41.

ON A REMARKABLE DISCOVERY IN THE THEORY OF CANONICAL FORMS AND OF HYPERDETERMINANTS.

[*Philosophical Magazine*, II. (1851), pp. 391—410.]

In a recently printed continuation* of a paper which appeared in the *Cambridge and Dublin Mathematical Journal*, I published a complete solution of the following problem. A homogeneous function of x, y of the degree $2n+1$ being given, required to represent it as the sum of $n+1$ powers of linear functions of x, y. I shall prepare the way for the more remarkable investigations which form the proper object of this paper, by giving a new and more simple solution of this linear transformation.

Let the given function be

$$a_0x^{2n+1}+(2n+1)\,a_1x^{2n}y+\tfrac{1}{2}\,(2n+1)\,(2n)\,a_2x^{2n-1}y^2+\ldots+a_{2n+1}y^{2n+1},$$

and suppose that this is identical with

$$(p_1x+q_1y)^{2n+1}+(p_2x+q_2y)^{2n+1}+\text{\&c.}+(p_{n+1}x+q_{n+1}y)^{2n+1}.$$

The problem is evidently possible and definite, there being $2n+2$ equations to be satisfied, and $(2n+2)$ quantities p_1, q_1, &c. for satisfying the same.

In order to effect the solution, let

$$q_1=p_1\lambda_1,$$

$$q_2=p_2\lambda_2,$$

$$\text{\&c.}=\text{\&c.}$$

$$q_{n+1}=p_{n+1}\lambda_{n+1},$$

[* p. 203 above.]

we have then

$$\begin{array}{llll}
p_1^{2n+1} & +p_2^{2n+1} & +\ldots+p_{n+1}^{2n+1} & =a_0, \\
p_1^{2n+1}\lambda_1 & +p_2^{2n+1}\lambda_2 & +\ldots+p_{n+1}^{2n+1}\lambda_{n+1} & =a_1, \\
p_1^{2n+1}\lambda_1^2 & +p_2^{2n+1}\lambda_2^2 & +\ldots+p_{n+1}^{2n+1}\lambda^2_{n+1} & =a_2, \\
\multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
p_1^{2n+1}\lambda_1^n & +p_2^{2n+1}\lambda_2^n & +\ldots+p_{n+1}^{2n+1}\lambda^n_{n+1} & =a_n, \\
p_1^{2n+1}\lambda_1^{n+1} & +p_2^{2n+1}\lambda_2^{n+1} & +\ldots+p_{n+1}^{2n+1}\lambda_{n+1}^{n+1} & =a_{n+1}, \\
\multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
p_1^{2n+1}\lambda_1^{2n+1} & +p_2^{2n+1}\lambda_2^{2n+1} & +\ldots+p_{n+1}^{2n+1}\lambda_{n+1}^{2n+1} & =a_{2n+1}.
\end{array}$$

Eliminate $p_1, p_2 \ldots p_{n+1}$ between the 1st, 2nd, 3rd ... $(n+2)$th equations, and it is easily seen that we obtain

$$a_{n+1}-a_n\Sigma\lambda_1+a_{n-1}\Sigma\lambda_1\lambda_2 \ldots \pm a_0\lambda_1\lambda_2\ldots\lambda_{n+1}=0.$$

Again, eliminating in like manner $p_1^{2n+1}\lambda_1$, $p_2^{2n+1}\lambda_2 \ldots p_{n+1}^{2n+1}\lambda_{n+1}$ between the 2nd, 3rd ... $(n+3)$th equations, we obtain

$$a_{n+2}-a_{n+1}\Sigma\lambda_1+\ldots\mp a_1\lambda_1\lambda_2\ldots\lambda_{n+1}=0;$$

and proceeding in the same way until we come to the combination of the $(n+1)$th ... $(2n+2)$th equations, and writing

$$\begin{array}{ll}
\Sigma\lambda_1 & =s_1, \\
\Sigma\lambda_1\lambda_2 & =s_2, \\
\dots\dots\dots\dots & \\
\lambda_1\lambda_2\ldots\lambda_{n+1} & =s_{n+1},
\end{array}$$

we find

$$\begin{array}{l}
a_{n+1}-a_n\ \ s_1+a_{n-1}\ s_2\ldots\pm a_0s_{n+1}=0, \\
a_{n+2}-a_{n+1}s_1+a_n\ \ s_2\ldots\mp a_1s_{n+1}=0, \\
a_{n+3}-a_{n+2}s_1+a_{n+1}\ s_2\ldots\pm a_2s_{n+1}=0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
a_{2n+1}-a_{2n}\ \ s_1+a_{2n-1}s_2\ldots+a_ns_{n+1}=0^*.
\end{array}$$

Hence it is obvious that

$$(x+\lambda_1y)(x+\lambda_2y)\ldots(x+\lambda_{n+1}y)$$

is a constant multiple of the determinant

$$\left|\begin{array}{ccccc}
x^{n+1}, & -x^ny, & x^{n-1}y^2 & \ldots & \pm y^{n+1} \\
a_{n+1}, & a_n, & a_{n-1} & \ldots & a_0 \\
a_{n+2}, & a_{n+1}, & a_n & \ldots & a_1 \\
\multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\
a_{2n+1}, & a_{2n}, & a_{2n-1} & \ldots & a_n
\end{array}\right|.$$

* These equations in their simplified form arise from the ordinary result of elimination, in this case containing as a factor the product of the differences of the quantities $\lambda_1, \lambda_2, \ldots \lambda_{n+1}$.

Hence $\lambda_1, \lambda_2 \ldots \lambda_{n+1}$ are known, and consequently

$$p_1, p_2 \cdots p_{n+1}, q_1, q_2 \cdots q_{n+1}$$

are known, by the solution of an equation of the $(n+1)$th degree.

Thus suppose the given function to be

$$F = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + 10fy^5$$
$$= (p_1x + q_1y)^5 + (p_2x + q_2y)^5 + (p_3x + q_3y)^5,$$

we shall have, by an easy inference from what has preceded,

$$(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$$

equal to a numerical multiple of the determinant

$$\begin{vmatrix} x^3, & -x^2y, & xy^2, & -y^3 \\ d, & c, & b, & a \\ e, & d, & c, & b \\ f, & e, & d, & c \end{vmatrix}.$$

The solution of the problem given by me in the paper before alluded to presents itself under an *apparently* different and rather less simple form. Thus, in the case in question, we shall find according to that solution,

$$(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$$

equal to a numerical multiple of the determinant

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix}.$$

The two determinants, however, are in fact identical, as is easily verified, for the coefficients of x^3 and y^3 are manifestly alike; and the coefficient of x^2y in the second form will be made up of the three determinants,

$$\begin{vmatrix} a, & b, & d \\ b, & c, & e \\ c, & d, & f \end{vmatrix}, \quad \begin{vmatrix} a, & c, & c \\ b, & d, & d \\ c, & e, & e \end{vmatrix}, \quad \begin{vmatrix} b, & b, & c \\ c, & c, & d \\ d, & d, & e \end{vmatrix},$$

of which the latter two vanish, and the first is identical with the coefficient of x^2y in the first solution. The same thing is obviously true in regard of the coefficients of xy^2 in the two forms, and a like method may be applied to show that in all cases the determinant above given is identical with the determinant of my former paper, namely

$$\begin{vmatrix} a_0x + a_1y, & a_1x + a_2y & \ldots & a_nx + a_{n+1}y \\ a_1x + a_2y, & a_2x + a_3y & \ldots & a_{n+1}x + a_{n+2}y \\ \cdots & \cdots & \cdots & \cdots \\ a_nx + a_{n+1}y, & a_{n+1}x + a_{n+2}y & \ldots & a_{2n}x + a_{2n+1}y \end{vmatrix}.$$

Thus, then, we see that for odd-degreed functions, the reduction to their canonical form of the sum of $(n+1)$ powers depends upon the solution of one single equation of the $(n+1)$th degree, and can never be effected in more than one way.

This new form of the resolving determinant affords a beautiful criterion for a function of x, y of the degree $2n+1$ being composed of n instead of, as in general, $(n+1)$ powers. In order that this may be the case, it is obvious that two conditions must be satisfied; but I pointed out in my supplemental paper on canonical forms, that all the coefficients of the resolving determinant must vanish, which appears to give far too many conditions. Thus, suppose we have

$$ax^7 + 7bx^6y + 21cx^5y^2 + 35dx^4y^3 + 35ex^3y^4 + 21fx^2y^5 + 7gxy^6 + hy^7.$$

The conditions of catalecticism, that is, of its being expressible under the form of the sum of three (instead of, as in general, four) seventh powers, requires that all the coefficients of the different powers of x and y must vanish in the determinant

$$\begin{vmatrix} y^4, & -y^3x, & y^2x^2, & -yx^3, & x^4 \\ a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \\ d, & e, & f, & g, & h \end{vmatrix};$$

in other words, we must have five determinants,

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}, \quad \begin{vmatrix} a, & c, & d, & e \\ b, & d, & e, & f \\ c, & e, & f, & g \\ d, & f, & g, & h \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c, & e \\ b, & c, & d, & f \\ c, & d, & e, & g \\ d, & e, & f, & h \end{vmatrix},$$

$$\begin{vmatrix} a, & b, & d, & e \\ b, & c, & e, & f \\ c, & d, & f, & g \\ d, & e, & g, & h \end{vmatrix}, \quad \begin{vmatrix} b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \\ e, & f, & g, & h \end{vmatrix},$$

all separately zero. But by my homaloidal law*, all these five equations amount only to $(5-4)(5-3)$, that is, to 2. I may notice here, that a theorem substantially identical with this law, and another absolutely identical with the theorem of compound determinants given by me in this *Magazine*, and afterwards generalized in a paper also published† in this *Magazine*, entitled

[* p. 150 above.] [† p. 241 above.]

"On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Forms," have been subsequently published as original in a recent number of M. Liouville's journal.

The general condition of mere singularity, as distinguished from catalecticism, that is, of the function of the degree $2n+1$, being incapable of being expressed as the sum of $n+1$ powers, is that the resolving resultant shall have two equal roots; in other words, that its determinant shall be zero.

Mr Cayley has pointed out to me a very elegant mode of identifying the two forms of the resolving resultant, which I have much pleasure in subjoining. Take as the example a function of the fifth degree, we have by the multiplication of determinants,

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ x, & y, & 0, & 0 \\ 0, & x, & y, & 0 \\ 0, & 0, & x, & y \end{vmatrix}$$

$$= \begin{vmatrix} y^3, & a, & b, & c \\ 0, & ax+by, & bx+cy, & cx+dy \\ 0, & bx+cy, & cx+dy, & dx+ey \\ 0, & cx+dy, & dx+ey, & ex+fy \end{vmatrix},$$

which dividing out each side of the equation by y^3, immediately gives the identity required, and the method is obviously general.

Turn we now to consider the mode of reducing a biquadratic function of two letters to its canonical form, *videlicet*

$$(fx+gy)^4 + (hx+ky)^4 + 6m\,(fx+gy)^2\,(hx+ky)^2.$$

Let the given function be written

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Let $\quad g = f\lambda_1,\ \ k = h\lambda_2,\ \ mf^2h^2 = \mu,\ \ \lambda_1 + \lambda_2 = s_1,\ \ \lambda_1\lambda_2 = s_2,$

then we have

$$f^4 + h^4 + 6\mu = a,$$

$$4f^4\lambda_1 + 4h^4\lambda_2 + 6\mu\,(2s_1) = 4b,$$

$$6f^4\lambda_1^2 + 6h^4\lambda_2^2 + 6\mu\,(s_1^2 + 2s_2) = 6c,$$

$$4f^4\lambda_1^3 + 4h^4\lambda_2^3 + 6\mu\,(2s_1s_2) = 4d,$$

$$f^4\lambda_1^4 + h^4\lambda_2^4 + 6\mu s_2^2 = e.$$

Eliminating f and h between the first, second and third; the second, third and fourth; and the third, fourth and fifth equations successively, we obtain

$$as_2 - bs_1 + c - \mu(8s_2 - 2s_1^2) = 0,$$

$$bs_2 - cs_1 + d - \mu(4s_1s_2 - s_1^3) = 0,$$

$$cs_2 - ds_1 + e - \mu(8s_2^2 - 2s_1^2s_2) = 0.$$

Let now $$(2s_1^2 - 8s_2)\mu = \nu,$$

and we shall have $$as_2 - bs_1 + (c + \nu) = 0,$$

$$bs_2 - \left(c - \frac{\nu}{2}\right)s_1 + d = 0,$$

$$(c + \nu)s_2 - ds_1 + e = 0.$$

Hence ν will be found from the cubic equation

$$\begin{vmatrix} a, & b, & c+\nu \\ 2b, & 2c-\nu, & 2d \\ c+\nu, & d, & e \end{vmatrix} = 0,$$

that is, $$\nu^3 - \nu(ae - 4bd + 3c^2) + \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0,$$

in which equation it will not fail to be noticed that the coefficient of ν^2 is zero, and the remaining coefficients are the two well-known hyperdeterminants, or, as I propose henceforth to call them, the two Invariants of the form

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4;$$

be it also further remarked that

$$\nu = 8(\tfrac{1}{4}s_1^2 - s_2)\mu,$$

in which equation the coefficient of 8μ is the Determinant or Invariant of

$$x^2 + s_1xy + s_2y^2.$$

When ν is thus found, s_1, s_2, and μ, being given by the equations in terms of ν, are known, and by the solution of a quadratic λ_1, λ_2 become known in terms of s_1, s_2, and f, h in terms of $\lambda_1, \lambda_2, \mu$, and the problem is completely determined. The most symmetrical mode of stating this method of solution is to suppose the given function thrown under the form

$$(fx + gy)^4 + (f_1x + g_1y)^4 + 6\epsilon(fx + gy)^2(f_1x + g_1y)^2.$$

Then writing

$$(fx + gy)(f_1x + g_1y) = Lx^2 + Mxy + Ny^2$$

$-\nu$, the quantity to be found by the solution of the cubic last given, becomes

$$8\epsilon\left(LN-\frac{M^2}{4}\right).$$

I shall now proceed to apply the same method to the reduction of the function

$$a_0x^8+8a_1x^7y+28a_2x^6y^2+56a_3x^5y^3+70a_4x^4y^4+56a_5x^3y^5$$
$$+28a_6x^2y^6+8a_7xy^7+a_8y^8,$$

under the form of

$$(p_1x+q_1y)^8+(p_2x+q_2y)^8+(p_3x+q_3y)^8+(p_4x+q_4y)^8$$
$$+70\epsilon\,(p_1x+q_1y)^2(p_2x+q_2y)^2(p_3x+q_3y)^2(p_4x+q_4y)^2.$$

It will be convenient to begin, as in the last case, by taking

$$q_1=p_1\lambda_1,\quad q_2=p_2\lambda_2,\quad q_3=p_3\lambda_3,\quad q_4=p_4\lambda_4,$$
$$\epsilon p_1{}^2p_2{}^2p_3{}^2p_4{}^2=m,$$

and

$$(x+\lambda_1y)(x+\lambda_2y)(x+\lambda_3y)(x+\lambda_4y)=x^4+s_1x^3y+s_2x^2y^2+s_3xy^3+s_4y^4=U,$$

we shall then have nine equations for determining the nine unknown quantities of the general form

$$p_1{}^8\lambda_1{}^\iota+p_2{}^8\lambda_2{}^\iota+p_3{}^8\lambda_3{}^\iota+p_4{}^8\lambda_4{}^\iota+M_\iota m=a_\iota,$$

where ι has all values from 0 to 8 inclusive, and where

$$M_\iota=70.\frac{1.2\ldots\iota.1.2\ldots(8-\iota)}{1.2\ldots8}$$

multiplied into the coefficient of $y^\iota x^{8-\iota}$ in U^2.

Taking these nine equations in consecutive fives, beginning with the first, second, third, fourth, fifth, and ending with the fifth, sixth, seventh, eighth, ninth, we obtain the five equations following:—

$$a_0s_4-a_1s_3+a_2s_2-a_3s_1+a_4s_0-mN_1=0,$$
$$a_1s_4-a_2s_3+a_3s_2-a_4s_1+a_5s_0-mN_2=0,$$
$$a_2s_4-a_3s_3+a_4s_2-a_5s_1+a_6s_0-mN_3=0,$$
$$a_3s_4-a_4s_3+a_5s_2-a_6s_1+a_7s_0-mN_4=0,$$
$$a_4s_4-a_5s_3+a_6s_2-a_7s_1+a_8s_0-mN_5=0,$$

where

$$N_1=M_0s_4-M_1s_3+M_2s_2-M_3s_1+M_4,$$
$$N_2=M_1s_4-M_2s_3+M_3s_2-M_4s_1+M_5,$$
$$N_3=M_2s_4-M_3s_3+M_4s_2-M_5s_1+M_6,$$
$$N_4=M_3s_4-M_4s_3+M_5s_2-M_6s_1+M_7,$$
$$N_5=M_4s_4-M_5s_3+M_6s_2-M_7s_1+M_8.$$

Developing now U^2, we have

$$M_0 = 70, \quad M_1 = \frac{35}{2} s_1, \quad M_2 = 5s_2 + \frac{5}{2} s_1^2, \quad M_3 = \frac{5}{2} s_3 + \frac{5}{2} s_1 s_2,$$

$$M_4 = 2s_4 + 2s_1 s_3 + s_2^2, \quad M_5 = \frac{5}{2} s_1 s_4 + \frac{5}{2} s_2 s_3, \quad M_6 = 5s_2 s_4 + \frac{5}{2} s_3^2,$$

$$M_7 = \frac{35}{2} s_3 s_4, \quad M_8 = 70 s_4^2.$$

Hence
$$N_1 = 72s_4 - 18s_1 s_3 + 6s_2^2,$$

$$N_2 = 18s_1 s_4 - \frac{9}{2} s_1^2 s_3 + \frac{3}{2} s_1 s_2^2,$$

$$N_3 = 12s_2 s_4 - 3s_1 s_2 s_3 + s_2^3,$$

$$N_4 = 18s_3 s_4 - \frac{9}{2} s_1 s_3^2 + \frac{3}{2} s_2^2 s_3,$$

$$N_5 = 72s_4^2 - 18s_1 s_3 s_4 + 6s_2^2 s_4.$$

Hence we have

$$N_1 = 72I, \quad N_2 = 72I\frac{s_1}{4}, \quad N_3 = 72I\frac{s_2}{6}, \quad N_4 = 72I\frac{s_3}{4}, \quad N_5 = 72Is_4,$$

where it will be observed that I is the quadratic invariant of U.

Making now

$$72mI = \nu,$$

we shall have the five following equations:—

$$\begin{aligned}
&a_0 s_4 - a_1 s_3 + a_2 s_2 - a_3 s_1 + (a_4 - \nu) = 0,\\
&a_1 s_4 - a_2 s_3 + a_3 s_2 - \left(a_4 + \frac{\nu}{4}\right) s_1 + a_5 = 0,\\
&a_2 s_4 - a_3 s_3 + \left(a_4 - \frac{\nu}{6}\right) s_2 - a_5 s_1 + a_6 = 0,\\
&a_3 s_4 - \left(a_4 + \frac{\nu}{4}\right) s_3 + a_5 s_2 - a_6 s_1 + a_7 = 0,\\
&(a_4 - \nu) s_4 + a_5 s_3 - a_6 s_2 - a_7 s_1 + a_8 = 0;
\end{aligned}$$

so that the problem reduces itself to finding ν, which is found from the equation of the fifth degree:—

$$\begin{vmatrix}
a_0, & a_1, & a_2, & a_3, & a_4 - \nu\\
a_1, & a_2, & a_3, & a_4 + \frac{\nu}{4}, & a_5\\
a_2, & a_3, & a_4 - \frac{\nu}{6}, & a_5, & a_6\\
a_3, & a_4 + \frac{\nu}{4}, & a_5, & a_6, & a_7\\
a_4 - \nu, & a_5, & a_6, & a_7, & a_8
\end{vmatrix} = 0,$$

ν, it will be observed, being 72 times the quadratic invariant of

$$(p_1x+q_1y)(p_2x+q_2y)(p_3x+q_3y)(p_4x+q_4y),$$

the function being supposed to be thrown under the form of

$$\Sigma(p_1x+q_1y)^8+70\epsilon(p_1x+q_1y)^2(p_2x+q_2y)^2(p_3x+q_3y)^2(p_4x+q_4y)^2.$$

It is obvious that in the equation for finding ν, all the coefficients being functions of the invariable quantities p_1, q_1, &c., and ϵ, must be themselves invariants of the given function; so that the determinant last given will present under one point of view four out of the six invariants belonging to a function of the eighth degree, and these four will be of the degrees 2, 3, 4, 5 respectively*.

I shall now proceed to generalize this remarkable law, and to demonstrate the existence and mode of finding $2n$ consecutively-degreed independent invariants of any homogeneous function of the degree $4n$, and of $n+1$ consecutively-even-degreed independent invariants of any homogeneous function of the degree $4n+2$; a result, whether we look to the fact of such invariants existing, or to the simplicity of the formula for obtaining them, equally unexpected and important, and tending to clear up some of the most obscure, and at the same time interesting points in this great theory of algebraical transformations.

In the first place, let me recall to my readers in the simplest form what is meant by an invariant† of a homogeneous function, say of two variables x and y. If the coefficients of the function $f(x, y)$ be called $a, b, c \dots l$, and if when for x we put $lx+my$, and for y, $nx+py$, where $lp-mn=1$, the coefficients of the corresponding terms become $a', b' \dots l'$; and if

$$I(a, b \dots l)=I(a', b' \dots l'),$$

then I is defined to be an invariant of f.

Let now $f(x, y)$ be a homogeneous function in x, y of the 2ιth degree, and write

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^\iota f(x, y)+\lambda(\eta x-\xi y)^\iota=P,$$

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^\iota f(lx+my, nx+py)+\lambda(\eta x-\xi y)^\iota=P',$$

where ξ and η are independent of x, y, and $lp-mn=1$.

Let
$$x'=lx+my,$$
$$y'=nx+py,$$

then
$$\xi\frac{d}{dx}+\eta\frac{d}{dy}=\xi\frac{dx'}{dx}\frac{d}{dx'}+\xi\frac{dy'}{dx}\frac{d}{dy'}+\eta\frac{dx'}{dy}\frac{d}{dx'}+\eta\frac{dy'}{dy}\frac{d}{dy'},$$

* The reasoning in this paragraph seems of doubtful conclusiveness. It may be accepted, however, as a fact of observation confirmed and generalized by the subsequent theorem, that the coefficients are invariants.

† *Olim*, Hyperdeterminant, Constant derivative.

and if we now write

$$l\xi + m\eta = \xi',$$
$$n\xi + p\eta = \eta',$$

we find
$$\xi \frac{d}{dx} + \eta \frac{d}{dy} = \xi' \frac{d}{dx'} + \eta' \frac{d}{dy'}.$$

Again, from the equations between x', y', x, y, we find

$$x = \frac{px' - my'}{pl - mn} = px' - my',$$

$$y = \frac{ly' - nx'}{pl - mn} = ly' - nx';$$

therefore
$$\eta x - \xi y = (p\eta + n\xi)\, x' - (m\eta + l\xi)\, y' = \eta' x' - \xi' y'.$$

Hence
$$P' = \left(\xi' \frac{d}{dx'} + \eta' \frac{d}{dy'}\right)^\iota f(x', y') + \lambda\, (\eta' x' - \xi' y')^\iota.$$

Again,
$$\frac{d}{d\xi} = l \frac{d}{d\xi'} + n \frac{d}{d\eta'},$$

$$\frac{d}{d\eta} = m \frac{d}{d\xi'} + p \frac{d}{d\eta'}.$$

Hence

$$\left(\frac{d}{d\xi}\right)^\iota P' = l^\iota \left(\frac{d}{d\xi'}\right)^\iota P' + \iota l^{\iota-1} n \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' + \&\text{c.} + n^\iota \left(\frac{d}{d\eta'}\right)^\iota P',$$

$$\left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' = l^{\iota-1} m \left(\frac{d}{d\xi'}\right)^\iota P' + \{l^{\iota-1} p + (\iota - 1)\, l^{\iota-2} mn\} \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' + \&\text{c.}$$
$$+ n^{\iota-1} p \left(\frac{d}{d\eta'}\right)^\iota P',$$

....

$$\left(\frac{d}{d\eta}\right)^\iota P' = m^\iota \left(\frac{d}{d\xi'}\right)^\iota P' + \iota m^{\iota-1} p \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P + \&\text{c.} + p^\iota \left(\frac{d}{d\eta'}\right)^\iota P'.$$

But P' being of ι dimensions in ξ' and η', and also in x and y, each of the equations above written will be of ι dimensions in x and y, and of no dimensions in ξ', η'; in fact, the successive terms of the right-hand members of the above $\iota + 1$ equations will be multiples of the $(\iota + 1)$ quantities

$$(x')^\iota, \quad (x')^{\iota-1} y', \quad (x')^{\iota-2} y'^\iota \ldots (y')^\iota.$$

Consequently a linear resultant may be taken of

$$\left(\frac{d}{d\xi}\right)^\iota P', \quad \left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' \ldots \left(\frac{d}{d\eta}\right)^\iota P',$$

treating x'^ι, $x'^{\iota-1} y' \ldots y'^\iota$ as independent, and as quantities to be eliminated; and this, according to a well-known principle of elimination, will prove the linear resultant of the foregoing equations to be equal to the linear resultant of

$$\left(\frac{d}{d\xi'}\right)^{\iota} P', \quad \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' \ldots \left(\frac{d}{d\eta'}\right)^{\iota} P',$$

multiplied by the determinant

$$\begin{vmatrix} l^{\iota}, & \iota n l^{\iota-1}, \ \tfrac{1}{2}\iota(\iota-1) n^{\iota} l^{\iota-2}, \ldots, & n^{\iota} \\ l^{\iota-1}m, \ l^{\iota-1}p + (\iota-1) mnl^{\iota-2}, & \ldots, & n^{\iota-1}p \\ \ldots\ldots\ldots & & \\ m^{\iota}, & \iota m^{\iota-1}p, \ \tfrac{1}{2}\iota(\iota-1) m^{\iota}p^{\iota-2}, \ldots, & p^{\iota} \end{vmatrix}.$$

This last written determinant may be shown from the method of its formation to be equal to $(lp - mn)^{\frac{\iota(\iota+1)}{2}}$, that is, to unity, because $lp - mn = 1$. Again, since

$$x'^{\iota} = l^{\iota}x^{\iota} + \iota l^{\iota-1} m x^{\iota-1} y + \&c. + m^{\iota} y^{\iota},$$

$$x'^{\iota-1}y' = l^{\iota-1} n x^{\iota} + (l^{\iota-1} n + (\iota-1) l^{\iota-2} mn) x^{\iota-1} y + \&c. + m^{\iota-1} p y^{\iota},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y'^{\iota} = n^{\iota}x^{\iota} + \iota n^{\iota-1} p x^{\iota-1} y + \ldots + p^{\iota} y^{\iota},$$

the resultant of $\left(\frac{d}{d\xi}\right)^{\iota} P' \ldots \left(\frac{d}{d\eta}\right)^{\iota} P'$, obtained by treating $x^{\iota}, x^{\iota-1}y \ldots y^{\iota}$ as the eliminables, will be equal to the resultant of the same functions when $x'^{\iota}, x'^{\iota-1}y' \ldots y'^{\iota}$ are taken as the eliminables* multiplied by a power of the determinant

$$\begin{vmatrix} l^{\iota}, & \ldots, & m^{\iota} \\ l^{\iota-1}n, & \ldots, & m^{\iota-1}p \\ \ldots\ldots & & \\ n^{\iota}, & \ldots, & p^{\iota} \end{vmatrix},$$

which determinant, like the last, is unity. Thus, then, we have succeeded in showing that the resultant obtained by eliminating $x^{\iota}, x^{\iota-1}y \ldots y^{\iota}$ between

$$\left(\frac{d}{d\xi}\right)^{\iota} P', \quad \left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' \ldots \left(\frac{d}{d\eta}\right)^{\iota} P'$$

is equal to the resultant obtained by eliminating $(x')^{\iota}, x'^{\iota-1}y' \ldots y'^{\iota}$ between

$$\left(\frac{d}{d\xi'}\right)^{\iota} P', \quad \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' \ldots \left(\frac{d}{d\eta'}\right)^{\iota} P';$$

* For the statement of the general principle of the change of the variables of elimination, see my paper in the March Number, 1851, of the *Camb. and Dub. Math. Jour.* [p. 186 above].

or, which is evidently the same thing, the resultant obtained by eliminating $x^\iota, x^{\iota-1}y \dots y^\iota$ between

$$\left(\frac{d}{d\xi}\right)^\iota P, \quad \left(\frac{d}{d\xi}\right)^{\iota-1}\frac{d}{d\eta}P \dots \left(\frac{d}{d\eta}\right)^\iota P;$$

that is to say, this last resultant remains absolutely unaltered in value when for x, y we write respectively

$$lx + my,$$
$$nx + py,$$

provided that $lp - mn = 1$.

Hence by definition this resultant is an invariant $f(x, y)$, and λ being arbitrary, all the separate coefficients of the powers of λ in this resultant must also be invariants. I proceed to express this resultant in terms of λ and the coefficients of (x, y). Let $\varpi = 1 . 2 . 3 \dots \iota$ and

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^\iota P = \left(\frac{d}{dx}\right)^\iota f + \lambda(-y)^\iota \qquad = E_1,$$

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^{\iota-1}\frac{d}{d\eta}P = \left(\frac{d}{dx}\right)^{\iota-1}\frac{d}{dy}f + \lambda(-y)^{\iota-1}x \qquad = E_2,$$

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^{\iota-2}\left(\frac{d}{d\eta}\right)^2 P = \left(\frac{d}{dx}\right)^{\iota-2}\left(\frac{d}{dy}\right)^2 f + \lambda(-y)^{\iota-2}x^2 = E_3,$$

....

$$\frac{1}{\varpi}\left(\frac{d}{d\eta}\right)^\iota P = \left(\frac{d}{dy}\right)^\iota f + \lambda x^\iota \qquad = E_{\iota+1};$$

and

$$f(x, y) = a_0 x^{2\iota} + 2\iota a_1 x^{2\iota-1}y + \tfrac{1}{2}(2\iota)(2\iota-1)a_2x^{2\iota-2}y^2 + \text{\&c.} + a_{2\iota}y^{2\iota}.$$

We find, writing $\sigma\lambda$ for λ, where $\sigma = 2\iota(2\iota-1)\dots(\iota+1)$,

$$\frac{1}{\sigma}E_1 = a_0x^\iota + \iota a_1x^{\iota-1}y + \tfrac{1}{2}\iota(\iota-1)a_2x^{\iota-2}y^2 \dots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_{\iota-2}x^2y^{\iota-2} + \iota a_{\iota-1}xy^{\iota-1} + a_\iota y^\iota + \lambda(-y)^\iota,$$

$$\frac{1}{\sigma}E_2 = a_1x^\iota + \iota a_2x^{\iota-1}y + \tfrac{1}{2}\iota(\iota-1)a_3x^{\iota-2}y^2 \dots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_{\iota-1}x^2y^{\iota-2} + \iota a_\iota xy^{\iota-1} + a_{\iota+1}y^\iota + \lambda(-y)^{\iota-1}x,$$

$$\frac{1}{\sigma}E_3 = a_2x^\iota + \iota a_3x^{\iota-1}y \dots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_\iota x^2y^{\iota-2} + \iota a_{\iota+1}xy^{\iota-1} + a_{\iota+2}y^\iota + \lambda(-y)^{\iota-2}x^2,$$

....

$$\frac{1}{\sigma}E_{\iota+1} = a_\iota x^\iota + \text{\&c.} + \lambda x^\iota;$$

accordingly, by eliminating

$$x^\iota, \quad \iota x^{\iota-1}y, \quad \tfrac{1}{2}\iota(\iota-1)x^{\iota-2}y^2 \ldots y^\iota,$$

we obtain as the required resultant*,

$$\begin{vmatrix} a_\iota \pm \lambda, & a_{\iota-1}, & a_{\iota-2}, & \ldots\ a_0 \\ a_{\iota+1}, & a_\iota \mp \dfrac{\lambda}{\iota}, & a_{\iota-1}, & \ldots\ a_1 \\ a_{\iota+2}, & a_{\iota+1}, & a_\iota \pm \dfrac{\lambda}{\frac{1}{2}\iota(\iota-1)}, & \ldots\ a_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{2\iota}, & a_{2\iota-1}, & \cdots\cdots, & \ldots\ a_\iota + \lambda \end{vmatrix}.$$

Inasmuch as all the coefficients of λ in this expression are invariants of $f(x, y)$, and there are no invariants of the first order, it is clear that the coefficient of λ^ι must be always zero, which is easily verified.

Again, if ι is odd, the determinant remains unaltered if we write $-\lambda$ for λ; hence when $f(x, y)$ is of the degree $4\epsilon+2$, all the coefficients of the odd powers of λ disappear. Thus, then, our theorem at once demonstrates that a function of x, y of the degree 4ϵ has 2ϵ invariants of all degrees from 2 up to $2\epsilon+1$ inclusive, and that a function of x, y of the degree $4\epsilon+2$ has $\epsilon+1$ invariants whose degrees correspond to all the even numbers in the series from 2 to $2\epsilon+2$.

But in order that the proposition, as above stated, may be understood in its full import and value, it is necessary to show that these invariants are independent of one another, which is usually a most troublesome and difficult task in inquiries of this description, but which the peculiar form of our grand determinant enables us to accomplish with extraordinary facility. In order to make the spirit of the demonstration more apparent, take the case of a function of the twelfth degree, whose coefficients, divided by the successive binomial numbers 1, 12, $\dfrac{12.11}{2}$, &c. may be called

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ m.$$

* Mr Cayley has made the valuable observation, that λ (given by equating to zero the above determinant) may be defined by means of the equation

$$\left(\frac{d}{dx}\frac{d}{d\eta} - \frac{d}{dy}\frac{d}{d\xi}\right)^\iota \{f(x, y) \times \phi(\xi, \eta)\} = \lambda\phi(x, y),$$

ϕ being itself a certain rational integral form of a function of the ιth degree, the ratio of whose coefficients would be given by virtue of the above equations as functions of λ and the coefficients of $f(x, y)$.

Our grand determinant then takes the form

$$\begin{vmatrix} g+\lambda, & f, & e, & d, & c, & b, & a \\ h, & g-\frac{\lambda}{6}, & f, & e, & d, & c, & b \\ i, & h, & g+\frac{\lambda}{15}, & f, & e, & d, & c \\ j, & i, & h, & g-\frac{\lambda}{20}, & f, & e, & d \\ k, & j, & i, & h, & g+\frac{\lambda}{15}, & f, & e \\ l, & k, & j, & i, & h, & g-\frac{\lambda}{6}, & f \\ m, & l, & k, & j, & i, & h, & g+\lambda \end{vmatrix}.$$

Here it will be observed that

a and m appear only	1	time.
b and l ...	2	times.
c and k ...	3	...
d and j ...	4	...
e and i ...	5	...
f and h ...	6	...
g ...	7	...

Let now the coefficients be called

$$H_2,\ H_3,\ H_4,\ H_5,\ H_6,\ H_7,$$

H_2 and H_3 manifestly are independent.

Again, if possible, let $H_4 = pH_2^2$, then a and m would appear twice in H_4, contrary to the rule.

Hence H_4 is independent of H_2, H_3.

For a similar reason H_5 cannot depend on H_2, H_3.

Again, if possible, let

$$H_6 = pH_2^3 + qH_2H_4 + rH_3^2,$$

H_2^3 will contain b^6l^6, which by the rule cannot appear in H_2H_4 or in H_3^2.

Hence $p=0$.

Also H_4 will contain $b^2l^2 \times$ the coefficient of λ^3 in

$$\left(g+\frac{\lambda}{15}\right)\left(g-\frac{\lambda}{20}\right)\left(g+\frac{\lambda}{15}\right),$$

which is not zero. And H_2 also contains bl; hence H_2H_4 will contain b^3l^3. But H_3 will evidently not contain b^3 or l^3, or b^2l or bl^2, nor can H_6 contain b^3l^3; hence $q = 0$. Finally, H_3^2 will contain c^6 and k^6, but H_6 can only contain as to these letters the combination c^3k^3; hence $r = 0$.

Consequently H_6 does not depend on H_2, H_4, H_3. As regards H_2, H_3, H_4, H_5, H_6 not vanishing, this may be made at once apparent by making all the letters but g vanish; the H's then become identical with the coefficients of

$$(g+\lambda)^2\left(g-\frac{\lambda}{6}\right)^2\left(g+\frac{\lambda}{15}\right)^2\left(g-\frac{\lambda}{20}\right),$$

none of which are zero except that of λ^6. The same or a similar demonstration may be extended to H_7 and easily generalized; hence, then, this most unexpected and surprising law is fully made out*.

To return to the subject of canonical forms, I have not found the method so signally successful in its application to the 4th and 8th degrees, conduct to the solution of other degrees, such as the 6th, 12th, or 16th, of all of which I have made trial; possibly another canonical form must be substituted to meet the exigency of these cases†; and it may be remarked in general, that if we have a function of the $(2n)$th degree, the canonical form assumed may be taken,

$$\Sigma\,(p_1x+q_1y)^{2n}+V;$$

where V, in lieu of being the squared product of

$$(p_1x+q_1y),\,(p_2x+q_2y),\,\ldots,\,(p_nx+q_ny),$$

* This demonstration, however, does not extend to show that the coefficients of the powers of λ may not possibly be dependents, that is, explicit functions of one another combined with other invariants not included among their number, or of these latter alone. For example, in the case of the 12th degree, we know by Mr Cayley's law that there must be two invariants of the 4th order. Our determinant gives only one of these. Call the other one K_4; by the above reasoning it is not disproved but that we may have

$$H_6=pH_2^3+qH_2H_4+rH_3^2+sH_2K_4.$$

I believe, however, that the H's may be demonstrated without much difficulty to be primitive or fundamental invariants. The law of Mr Cayley here adverted to admits of being stated in the following terms:—The number of independent invariants of the 4th order belonging to a function of x, y of the nth degree is equal to the number of solutions in integers (not less than zero) of the equation $2x+3y=n-3$. *Vide* his memorable paper (in which several numerical errors occur against which the reader should be cautioned) "On Linear Transformations," vol. I. *Cambridge and Dublin Mathematical Journal*, new series. There is no great difficulty in showing, by aid of the doctrine of symmetrical functions, that there can never be *more* than one quadratic or one cubic invariant, and in what cases there is one or the other, or each, to any given function of two variables. The general law, however, for the number of invariants of any order other than 2, 3, 4 remains to be made out, and is a great desideratum in the theory of linear transformations.

† See the Postscript [p. 283] for a verification of this conjecture.

may be any hyperdeterminant, or (as I shall in future call such functions) covariant of this product, understanding $P(x, y)$ to be a covariant of $f(x, y)$ when $P(lx+my, nx+py)$ stands in precisely the same relation to $f(lx+my, nx+py)$ as $P(x, y)$ to $f(x, y)$, provided only that $lp-mn=1$. For the relation and distinction between covariants and contravariants, see a short article of mine* in the *Cambridge and Dublin Mathematical Journal* for this month. In endeavouring to apply the method of the text to the Sextic Function

$$ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6,$$

thrown under the form

$$\Sigma\,(px+qy)^6 + 20\epsilon U^2,$$

where

$$U = (p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y) = s_0x^3 + s_1x^2y + s_2xy^2 + s_3y^3,$$

I obtain the following equations:

$$as_3 - bs_2 + cs_1 - ds_0 = \epsilon\,(162s_0^2s_3 - 54s_0s_1s_2 + 12s_1^3),$$

$$bs_3 - cs_2 + ds_1 - es_0 = \epsilon\,(54s_0s_1s_3 + 6s_1^2s_2 - 36s_0s_2^2),$$

$$cs_3 - ds_2 + es_1 - fs_0 = \epsilon\,(-54s_0s_2s_3 - 6s_1s_2^2 + 36s_3s_1^2),$$

$$ds_3 - es_2 + fs_1 - gs_0 = \epsilon\,(-162s_0s_3^2 + 54s_1s_2s_3 + 12s_2^3).$$

In these equations, if we call the quantities multiplied by ϵ respectively L, M, N, P, we shall find

$$s_3L - \frac{1}{3}s_2M - \frac{1}{3}s_1N + s_0P = 0,$$

and

$$s_3L - s_2M - s_1N + s_0P = I;$$

where I denotes the determinant, or, as I shall in future call such function (in order to avoid the obscurity and confusion arising from employing the same word in two different senses), the Discriminant†, which is the biquadratic (and of course sole) invariant of the cubic function

$$s_0x^3 + s_1x^2y + s_2xy^2 + s_3y^3.$$

The reduction of the function of the fourth degree to its canonical form may be effected very easily by means of the properties of the invariants of

[* p. 200 above.]

† "Discriminant," because it affords the *discrimen* or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally of the existence or otherwise of multiple points in the locus represented or characterized by any algebraical function, the most obvious and first observed species of singularity in such function or locus. Progress in these researches is impossible without the aid of clear expression; and the first condition of a good nomenclature is that different things shall be called by different names. The innovations in mathematical language here and elsewhere (not without high sanction) introduced by the author, have been never adopted except under actual experience of the embarrassment arising from the want of them, and will require no vindication to those who have reached that point where the necessity of some such additions becomes felt.

the canonical form, as I have shown in the *Cambridge and Dublin Mathematical Journal.* Accordingly I have endeavoured to ascertain whether the reduction of the sixth degree might not be effected by a similar method.

If we start with the form $ax^6+by^6+cz^6+90mx^2y^2z^2$, where $x+y+z=0$, which is only another mode of representing the canonical form previously given, we shall find that there are four independent invariants, of the second, fourth, sixth and tenth degrees. Calling these H_2, H_4, H_6, H_{10}, and writing s_1, s_2, s_3 for $a+b+c$, $ab+ac+bc$, abc it will be found, after performing some extremely elaborate computations, that

$$H_2 = s_2 - 270m^2,$$

$$H_4 = 6ms_3 + 45m^2s_2 + 216m^3s_1 + 891m^4,$$

$$\begin{aligned}H_6 = 4s_3^2 + 120s_2s_3m &- \{684s_2^2 + 432s_1s_3\}\, m^2 \\ &+ (13 . 27 . 64s_3 - 64 . 81s_1s_2)\, m^3 + 8 . 81 . 169s_2m^4 \\ &+ 7 . 128 . 729s_1m^5 + 16 . 729 . 239m^6.\end{aligned}$$

H_{10} is too enormously long to attempt to compute; but we can easily prove its independent existence by making $m=0$, in which case the (determinant, or, to use the new term proposed, the) discriminant of $ax^6+by^6+cz^6$ becomes the product of the twenty-five forms of the expression

$$(ab)^{\frac{1}{5}} + (ac)^{\frac{1}{5}} . 1^{\frac{1}{5}} + (bc)^{\frac{1}{5}} . 1^{\frac{1}{5}}*.$$

Now in general the value of such a product for $\alpha^{\frac{1}{5}}+\beta^{\frac{1}{5}}.1^{\frac{1}{5}}+\gamma^{\frac{1}{5}}.1^{\frac{1}{5}}$ is obviously of the form

$$(\alpha+\beta+\gamma)^5 + \alpha\beta\gamma\,\{f(\alpha+\beta+\gamma)^2 + g(\alpha\beta+\alpha\gamma+\beta\gamma)\};$$

for when $\alpha=0$ or $\beta=0$ or $\gamma=0$, the product must become respectively $(\beta+\gamma)^5$, $(\gamma+\alpha)^5$ and $(\alpha+\beta)^5$. Moreover, without caring to calculate f, g†, it is enough for our present purpose to satisfy ourselves that g cannot be zero, as then the product would have a factor $(\alpha+\beta+\gamma)^2$. Hence, then, on putting

* Such a product in the language of the most modern continental analysis is, I believe, termed a Norm. If we suppose the general function of x, y of the 4th degree thrown under the form $Au^4+Bv^4+Cw^4$, where $u+v+w=0$, and the general function of x, y, z of the 3rd degree thrown under the form $Au^3+Bv^3+Cw^3+D\theta^3$, where $u+v+w+\theta=0$, the theory of norms will afford an instantaneous and, so to speak, intuitive demonstration of the respective related theorems, and the discriminant (*aliter* determinant) of each such function is decomposable into the sum of a square and a cube. Each of these forms is indeterminate, in either case there being but two relations fixed between the coefficients A, B, C; A, B, C, D; and we may easily establish the following singular species of algebraical porism. In the first case

$$(ABC)^2 : (AB+AC+BC)^3,$$

and in the second case

$$(ABCD)^3 : (\Sigma A^2B^2C^2 - 2ABCD\Sigma AB)^2$$

are *invariable ratios.*

† $f=-625$, $g=3125$.

$\alpha = bc$, $\beta = ac$, $\gamma = ab$, we see that the discriminant, when m is 0, will be of the form

$$s_2^5 + fs_2^2 s_3^2 + gs_3^3 s_1.$$

But when m is 0, H_4 vanishes, and there is no term s_1 or s_3 in H_2. Hence evidently the discriminant H_{10} just found cannot be dependent on H_2, H_4, or H_6; nor is it possible to make

$$H_{10} + pH_2^5 + qH_2^2 H_6,$$

that is, $$(p+1) s_2^5 + fs_2^2 s_3^2 + gs_3^3 s_1$$

a perfect square on account of g not vanishing; so there is no H_5 upon which H_{10} can depend. Hence, admitting, as there seems every reason to do, that the number of invariants of a function of x, y of the degree m is $m-2$, we find that the four invariants in the case of the first degree are respectively of the second, fourth, sixth, and tenth dimensions, a determination in itself, as a step to the completion of the theory of invariants, of no minor importance.

But it seems hopeless by means of these forms to arrive at the desired canonical reduction. The forms, however, of H_2, H_4, H_6 are very remarkable as not rising above the first, first and second degrees respectively in s_1, s_2, s_3. Also H_4 vanishes when $m=0$ and H_4 has been obtained by putting

$$ax^6 + by^6 + cz^6 + 90mx^2y^2z^2$$

under the form of

$$Ax^6 + 6Bx^5y + 15Cx^4y^2 + 20Dx^3y^3 + 15Ex^2y^4 + 6Fxy^5 + Gy^6,$$

and taking the determinant

$$\begin{vmatrix} A & B & C & D \\ B & C & D & E \\ C & D & E & F \\ D & E & F & G \end{vmatrix}.$$

Consequently *in general* the vanishing of the above-written determinant will express the condition that a function of the sixth degree may be decomposable into three sixth powers. This also is true more generally. If $F(x, y)$ be a function of $2i$ dimensions, the vanishing of the resultant in respect to x^i, $x^{i-1}y \ldots y^i$ (taken dialytically) of

$$\left(\frac{d}{dx}\right)^i F, \ \left(\frac{d}{dx}\right)^{i-1} \frac{d}{dy} F \ldots \left(\frac{d}{dy}\right)^i F$$

will indicate that F admits of being decomposed into i powers of linear functions of x, y*.

In consequence of the greater interest, at least to the author, of the preceding investigations, I have delayed the insertion of the promised continuation of my paper on extensions of the dialytic method, which will

* Such a function so decomposable may be termed meio-catalectic. Meio-catalecticism for even-degreed functions is the analogue of singularity for odd-degreed functions.

appear in a subsequent Number. I take this opportunity of correcting a trifling slip of the pen which occurs towards the end* of the paper alluded to. The values of $\frac{x}{z}$ and $\frac{y}{z}$ become zero, and not infinite, when $N = 0$; and the antepenultimate paragraph should end with the words "an incomplete resultant." The theorem also, in the last paragraph but one, should be stated more distinctly as subject to an important exception as follows.

Whenever the resultant of a system of equations $F = 0$, $G = 0$, &c. contains a factor R'^m, this will indicate that, on making $R' = 0$, the given system of equations will admit of being satisfied by m algebraically distinct systems of values of the variables, except in those cases where there is a singularity in the forms of F, G, &c., taken either separately, or in partial combination with one another. An example will serve to make the meaning of the exception apparent. Let F, G, H denote three quadratic equations in x and y, so that $F = 0$, $G = 0$, $H = 0$ may be conceived as representing three conic sections. Let R be the resultant of F, G, H, and suppose the relations of the coefficients in F, G, H to be such that $R = R'^2$; then $R' = 0$ will imply the existence of one or the other of the three following conditions: namely, either that the three conics have a chord in common, which is the most general inference; or, which is less general, that two of the conics touch one another; or, which is the most special case of all, that one of the conics is a pair of right lines.

So, again, if we have two equations in x, and their resultant contains F^2, this may arise either from one of the functions containing a square factor, or from their being susceptible, on instituting one further condition, namely of $F = 0$, of having a quadratic factor in common between them.

P.S. The conjecture made in the preceding pages has been since confirmed by the discovery of a modification in the canonical form applicable to functions of the sixth degree, which simplifies the theory in a remarkable manner. Assume $f(x, y)$, a function of the sixth degree, as equal to

$$au^6 + bv^6 + cw^6 \pm muvw(u - v)(v - w)(w - u),$$

where u, v, w, linear functions of x and y, satisfy the equation

$$u + v + w = 0;$$

then will the product of uvw be capable of being determined by means of the solution of a quadratic equation, of the square root of whose roots the coefficients of uvw will be known linear functions. Thus by an affected quadratic, a pure quadratic, and a cubic equation, the values of u, v, w may be completely ascertained. The discussion of this theory, and of a general inverse method for assigning the true (in the sense of the most manageable) Canonical Form for functions of any even degree, will form the subject of a subsequent communication.

[* p. 264 above.]