form to that of the residues, that is, as complete functions of the factors and differences of the roots of fx, I have pleasure in submitting the result to the notice of the Mathematical Section of the British Association.

Let $h_1, h_2, h_3 \dots h_n$ be the n roots of fx.

Let $\zeta(a, b, c \dots l)$ in general denote the squared product of the differences of $a, b, c \dots l$.

Let Z_i denote in general $\Sigma\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_i})$, where $\theta_1, \theta_2 \dots \theta_i$ indicate any combination of i out of the n quantities $a, b, c, \dots l$, with the convention that $Z_0 = 1$, $Z_1 = n$; and let (i) denote $\frac{1}{2}\{1 + (-1)^i\}$, being zero when i is odd, and unity when i is even; then I find that the ith quotient Q_i may be written under the form

$$Q_i = {}_iP_1{}^2(x - h_1) + {}_iP_2{}^2(x - h_2) + \dots + {}_iP_n{}^2(x - h_n),$$

where in general

$$\begin{aligned}{}_iP_e = {} & \frac{Z_{i-1}}{Z_i}\frac{Z^2{}_{i-3}}{Z^2{}_{i-2}}\frac{Z^2{}_{i-5}}{Z^2{}_{i-4}}\cdots\frac{Z^2{}_{(i)}}{Z^2{}_{(i)+1}} \\ & \times \Sigma\{\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_{i-1}}) \times (h_e - h_{\theta_1})(h_e - h_{\theta_2}) \dots (h_e - h_{\theta_{i-1}})\}.\end{aligned}$$

If we suppose $\frac{f'x}{fx}$, by means of the common measure process, to be expanded under the form of an improper continued fraction, the successive quotients will be the values of $Q_1, Q_2 \dots Q_n$ above found, that is

$$\frac{f'x}{fx} = \frac{1}{Q_1 -} \frac{1}{Q_2 -} \frac{1}{Q_3 -} \dots \frac{1}{Q_n};$$

the successive convergents of this fraction will be

$$\frac{1}{Q_1}, \; \frac{Q_2}{Q_1 Q_2 - 1}, \; \frac{Q_2 Q_3 - 1}{Q_1 Q_2 Q_3 - Q_1 - Q_3}, \dots, \frac{f'x}{fx}.$$

The numerators and denominators of these convergents will consequently also be functions of the factors exclusively. They are the quantities the sum of the products of which multiplied respectively by fx and $f'x$ produce (to constant factors *près*) the residues. The denominators are expressible very simply in terms of the factors and the differences of the roots; and their values under such forms were published by me about the same time as the values of the residues in the *Philosophical Magazine*; the expression for the numerators is much more complicated, but is given in my paper, "The Syzygetic Relations," &c., in the *Philosophical Transactions*. [p. 429 below.]

By comparing the expression for any quotient with the expressions for the two residues from which it may be derived, we obtain the following remarkable identity: $Z_{i-1} \times Z_i$, that is

$$\Sigma\zeta(h_1 h_2 \dots h_{i-1}) \times \Sigma\zeta(h_1 h_2 \dots h_i) = {}_iP_1^2 + {}_iP_2^2 + {}_iP_3^2 + \dots + {}_iP_n^2.$$

When the roots are all real, we have thus the product of one sum of squares by the product of another sum of squares (the number in each sum depending upon the arbitrary quantity i), brought under the form of a sum of a constant number n of squares, which in itself is an interesting theorem.

The expression above given for Q_i leads to a remarkable relation between the quotients and convergents to $\frac{f'x}{fx}$.

Let it be supposed, as before, that

$$\frac{f'x}{fx} = \frac{1}{Q_1 x -} \frac{1}{Q_2 x -} \frac{1}{Q_3 x -} \dots \frac{1}{Q_n x},$$

and let the successive convergents to this continued fraction be

$$\frac{N_1(x)}{D_1(x)}, \; \frac{N_2(x)}{D_2(x)}, \; \frac{N_3(x)}{D_3(x)}, \dots \frac{N_n(x)}{D_n(x)},$$

where the numerators and denominators are not supposed to undergo any reductions, but are retained in their crude forms as deduced from the law

$$N_i = Q_i N_{i-1} - N_{i-2},$$

$$D_i = Q_i D_{i-1} - D_{i-2}.$$

$N_1(x)$ being 1, and $D_1(x)$ being $Q_1(x)$; then it may be deduced from the published results above adverted to that

$$D_i(x) = \frac{Z^2_{i-1} Z^2_{i-3} \dots Z^2_{(i)}}{Z_i^2 Z^2_{i-2} \dots Z^2_{(i)+1}} \{\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_i})(x - h_{\theta_1})(x - h_{\theta_2}) \dots (x - h_{\theta_i})\}.$$

Hence $$\Sigma\{\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_{i-1}}) \times (h_e - h_{\theta_1})(h_e - h_{\theta_2}) \dots (h_e - h_{\theta_{i-1}})\}$$
$$= \frac{Z^2_{i-1} Z^2_{i-3} \dots Z^2_{(i-1)+1}}{Z^2_{i-2} Z^2_{i-4} \dots Z^2_{(i-1)}} D_{i-1}(h_e);$$

and we have therefore

$$_iP_e = \frac{Z^3_{i-1}}{Z_i} \frac{Z^4_{i-3}}{Z^4_{i-4}} \frac{Z^4_{i-5}}{Z^4_{i-6}} \dots \frac{Z^4_{(i)}}{Z^4_{(i)+1}} D_{i-1}(h_e),$$

and consequently

$$Q_i = \frac{Z^6_{i-1}}{Z_i^2} \frac{Z^8_{i-3}}{Z^8_{i-4}} \dots \frac{Z^8_{(i)}}{Z^8_{(i)+1}} \Sigma\{(D_{i-1}(h_e))^2 (x - h_e)\},$$

which is the general equation connecting the form of each quotient with that of the denominator to the immediately preceding unreduced convergent in the expansion of $\frac{f'x}{fx}$ under the form of an improper continued fraction.

If instead of the denominator of the unreduced convergents, the denominators of the convergents reduced to their simplest forms be employed, the powers of Z in the constant factor will undergo a diminution. The essential part of this theorem admits of being stated in general terms as follows:—

"If the quotient of an algebraical function of x by its first differential coefficient be expressed under the form of a continued fraction whose successive partial quotients are linear functions of x, any one of these quotients may be found (to a constant factor *près*) by taking the sum of the products formed by multiplying each factor $(x - h)$ of the given function by the square of what the denominator of the immediately antecedent convergent fraction becomes after substituting in it for x the root corresponding to such factor."

P.S. Since the above was read before the British Association, the theory has been extended by the author to comprise the general case of the expansion of any two algebraical functions under the form of a continued fraction, and has been incorporated into the paper in the *Philosophical Transactions* above referred to.