45.

ON A REMARKABLE THEOREM IN THE THEORY OF EQUAL ROOTS AND MULTIPLE POINTS.

[*Philosophical Magazine*, III. (1852), pp. 375—378.]

In order that the theorem which I propose to state may be the more easily understood, and with the least ambiguity expressed, I shall commence with the case of a homogeneous function of two variables only, x and y.

Let

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots + nb'xy^{n-1} + a'y^n,$$

and let the result of operating with the symbol

$$x^n \frac{d}{da} + x^{n-1}y\frac{d}{db} + \ldots + y^{n-1}x\frac{d}{db'} + y^n\frac{d}{da'},$$

on any function of a, b, $c \ldots b'$, a' be called the Evectant of such function, and the result of repeating this process r times the rth Evectant.

Understand by the multiplicity of the equation the number of equalities between the roots that exist; so that a pair of equal roots will signify a multiplicity 1, two pairs of equal roots, or three equal roots a multiplicity 2; a pair of equal roots and a set of three equal roots, a multiplicity $1+2$ or 3, and so on. Now suppose the total multiplicity of ϕ to be m: the first part of the proposition consists in the assertion that the 1st, 2nd, 3rd ... $(m-1)$th Evectants of the discriminant of ϕ, that is of the result of eliminating x and y between $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$ (as well as the discriminant itself), will all vanish in whatever way the multiplicity is distributed; the second part of the proposition about to be stated requires that the mode should be taken into account of the manner in which the multiplicity (m) is made up. Suppose, then, that there are r groups of roots, for one of which the multiplicity is m_1, for the second m_2, &c., and for the rth m_r, so that $m_1 + m_2 + \ldots + m_r = m$. Then, I say, that the mth evectant of the determinant of ϕ is of the form

$$(a_1x + b_1y)^{m_1n}\,(a_2x + b_2y)^{m_2n} \ldots (a_rx + b_ry)^{m_rn},$$

where $a_1 : b_1$, $a_2 : b_2 \ldots a_r : b_r$ are the ratios of $x : y$ corresponding to the several sets of equal roots.

This latter part of the theorem for the case of $m = 1$ was discovered inductively by Mr Cayley, by considering the cases when ϕ is a cubic, or a biquadratic function. I extended the theory to functions of any number of variables, and supplied a demonstration, that is for the case of one pair of equal roots. Mr Salmon showed that my demonstration could be applied to the case of two pairs of equal roots, or two double points, &c., and very nearly at the same time I made the like extension to the case of three equal roots, cusps, &c., and almost immediately after I obtained a demonstration for the theorem in its most general form. This demonstration reposes upon a very refined principle, which I had previously discovered but have not yet published, in the Theory of Elimination.

I have here anticipated a little in speaking of the theorem as applicable to curves and other loci.

Suppose $\phi(x, y, z) = 0$ to be the equation to a curve expressed homogeneously.

Let

$$\begin{aligned}\phi(x, y, z) &= ax^n + (na'x^{n-1}y + nb'x^{n-1}z)\\ &\quad + \tfrac{1}{2}n(n-1)\,a''x^{n-2}y^2 + n(n-1)\,b''x^{n-2}yz + \tfrac{1}{2}n(n-1)\,c''x^{n-2}z^2,\\ &\quad + \text{\&c.} \qquad \text{\&c.},\end{aligned}$$

and understand by the evectant of any quantity the result of operating upon it with the symbol

$$x^n \frac{d}{da} + x^{n-1}y\,\frac{d}{da'} + x^{n-1}z\,\frac{d}{db'} + x^{n-2}y^2\frac{d}{da''} + \text{\&c.}$$

Suppose, now, the curve to have double points, the $(r-1)$th evectant (and of course all the inferior evectants) of the discriminant of ϕ (meaning thereby the result of eliminating x, y, z between $\dfrac{d\phi}{dx}$, $\dfrac{d\phi}{dy}$, $\dfrac{d\phi}{dz}\Big)$ will all vanish, and the rth evectant will be of the form

$$(a_1x + b_1y + c_1z)^n \times (a_2x + b_2y + c_2z)^n \ldots \times (a_rx + b_ry + c_rz)^n,$$

where $a_1 : b_1 : c_1$, $a_2 : b_2 : c_2 \ldots a_r : b_r : c_r$ are the ratios of the coordinates at the respective double points. If there be cusps the multiplicity of each such will be 2; and calling the total multiplicity m, to every cusp will correspond a factor of the $2n$th power in the mth evectant; and so on in general for various degrees of multiplicity at the singular points respectively. The like theorem extends to conical and other singular points of surfaces; so that there exists a method, when a locus is given having any degree of multiplicity, of at once detecting the amount and distribution of this multiplicity, and the positions of the one or more singular points. In conclusion I may state, that precisely analogous results (*mutatis mutandis*) obtain, when, in place of a single function having multiplicity, we take the more general supposition of any number of homogeneous functions being subject to the condition of pluri-simultaneity, that is being capable of being made to vanish by each of several different systems of values for the ratios between the variables. Multiplicity in a single function is, in fact, nothing more nor less than pluri-simultaneity existing between the functions derived from it by differentiating with respect to each of the given variables successively. But as I purpose to give these theorems and their demonstration, which I have already imparted to my mathematical correspondents, in a paper destined for reading before the Royal Society, I need not further enlarge upon them on the present occasion.

P.S. In the above statement I have spoken only of cusps of curves which are the precise and unambiguous analogues of three coincident points in point-systems, in order to avoid the necessity of entering into any disquisition as to the species of singularity in curves or other loci corresponding to higher degrees of multiplicity in point-systems, a subject which has not hitherto been completely made out. I may here also add a remark, which gives a still higher interest to the theory, which is (to confine ourselves, for the sake of brevity, to functions of two variables), that if any root of $x:y$, say $a:b$, occur $1+\mu$ times, the total multiplicity of the equation being supposed m, and its degree n, then taking ι any integer number not exceeding μ, the $(m+\iota)$th evectant of the discriminant will contain the factor $(ax+by)^{(\mu-\iota)n}$. So that, for instance, if there be but a single group of equal roots, and they be $1+\mu$ in number, every evectant up to the $(\mu-1)$th inclusive will vanish, and from the μth to the $(2\mu-\iota)$th will contain a power of $(ax+by)^n$.