

ON A CERTAIN FUNDAMENTAL THEOREM OF
DETERMINANTS.

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THE subjoined theorem, which is one susceptible of great extension and generalization, appears to me, and indeed from use and acquaintance (it having been long in my possession) I know to be so important and fundamental, as to induce me to extract it from a mass of memoranda on the same subject; and as an act of duty to my fellow-labourers in the theory of determinants, more or less forestall time (the sure discoverer of truth) by placing it without further delay on record in the pages of this *Magazine*. Its developments and applications must be reserved for a more convenient occasion, when the interest in the New Algebra (for such, truly, it is the office of the theory of determinants to establish), and the number of its disciples in this country, shall have received their destined augmentation. In a recent letter to me, M. Hermite well alludes to the theory of determinants as "That vast theory, transcendental in point of difficulty, elementary in regard to its being the basis of researches in the higher arithmetic and in analytical geometry."

The theorem is as follows:—Suppose that there are two determinants of the ordinary kind, each expressed by a square array of terms made up of n lines and n columns, so that in each square there are n^2 terms. Now let n be broken up in any given manner into two parts p and q , so that $p + q = n$. Let, firstly, one of the two given squares be divided in a given *definite* manner into two parts, one containing p of the n given lines, and the other part q of the same; and secondly, let the other of the two given squares be divided *in every possible way* into two parts, consisting of q and p lines respectively, so that on tacking on the part containing q lines of the second square to the part containing p lines of the first square, and the part containing p lines of the second square to the part containing q of the first, we

get back a new couple of squares, each denoting a determinant different from the two given determinants; the number of such new couples will evidently be

$$\frac{n(n-1)\dots(n-p+1)}{1 \cdot 2 \dots p};$$

and my theorem is, that *the product of the given couple of determinants is equal to the sum of the products (affected with the proper algebraical sign) of each of the new couples formed as above described.* Analytically the theorem may be stated as follows.

Let
$$\left\{ \begin{matrix} a_1, & a_2 & \dots & a_n \\ b_1, & b_2 & \dots & b_n \end{matrix} \right\}, \quad \left\{ \begin{matrix} \alpha_1, & \alpha_2 & \dots & \alpha_n \\ \beta_1, & \beta_2 & \dots & \beta_n \end{matrix} \right\},$$

according to the notation heretofore* employed by me in the preceding numbers of this *Magazine*, denote any two common determinants, each of the n th order, and let the numbers $\theta_1, \theta_2 \dots \theta_n$ be disjunctively equal to the numbers $1, 2 \dots n$ and $p+q=n$; then will

$$\begin{aligned} & \left\{ \begin{matrix} a_1, & a_2 & \dots & a_n \\ b_1, & b_2 & \dots & b_n \end{matrix} \right\} \times \left\{ \begin{matrix} \alpha_1, & \alpha_2 & \dots & \alpha_n \\ \beta_1, & \beta_2 & \dots & \beta_n \end{matrix} \right\} \\ = \Sigma \pm & \left\{ \begin{matrix} a_1, & a_2 & \dots & a_n \\ b_1, & b_2 & \dots & b_p, & \beta_{\theta_{p+1}}, & \beta_{\theta_{p+2}} & \dots & \beta_{\theta_n} \end{matrix} \right\} \times \left\{ \begin{matrix} \alpha_1, & \alpha_2 & \dots & \alpha_n \\ \beta_{\theta_1}, & \beta_{\theta_2} & \dots & \beta_{\theta_p}, & b_{p+1}, & b_{p+2} & \dots & b_n \end{matrix} \right\}. \end{aligned}$$

The general term under the sign of summation may be represented by aid of the disjunctive equations

$$\begin{aligned} \phi_1, \phi_2 \dots \phi_n &= 1, 2 \dots n, \\ \psi_1, \psi_2 \dots \psi_n &= 1, 2 \dots n, \end{aligned}$$

under the form of

$$\begin{aligned} & (a_{\phi_1} \cdot b_1 \times a_{\phi_2} \cdot b_2 \times \dots \times a_{\phi_p} \cdot b_p) (a_{\psi_{p+1}} \cdot b_{p+1} \times a_{\psi_{p+2}} \cdot b_{p+2} \times \dots \times a_{\psi_n} \cdot b_n) \\ & \times (a_{\phi_{p+1}} \cdot \beta_{\theta_{p+1}} \times a_{\phi_{p+2}} \cdot \beta_{\theta_{p+2}} \times \dots \times a_{\phi_n} \cdot \beta_{\theta_n}) (a_{\psi_1} \cdot \beta_{\theta_1} \times a_{\psi_2} \cdot \beta_{\theta_2} \times \dots \times a_{\psi_p} \cdot \beta_{\theta_p}). \end{aligned}$$

1st. When $\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p$, it will readily be seen, that for given values of $\phi_1, \phi_2 \dots \phi_p$, the product of the third and fourth factors becomes *substantially* identical with the general term of the determinant

$$\left\{ \begin{matrix} \alpha_1, & \alpha_2 & \dots & \alpha_n \\ \beta_1, & \beta_2 & \dots & \beta_n \end{matrix} \right\},$$

and consequently, making the system $\phi_1, \phi_2 \dots \phi_p$ (or, which is the same thing, its equivalent $\psi_1, \psi_2 \dots \psi_p$) go through all its values, we get back for the sum of the terms corresponding to the equation

$$\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p,$$

[* p. 242 above.]

the product of the determinants

$$\left\{ \begin{matrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{matrix} \right\} \text{ and } \left\{ \begin{matrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{matrix} \right\}.$$

2nd. When we have not the equality above supposed between the ϕ 's and the ψ 's, let

$$\phi_{p-h} = \psi_{p+k} \text{ and } \phi_{p+\eta} = \psi_{p-\zeta};$$

the corresponding term included under the Σ will contain the factor

$$\alpha_{\phi_{p+\eta}} \cdot \beta_{\theta_{p+\eta}} \times \alpha_{\psi_{p-\zeta}} \cdot \beta_{\theta_{p-\zeta}}.$$

Now leaving $\phi_1, \phi_2 \dots \phi_p$, and $\psi_1, \psi_2 \dots \psi_p$ unaltered, we may take a system of values $\theta'_1, \theta'_2 \dots \theta'_n$, such that

$$\theta'_{p+\eta} = \theta_{p-\zeta},$$

and

$$\theta'_{p-\zeta} = \theta_{p+\eta},$$

and for all other values of q except $p + \eta$, or $p - \zeta$, $\theta'_q = \theta_q$. The corresponding new value of the general term so formed by the substitution of the θ' for the θ series, will be identical with that of the term first spoken of, but will have the contrary algebraical sign, because the θ' arrangement of the figures 1, 2, 3 ... p is deducible by a single interchange from the θ arrangement of the same, the rule for the imposition of the algebraical sign plus or minus being understood to be, that the term in which

$$\beta_{\theta_{p+1}}, \beta_{\theta_{p+2}} \dots \beta_{\theta_n}; \beta_{\theta_1}, \beta_{\theta_2} \dots \beta_{\theta_p}$$

enter into the symbolical forms of the respective derived couples of determinants, has the same sign as, or the contrary sign to, that in which

$$\beta_{\theta'_{p+1}}, \beta_{\theta'_{p+2}} \dots \beta_{\theta'_n}; \beta_{\theta'_1}, \beta_{\theta'_2} \dots \beta_{\theta'_p}$$

so enter, according as an odd or an even number of interchanges is required to transform the arrangement

$$\theta_{p+1}, \theta_{p+2} \dots \theta_n; \theta_1, \theta_2 \dots \theta_p$$

into the arrangement

$$\theta'_{p+1}, \theta'_{p+2} \dots \theta'_n; \theta'_1, \theta'_2 \dots \theta'_p.$$

I have therefore shown that all the terms arising from the expansion of the products included under the sign of summation, for which the disjunctive identity $\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p$ does not exist, enter into the final sum in pairs, equal in quantity and differing in sign, which consequently mutually destroy, and that the terms for which the said identity does exist together make up the sum

$$\left\{ \begin{matrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{matrix} \right\} \times \left\{ \begin{matrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{matrix} \right\};$$

which proves, upon first principles drawn direct from that notion of polar dichotomy of permutation systems which rests at the bottom of the whole theory of the subject, the fundamental, and, as I believe, perfectly new theorem, which it is the object of this communication to establish.

In applying the theorem thus analytically formulized, it is of course to be understood that, under the sign Σ , *permutations* within the separate parts of a given arrangement,

$$\theta_{p+1}, \theta_{p+2} \dots \theta_n; \theta_1, \theta_2 \dots \theta_p,$$

are inadmissible, the total number of terms so included being restricted to

$$\frac{n(n-1) \dots (n-p+1)}{1 \cdot 2 \dots p}.$$

The theorem may be extended so as to become a theorem for the expansion of the product of any number of determinants, and adapted so as to take in that far more general class of functions known to Mr Cayley and myself under the new name of commutants, of which determinants present only a particular, and that the most limited instance.