3.

ON THE MOTION AND REST OF RIGID BODIES.

[*Philosophical Magazine,* XIV. (1839), pp. 188—190.]

In the subjoined investigation, which, as far as I know, is my own, I apply the same method to rigid as in the preceding paper I applied to fluid systems.

Let x, y, z be the coordinates of any particle in a rigid body; x', y', z' the coordinates of some other particle, and let

$$x' = x + h, \quad y' = y + k, \quad z' = z + l.$$

Call $\Delta x, \Delta y, \Delta z$ the increments which x, y, z receive after the lapse of a small interval of time; so that terms in which they enter in two or more dimensions may be neglected.

Then
$$\Delta(x') = \Delta x + \frac{d\Delta x}{dx} h + \frac{d\Delta x}{dy} k + \frac{d\Delta x}{dz} l + P,$$

$$\Delta(y') = \Delta y + \frac{d\Delta y}{dx} h + \frac{d\Delta y}{dy} k + \frac{d\Delta y}{dz} l + Q,$$

$$\Delta(z') = \Delta z + \frac{d\Delta z}{dx} h + \frac{d\Delta z}{dy} k + \frac{d\Delta z}{dz} l + R,$$

P, Q, R containing binary and higher combinations of h, k, l, which we shall have no occasion to express.

At the commencement of the interval the squared distance of the two particles was $(x'-x)^2 + (y'-y)^2 + (z'-z)^2$; at the end of the interval the distance squared is

$$(x' - x + \Delta(x') - \Delta x)^2 + (y' - y + \Delta(y') - \Delta y)^2 + (z' - z + \Delta(z') - \Delta z)^2,$$

and these two expressions must be the same by the conditions of rigidity whatever $h, k,$ and l may be; that is

$$h^2 + k^2 + l^2 = \left(h + \frac{d\Delta x}{dx} h + \frac{d\Delta x}{dy} k + \frac{d\Delta x}{dz} l + P\right)^2$$
$$+ \left(k + \frac{d\Delta y}{dx} h + \frac{d\Delta y}{dy} k + \frac{d\Delta y}{dz} l + Q\right)^2$$
$$+ \left(l + \frac{d\Delta z}{dx} h + \frac{d\Delta z}{dy} k + \frac{d\Delta z}{dz} l + R\right)^2,$$

for all values of $h, k,$ and l.

Hence rejecting infinitesimals of the second order and equating to zero separately the coefficients of h^2, k^2, l^2, and of kl, lh, hk, we have

$$\frac{d\Delta x}{dx} = 0. \quad (a) \qquad \frac{d\Delta y}{dz} + \frac{d\Delta z}{dy} = 0. \quad (d)$$

$$\frac{d\Delta y}{dy} = 0. \quad (b) \qquad \frac{d\Delta z}{dx} + \frac{d\Delta x}{dz} = 0. \quad (e)$$

$$\frac{d\Delta z}{dz} = 0. \quad (c) \qquad \frac{d\Delta x}{dy} + \frac{d\Delta y}{dx} = 0. \quad (f)$$

By differentiating (*d*), (*e*), (*f*) with respect to z, x, y respectively, and substituting from (*a*), (*b*), (*c*), we obtain

$$\frac{d^2\Delta y}{dz^2} = 0, \quad \frac{d^2\Delta z}{dx^2} = 0, \quad \frac{d^2\Delta x}{dy^2} = 0.$$

By differentiating the same with respect to y, z, x respectively, and proceeding as before, we have

$$\frac{d^2\Delta z}{dy^2} = 0, \quad \frac{d^2\Delta x}{dz^2} = 0, \quad \frac{d^2\Delta y}{dx^2} = 0.$$

Thus, then, we have

$$\frac{d\Delta x}{dx} = 0, \quad \frac{d^2\Delta x}{dy^2} = 0, \quad \frac{d^2\Delta x}{dz^2} = 0,$$

$$\frac{d\Delta y}{dy} = 0, \quad \frac{d^2\Delta y}{dz^2} = 0, \quad \frac{d^2\Delta y}{dx^2} = 0,$$

$$\frac{d\Delta z}{dz} = 0, \quad \frac{d^2\Delta z}{dx^2} = 0, \quad \frac{d^2\Delta z}{dy^2} = 0,$$

therefore

$$\Delta x = A + By + Cz, \qquad (o)$$

$$\Delta y = D + Ez + Fx, \qquad (p)$$

$$\Delta z = G + Hx + Ky, \qquad (q)$$

A, B, C, D, E, F, being constant for a *given instant* of time; between which by virtue of the equations (*d*), (*e*), (*f*), we have the relations

$$E + K = 0, \qquad H + C = 0, \qquad B + F = 0.$$

If we call u, v, w the three component velocities of the particles at x, y, z parallel to the three axes, and X_1, Y_1, Z_1, the three internal forces, it is at once seen that u, v, w, as also ΔX_1, ΔY_1, ΔZ_1 must be subject to the same equations as limit Δx, Δy, Δz; so that

$$u = a + \gamma y - \beta z, \qquad (1)$$

$$v = b + \alpha z - \gamma x, \qquad (2)$$

$$w = c + \beta x - \alpha y, \qquad (3)$$

$$\Delta X_1 = a_1 + \gamma_1 y - \beta_1 z, \qquad (h)$$

$$\Delta Y_1 = b_1 + \alpha_1 z - \gamma_1 x, \qquad (j)$$

$$\Delta Z_1 = c_1 + \beta_1 x - \alpha_1 y. \qquad (k)$$

Also if X, Y, Z be the impressed forces, we have

$$X_1 + X = \frac{du}{dt}, \tag{4}$$

$$Y_1 + Y = \frac{dv}{dt}, \tag{5}$$

$$Z_1 + Z = \frac{dw}{dt}. \tag{6}$$

And by Gauss's principle, calling m the mass of the particle at x, y, z,

$$\Delta\Sigma m\,(X_1^2 + Y_1^2 + Z_1^2) = 0.$$

Hence equating separately to zero the coefficients of a_1, b_1, c_1 and of α_1, β_1, γ_1 in the quantity $\Sigma m\,(X_1\Delta X_1 + Y_1\Delta Y_1 + Z_1\Delta Z_1)$ we have

$$\left.\begin{aligned} \Sigma m\, X_1 &= 0\\ \Sigma m\, Y_1 &= 0\\ \Sigma m\, Z_1 &= 0\\ \Sigma m\,(Z_1 y - Y_1 z) &= 0\\ \Sigma m\,(X_1 z - Z_1 x) &= 0\\ \Sigma m\,(Y_1 x - X_1 y) &= 0 \end{aligned}\right\}. \tag{7—12}$$

Lastly, we have the equations

$$u = \frac{dx}{dt}, \tag{13}$$

$$v = \frac{dy}{dt}, \tag{14}$$

$$w = \frac{dz}{dt}. \tag{15}$$

From the fifteen equations marked (1) to (15), the motion may be determined by assigning the position of each particle at the end of the time t in terms of its three initial coordinates, its three initial velocities, and the initial values of the nine quantities

$$\begin{matrix} \Sigma mx, & \Sigma myz, & \Sigma mx^2,\\ \Sigma my, & \Sigma mzx, & \Sigma my^2,\\ \Sigma mz, & \Sigma mxy, & \Sigma mz^2. \end{matrix}$$

In the case of rest $X_1 = -X$, $Y_1 = -Y$, $Z_1 = -Z$, and the equations (7) to (12) inclusively taken, express the conditions of equilibrium.

The equations (o), (p), (q), which have been obtained from conditions *purely geometrical*, establish the well-known but interesting and *not obvious* fact, that any *small* motion of a rigid body may be conceived as made up of a motion of translation and a motion about *one* axis.