48.

ON STAUDT'S THEOREMS CONCERNING THE CONTENTS OF POLYGONS AND POLYHEDRONS, WITH A NOTE ON A NEW AND RESEMBLING CLASS OF THEOREMS.

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THE beautiful and important geometrical theorems of Staudt are, I believe, little, if at all, known to English mathematicians. They originally appeared in *Crelle's Journal* for the year 1843, and have been recently reproduced in M. Terquem's *Nouvelles Annales* for the August Number of the present year.

These theorems may be summed up, in a word, as intended to show the possibility and method of expressing the product of any two polygons or any two polyhedrons as entire functions of the squares of the distances of the angular points of the two figures from one another. The well-known expression for the square of the area of a triangle in terms of the sides (in which, when expanded, only even powers of the lengths of the sides appear), is but a particular case of Staudt's theorem for polygons, for it may be considered as the case of two equal and similar triangles whose angular points coincide. So in like manner, as observed by Staudt, a similar expression in terms of its sides may be found for the square of a pyramid. This expression had, however, been previously given (although, by a strangle negligence, not named for what it was) by Mr Cayley in the *Cambridge Mathematical Journal* for the year 1841*, in his paper on the relations between the mutual distances to one another of four points in a plane and five points in space; the singularly ingenious (and as singularly undisclosed) principle of that paper consisting in obtaining an expression for the volume of a pyramid in terms of its sides, and equating this, or rather its square, to zero as the conditions of the four angular points lying in the same plane.

* Query, Is not this expression for the volume of a pyramid in terms of its sides to be found in some previous writer? It can hardly have escaped inquiry.

The analogous condition for five points in space is virtually deduced by going out into rational space of four dimensions, and equating to zero the expression obtained for the volume of a plupyramid; meaning thereby the figure which stands in the same relation to space of four as a pyramid to space of three dimensions. Mr Cayley's method, if it had been pursued a step further, would have led him to a complete anticipation of the principal part of Staudt's discovery. The method here given is not substantially different from Mr Cayley's, but is made to rest upon a more general principle of transformation than that which he has employed. As to Staudt's own method, it is as clumsy and circuitous as his results are simple and beautiful. Geometry, trigonometry and statics, are laid under contribution to demonstrate relations which will be seen to flow as immediate and obvious consequences from the most elementary principles in the algorithm of determinants. Perhaps, however, M. Staudt's method is as good as could be found in the absence of the application of the method of determinants, the powers of which, even so recently as ten years ago, were not so well understood or so freely applied as at the present day.

The following new but simple theorem, of which I shall have occasion to make use, will be found to be a very useful addition to the ordinary method for the multiplication of determinants. "If the determinants represented by two square matrices are to be multiplied together, *any number of columns may be cut off* from the one matrix, and a corresponding number of columns from the other. Each of the lines in either one of the matrices so reduced in width as aforesaid being then multiplied by each line of the other, and the results of the multiplication arranged as a square matrix and bordered with the two respective sets of columns cut off arranged symmetrically (the one set parallel to the new columns, the other set parallel to the new lines), the complete determinant represented by the new matrix so bordered (abstraction made of the algebraical sign) will be the product of the two original determinants."

Thus $\begin{pmatrix} ab \\ cd \end{pmatrix} \times \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix}$ may be put under any one of the three following forms:—

$$\begin{vmatrix} a\alpha + b\beta, & a\gamma + b\delta \\ c\alpha + d\beta, & c\gamma + d\delta \end{vmatrix}$$

or
$$\begin{vmatrix} a\alpha, & a\gamma, & b \\ c\alpha, & c\gamma, & d \\ \beta, & \delta, & 0 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 2, & 2, & a, & b \\ 2, & 2, & c, & d \\ \alpha, & \beta, & 0, & 0 \\ \gamma, & \delta, & 0, & 0 \end{vmatrix}\text{*.}$$

* Any quantities might be substituted instead of 2 in the places occupied by the figure in the above determinant, as such terms do not influence the result; this figure is probably, however, the proper quantity arising from the application of the rule, because (as all who have calculated with determinants are aware) the value of the determinant represented by a matrix of *no places* is not zero but unity.

And in general for two matrices of n^2 terms each, this rule of multiplication will give $(n+1)$ distinct forms representing their products.

Thus, as a further example,

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \times \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix},$$

besides the first and last forms, will be representable by the two intermediate forms

$$-\begin{vmatrix} a\alpha+b\beta, & a\alpha'+b\beta', & a\alpha''+b\beta'', & c \\ a'\alpha+b'\beta, & a'\alpha'+b'\beta', & a'\alpha''+b'\beta'', & c' \\ a''\alpha+b''\beta, & a''\alpha'+b''\beta', & a''\alpha''+b''\beta'', & c'' \\ \gamma, & \gamma', & \gamma'', & 0 \end{vmatrix}$$

and

$$+\begin{vmatrix} a\alpha, & a\alpha', & a\alpha'', & b, & c \\ a'\alpha, & a'\alpha', & a'\alpha'', & b', & c' \\ a''\alpha, & a''\alpha', & a''\alpha'', & b'', & c'' \\ \beta, & \beta', & \beta'', & 0, & 0 \\ \gamma, & \gamma', & \gamma'', & 0, & 0 \end{vmatrix}.$$

To arrive, for instance, at the latter of these two forms, we have only to write the two given matrices under the respective forms

$$\begin{vmatrix} a, & b, & c, & 0, & 0 \\ a', & b', & c', & 0, & 0 \\ a'', & b'', & c'', & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} \alpha, & 0, & 0, & \beta, & \gamma \\ \alpha', & 0, & 0, & \beta', & \gamma' \\ \alpha'', & 0, & 0, & \beta'', & \gamma'' \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \end{vmatrix}$$

and then apply the ordinary rule of multiplication. So, again, to arrive at the first of the above written two forms, we must write the two given matrices under the respective forms

$$\begin{vmatrix} a, & b, & c, & 0 \\ a', & b', & c', & 0 \\ a'', & b'', & c'', & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \text{ and } -\begin{vmatrix} \alpha, & \beta, & 0, & \gamma \\ \alpha', & \beta', & 0, & \gamma' \\ \alpha'', & \beta'', & 0, & \gamma'' \\ 0, & 0, & 1, & 0 \end{vmatrix}$$

and proceed as before.

This rule is interesting as exhibiting, as above shown, a complete scale whereby we may descend from the ordinary mode of representing the product of two determinants to the form, also known, where the two original determinants are made to occupy opposite quadrants of a square whose places in one of the remaining quadrants are left vacant, and shows us that under one aspect at least this latter form may be regarded as a matrix *bordered* by the two given matrices.

A second but obvious theorem requiring preliminary notice is the following, namely that the value of the determinant to the matrix

$$\begin{matrix} a_{1,1}, & a_{1,2} \ldots a_{1,n}, & 1, \\ a_{2,1}, & a_{2,2} \ldots a_{2,n}, & 1, \\ \ldots\ldots & \ldots\ldots & \ldots \\ a_{n,1}, & a_{n,2} \ldots a_{n,n}, & 1, \\ 1, & 1, \ldots 1, & 0, \end{matrix}$$

is the same as the value of the determinant to the matrix

$$\begin{matrix} A_{1,1}, & A_{1,2} \ldots A_{1,n}, & 1, \\ A_{2,1}, & A_{2,2} \ldots A_{2,n}, & 1, \\ \ldots\ldots & \ldots\ldots & \ldots \\ A_{n,1}, & A_{n,2} \ldots A_{n,n}, & 1, \\ 1, & 1, \ldots 1, & 0, \end{matrix}$$

where in general

$$A_{r,s} = a_{r,s} + h_r + k_s,$$

$h_1, h_2 \ldots h_n$ and $k_1, k_2 \ldots k_n$ being any two perfectly arbitrary series of quantities. This simple transformation is of course derived by adding to the respective columns in the first matrix the last column (consisting of units) multiplied respectively by $h_1, h_2 \ldots h_n, 0$; and to the respective lines, the last line (consisting of units) multiplied respectively by $k_1, k_2 \ldots k_n, 0$.

Suppose, now, that we have two tetrahedrons whose volumes are represented respectively by one-sixth of the respective determinants

$$\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}, \quad \begin{vmatrix} \xi_1, & \eta_1, & \zeta_1, & 1 \\ \xi_2, & \eta_2, & \zeta_2, & 1 \\ \xi_3, & \eta_3, & \zeta_3, & 1 \\ \xi_4, & \eta_4, & \zeta_4, & 1 \end{vmatrix},$$

x_r, y_r, z_r representing the orthogonal coordinates of the point r in one tetrahedron, and ξ_r, η_r, ζ_r the same for the point r in the other.

By the first theorem their product may be represented (striking off the last column only from each matrix) by the matrix

$$\begin{vmatrix} \Sigma x_1\xi_1, & \Sigma x_1\xi_2, & \Sigma x_1\xi_3, & \Sigma x_1\xi_4, & 1 \\ \Sigma x_2\xi_1, & \Sigma x_2\xi_2, & \Sigma x_2\xi_3, & \Sigma x_2\xi_4, & 1 \\ \Sigma x_3\xi_1, & \Sigma x_3\xi_2, & \Sigma x_3\xi_3, & \Sigma x_3\xi_4, & 1 \\ \Sigma x_4\xi_1, & \Sigma x_4\xi_2, & \Sigma x_4\xi_3, & \Sigma x_4\xi_4, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

where, in general, any such term as $\Sigma x_r \xi_s$ represents

$$x_r \xi_s + y_r \eta_s + z_r \zeta_s.$$

Again, by virtue of the second theorem, adding

$$-\tfrac{1}{2}\Sigma x_1^2, \quad -\tfrac{1}{2}\Sigma x_2^2, \quad -\tfrac{1}{2}\Sigma x_3^2, \quad -\tfrac{1}{2}\Sigma x_4^2$$

to the respective lines, and

$$-\tfrac{1}{2}\Sigma \xi_1^2, \quad -\tfrac{1}{2}\Sigma \xi_2^2, \quad -\tfrac{1}{2}\Sigma \xi_3^2, \quad -\tfrac{1}{2}\Sigma \xi_4^2$$

to the respective columns, the above matrix becomes (after a change of signs not affecting the result) the $-\frac{1}{8}$th of

$$\begin{vmatrix} \Sigma(x_1-\xi_1)^2, & \Sigma(x_1-\xi_2)^2, & \Sigma(x_1-\xi_3)^2, & \Sigma(x_1-\xi_4)^2, & 1 \\ \Sigma(x_2-\xi_1)^2, & \Sigma(x_2-\xi_2)^2, & \Sigma(x_2-\xi_3)^2, & \Sigma(x_2-\xi_4)^2, & 1 \\ \Sigma(x_3-\xi_1)^2, & \Sigma(x_3-\xi_2)^2, & \Sigma(x_3-\xi_3)^2, & \Sigma(x_3-\xi_4)^2, & 1 \\ \Sigma(x_4-\xi_1)^2, & \Sigma(x_4-\xi_2)^2, & \Sigma(x_4-\xi_3)^2, & \Sigma(x_4-\xi_4)^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

or calling the angular points of the one tetrahedron a, b, c, d, and of the other p, q, r, s, 8×36, that is 288 times, their product is representable by $-1 \times$ the determinant

$$\begin{vmatrix} (ap)^2, & (aq)^2, & (ar)^2, & (as)^2, & 1 \\ (bp)^2, & (bq)^2, & (br)^2, & (bs)^2, & 1 \\ (cp)^2, & (cq)^2, & (cr)^2, & (cs)^2, & 1 \\ (dp)^2, & (dq)^2, & (dr)^2, & (ds)^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

and of course if p, q, r, s coincide respectively with a, b, c, d, 576 times the square of the tetrahedron $abcd$ will be represented under Mr Cayley's form,

$$\begin{vmatrix} 0, & (ab)^2, & (ac)^2, & (ad)^2, & 1 \\ (ba)^2, & 0, & (bc)^2, & (bd)^2, & 1 \\ (ca)^2, & (cb)^2, & 0, & (cd)^2, & 1 \\ (da)^2, & (db)^2, & (dc)^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} *,$$

four out of the sixteen distances vanishing, and the remaining twelve reducing to six pairs of equal distances. The demonstration of Staudt's

* The corresponding quantity to the above determinant for the case of the triangle (hereafter given) is identical with the Norm to the sum of the sides. I have succeeded in finding the Factor (of ten dimensions in respect of the edges), which, multiplied by the above Determinant itself, expresses the Norm to the sum of the Faces, that is, the superficial area of the Tetrahedron.

theorem for triangles is obtained in precisely the same way by throwing the product of the two determinants

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \xi_1, & \eta_1, & 1 \\ \xi_2, & \eta_2, & 1 \\ \xi_3, & \eta_3, & 1 \end{vmatrix}$$

under the form of $-\frac{1}{4}$th of

$$\begin{vmatrix} \Sigma(x_1-\xi_1)^2, & \Sigma(x_1-\xi_2)^2, & \Sigma(x_1-\xi_3)^2, & 1 \\ \Sigma(x_2-\xi_1)^2, & \Sigma(x_2-\xi_2)^2, & \Sigma(x_2-\xi_3)^2, & 1 \\ \Sigma(x_3-\xi_1)^2, & \Sigma(x_3-\xi_2)^2, & \Sigma(x_3-\xi_3)^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}.$$

When the two triangles coincide, calling their angular points a, b, c the above written determinant becomes

$$\begin{vmatrix} 0, & (ab)^2, & (ac)^2, & 1 \\ (ba)^2, & 0, & (bc)^2, & 1 \\ (ca)^2, & (cb)^2, & 0, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

or

$$(ab)^4+(ac)^4+(bc)^4-2(ab)^2.(ac)^2-2(ab)^2.(bc)^2-2(ac)^2.(bc)^2,$$

the negative of which is the well-known form expressing the square of four times the area of the triangle abc.

There is another and more general theorem of Staudt for two triangles not in the same plane, which may be obtained with equal facility. In fact, if we start from the determinant

$$\begin{vmatrix} (a\alpha)^2, & (a\beta)^2, & (a\gamma)^2, & 1 \\ (b\alpha)^2, & (b\beta)^2, & (b\gamma)^2, & 1 \\ (c\alpha)^2, & (c\beta)^2, & (c\gamma)^2, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

and add to each column respectively the last column multiplied by $e\xi_1^2$, $e\xi_2^2$, $e\xi_3^2$ respectively, we arrive at the form

$$\begin{vmatrix} (a\alpha)^2+e\xi_1^2, & (a\beta)^2+e\xi_2^2, & (a\gamma)^2+e\xi_3^2, & 1 \\ (b\alpha)^2+e\xi_1^2, & (b\beta)^2+e\xi_2^2, & (b\gamma)^2+e\xi_3^2, & 1 \\ (c\alpha)^2+e\xi_1^2, & (c\beta)^2+e\xi_2^2, & (c\gamma)^2+e\xi_3^2, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

and considering ξ_1, η_1; ξ_2, η_2; ξ_3, η_3 as the coordinates of α, β, γ, the projections upon the plane of abc of a triangle ABC, whose plane intersects the former plane in the axis of y, and makes with that plane an angle whose tangent is e, it is easily seen that this determinant is term for term identical with the determinant

$$\begin{vmatrix} (aA)^2, & (aB)^2, & (aC)^2, & 1 \\ (bA)^2, & (bB)^2, & (bC)^2, & 1 \\ (cA)^2, & (cB)^2, & (cC)^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix},$$

which therefore expresses -16 times the product of the triangles abc and $\alpha\beta\gamma$, that is $abc \times ABC \times$ cosine of the angle between the two. A similar method, if we ascend from sensible to rational geometry, may be given for expressing in terms of the distances the product of *any* two pyramids (in a hyperspace) by the cosine of the angle included between the two infinite spaces* in which they respectively lie. To pass from the cases which have been considered of two triangles to two polygons, or of two tetrahedrons to two polyhedrons, generally presents no difficulty; and for Professor Staudt's method of doing so, which is simple and ingenious, and does not admit of material improvement, the reader is referred to the memoir in Crelle's *Journal* or Terquem's *Annales* already adverted to. It is, however, to be remarked (and this does not appear to be sufficiently noticed in the memoirs referred to), that whilst the expression for the product of any two polygons in terms of the distances given by Staudt's theorem is unique, that for the product of two polyhedrons given by the same is not so, but will admit of as many varieties of representation as there are units in the product of the numbers respectively expressing the number of ways in which each polygonal face of each polyhedron admits of being mapped out into triangles. I cannot help conjecturing (and it is to be wished that Professor Staudt or some other geometrician would consider this point) that in every case there exists, linearly derivable from Staudt's optional formulæ (but not coincident with any one of them), some unique and best, because most symmetrical, formula for expressing the product of two polyhedrons in terms of the distances of the angular points of the one from those of the other. In conclusion I may observe, that there is a theorem for distances measured on a given straight line, which, although not mentioned by Staudt, belongs to precisely the same class as his theorems for areas in a plane and volumes in space; namely a theorem which expresses twice the rectangle of any two such distances under the form of an aggregate of four squares, two taken positively and two

* In rational or universal geometry, that which is commonly termed infinite space (as if it were something absolute and unique, and to which, by the conditions of our being, the representative power of the understanding is limited), is regarded as a single homaloid related to a plane, precisely in the same way as a plane is to a right line. Universal geometry brings home to the mind with an irresistible force of conviction the truth of the Kantian doctrine of locality.

negatively; that is to say, if A, B, C, D be any four points on a right line $2AB \times CD = AD^2 + BC^2 - AC^2 - BD^2$. I know not whether this theorem be new, but it is one which evidently must be of considerable utility to the practical geometer.

Note on the above.

The fundamental theorem in determinants, published by me in the *Philosophical Magazine* in the course of last year*, leads immediately to a class of theorems strongly resembling, and doubtless intimately connected with, those of Staudt.

Thus for triangles we have by this fundamental theorem

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ \eta_1, & \eta_2, & \eta_3 \\ 1, & 1, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1, & \xi_1, & \xi_2 \\ y_1, & \eta_1, & \eta_2 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_3, & x_2, & x_3 \\ \eta_3, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} + \begin{vmatrix} x_1, & \xi_2, & \xi_3 \\ y_1, & \eta_2, & \eta_3 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_1, & x_2, & x_3 \\ \eta_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} x_1, & \xi_3, & \xi_1 \\ y_1, & \eta_3, & \eta_1 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_2, & x_2, & x_3 \\ \eta_2, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}$$

and consequently, if ABC, DEF be any two triangles,

$$ABC \times DEF = ADE \times FBC + AEF \times DBC + AFD \times BCE.$$

This may be considered a theorem relating to two ternary systems of points in a plane. The analogous and similarly obtainable theorem for two binary systems of points in the same right line is

$$AB \times CD = AC \times DB - AD \times CB.$$

As in applying this last theorem to obtain correct numerical results we must give the same algebraical sign to any two lengths denoted by the two *arrangements* XY, ZT, according as the direction from X to Y is the same as that from Z to T, or contrary to it, so in the theorem for the products of triangles, the areas denoted by any two ternary arrangements XYZ, TUV must be taken with the like or the contrary sign, according as the direction of the rotation XYZ is consentient with or contrary to that of TUV; so that three of the six possible arrangements of XYZ may be used indifferently for one another, but the other three would imply a change of sign. If we

[* See pp. 249, 253 above.]

analyse what we mean by fixing the direction of the rotation of XYZ, and reduce this form of speech to its simplest terms, we easily see that it amounts to ascertaining on which side of B, C lies, that is whether to its right or left, to a spectator stationed at A on a given side of the plane ABC.

Let us now pass to the corresponding theorems for two tetrahedrons put respectively under the forms

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix}, \quad \begin{vmatrix} \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ \zeta_1, & \zeta_2, & \zeta_3, & \zeta_4 \\ 1, & 1, & 1, & 1 \end{vmatrix}.$$

We may represent this product in either of two ways by the application of our fundamental theorem, namely as

$$\begin{vmatrix} x_1, & \xi_1, & \xi_2, & \xi_3 \\ y_1, & \eta_1, & \eta_2, & \eta_3 \\ z_1, & \zeta_1, & \zeta_2, & \zeta_3 \\ 1, & 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_4, & x_2, & x_3, & x_4 \\ \eta_4, & y_2, & y_3, & y_4 \\ \zeta_4, & z_2, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix} + \&c.$$

or as

$$\begin{vmatrix} x_1, & x_2, & \xi_1, & \xi_2 \\ y_1, & y_2, & \eta_1, & \eta_2 \\ z_1, & z_2, & \zeta_1, & \zeta_2 \\ 1, & 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_3, & \xi_4, & x_3, & x_4 \\ \eta_3, & \eta_4, & y_3, & y_4 \\ \zeta_3, & \zeta_4, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix} + \&c.$$

there being four products to be added together in the first expression and six in the latter; and the rule, if we wish that all the products may be additive, being that on removing the sign of multiplication the determinant to the square matrix formed by the Greek letters *in situ* shall always preserve the same sign. Hence we derive two geometrical formulæ concerning the products of polyhedrons, namely

$$(1) \quad ABCD \times EFGH = ABCE \times FGHD - ABCF \times GHED + ABCG \times HEFD - ABCH \times FGED.$$

$$(2) \quad ABCD \times EFGH = ABEF \times GHCD + ABGH \times EFCD + ABEG \times HFCD + ABHF \times EGCD + ABEH \times FGCD + ABFG \times EHCD.$$

These formulæ give rise to an exceedingly interesting observation. In order that they shall be numerically true, we must have a rule for fixing the sign to be given to the solid content represented by any reading off of the four points of a tetrahedron, that is we must have a rule for determining the sign of solid contents of figures situated anywhere in space analogous to that which, as applied to linear distances reckoned on a given right line, is the true foundation of the language of trigonometry, and the condition precedent for the possibility of any system of analytical geometry such as exists, and which, not altogether without surprise, I have observed in the pages of this *Magazine* one of the learned contributors has thought it necessary to vindicate the propriety of importing into his theory of quaternions.

Various rules may be given for fixing the sign of a tetrahedron denoted by a given order of four letters. One is the following: the content of $ABCD$ is to be taken positive or negative, according as to a spectator at A the rotation of BCD is positive or negative. Another, again, is to consider AB and CD as representing, say two electrical currents, and to suppose a spectator so placed that the current AB shall pass through the longitudinal axis of his body from the head towards the feet, and looking towards the other current CD; the sign of the solid content of the tetrahedron (and, indeed, also the effect, in a general sense, of the action of the two currents upon one another) will depend upon the circumstance of this latter current appearing to flow from the right to the left, or contrariwise in respect of the spectator. Last and simplest mode of all, the sign of the solid content of $ABCD$ will depend upon the *nature* (in respect to its being a right-handed or left-handed-screw) of any regular screw-line (whether the common helix or one in which the increase or decrease of the inclination is always in the same direction) terminating at B and C, and so taken that BA shall be the direction of the tangent produced at B, and CD the direction of the tangent produced at C. Inasmuch as of the twenty-four permutations of a quaternary arrangement a defined twelve have one sign, and the other twelve the contrary sign, these various definitions of the direction, or, as it may be termed, polarity, of a tetrahedron corresponding to a given reading, whether as taken each in itself or compared one with another, give rise to, or rather imply a considerable number of interesting theorems included in our intuitions of space, and probably belonging to the, in my belief, inexhaustible class of primary and indemonstrable truths of the understanding.