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Normal shock reflection in rubber-like elastic material

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USING a semi-inverse method, a normal reflection of a finite elastic plane shock wave at a plane boundary of a special elastic incompressible material is examined. Three kinds of boundary conditions are considered: 1) frictionless-rigid boundary, 2) clamped boundary and 3) free boundary. In all cases the reflection solution is, depending on the property of the material, either asingle simple wave or a shock wave, in cases 1), 3) of unloading type, in case 2) of loading type.

Płaska fala uderzeniowa propaguje się w nieściśliwym materiale sprężystym scharakteryzowanym trzema stałymi materiałowymi i pada prostopadle na jego brzeg. Na podstawie metody półodwrotnej rozpatruje się odbicie dla warunków brzegowych odpowiadających: 1) beztarciowemu podparciu sztywnemu, 2) brzegowi zamocowanemu oraz 3) brzegowi swobodnemu. We wszystkich trzech przypadkach w rezultacie odbicia powstaje w zależności od własności materiału albo fala prosta albo fala uderzeniowa; w przypadkach 1) i 3) fala odciążenia, a w przypadku 2) fala obciążenia

Плоская ударная волна распространяется в несжимаемом упругом материале, охарактеризованном тремя материальными постоянными, и падает перпендикулярно на его границу. Опираясь на полуобратный метод, рассматривается отражение для граничных условий, отвечающих: 1) жесткому опиранию без трения, 2) закрепленной границе, 3) свободной границе. Во всех трех случаях, в результате отражения, возникает, в зависимости от свойств материала, или простая волна, или ударная волна; в случаях 1 и 3 — волна разгрузки, а в случае 2 — волна нагрузки.

1. Introduction

WRIGHT in his paper on reflection of oblique shocks [7] presented a semi-inverse method, based on strictly mechanical considerations, of finding the reflected waves for a shock wave incident on a plane boundary of a nonlinear elastic body. In this method the incident shock wave is given *a priori*. The medium ahead of the shock has a fixed state; this means that the state behind the shock is also known. Then it is assumed that the state behind the wave and the state at the boundary (compatible with the boundary conditions) are connected by means of a sequence of one parameter families of reflected simple waves centered at a point of reflection. If the assumed reflection pattern fails the admissibility test, it is modified to include shocks or a combination of shocks and simple waves.

In this paper the semi-inverse method is used to examine a problem of reflection at a plane boundary of a plane transverse shock wave propagating through an elastic rubber-like ZAHORSKI's material [8] in a direction perpendicular to the boundary. The material region ahead of the incident shock is unstrained and at rest. The strength and direction of polarization of the shock are given; then the basic assumption is made that the reflection solution is a simple wave, with a fixed direction of propagation perpendicular to the boundary. The reflection problem then reduced to an initial boundary value problem for ordinary differential equations governing the variation of the deformation gradient and velocity in the region of a simple wave. Since the motion is restricted to one dimension, the reflection pattern includes a single (instead of two for incompressible materials) reflected wave only. If the obtained reflection solution fails the admissibility test, the shock wave replaces the simple wave in the assumed solution pattern. The differential equations for simple waves are then replaced by the jump conditions.

Section 2 contains a summary of the necessary theory and derivation of the propagation condition for simple waves in incompressible materials. Since the reflected wave can be a simple wave or a shock, we present in Sect. 3 differential equations for simple waves and jump conditions for shock waves. Three types of boundary conditions and the corresponding initial-boundary value problems are considered in Sect. 4.

2. Basic equations

The motion of a continuum is given by $x_i = x_i(X_\alpha, t)$ where x_i and X_α are the Cartesian coordinates of a material particle in the present configuration B and the reference configuration B_R, respectively. The deformation gradient $x_{i\alpha}$, its inverse $X_{\alpha i}$ and the velocity are defined by

$$x_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}, \quad X_{\alpha i} = \frac{\partial X_\alpha}{\partial x_i}, \quad \dot{x}_i = u_i = \frac{\partial x_i}{\partial t}. \tag{2.1}$$

It is assumed that the material is homogeneous, elastic and incompressible. The incompressibility constraint requires that

$$I = \det(x_{i\alpha}) = 1. \tag{2.2}$$

The Piola–Kirchhoff stress tensor for such a material is

$$T_{i\alpha} = \varrho_R \frac{\partial \sigma}{\partial x_{i\alpha}} + pX_{\alpha i}, \tag{2.3}$$

where σ denotes internal energy per unit mass in B_R, $\varrho_R = \varrho$ is the density and $p = p(X_\alpha)$ is an arbitrary scalar function (hydrostatic pressure).

If the stress and velocity fields are differentiable, then the equations expressing balance of momentum and moment of momentum are

$$T_{i\alpha,\alpha} = \varrho \dot{u}_i, \quad x_{i\alpha} T_{j\alpha} = x_{j\alpha} T_{i\alpha}. \tag{2.4}$$

If the functions $x_i(X_\alpha, t)$ are continuous everywhere but have discontinuous first derivatives on some propagating surface $S = 0$, Eqs. (2.4) must be completed by jump conditions on this surface:

$$\begin{gathered} [\![T_{i\alpha}]\!] N_\alpha = -\varrho [\![u_i]\!] V, \\ [\![x_{i\alpha}]\!] = a_i N_\alpha, \quad [\![u_i]\!] = -a_i V. \end{gathered} \tag{2.5}$$

Such a surface is called a shock wave. The vector $\mathbf{N}$ is a unit normal to the wave. V is the speed of propagation along $\mathbf{N}$ and $\mathbf{a}$ is the amplitude vector of the jump. The bold square brackets indicate the jump in the quantity enclosed across S; thus

$$[\![\cdot]\!] = (\cdot)^B - (\cdot)^F,$$

where the letters F and B refer to the limit values taken in the front and rear sides of S, respectively.

Simple waves [7] are defined to be regions of space-time in which all field quantities are continuous functions of a single parameter, say, $\lambda = \psi(X_\alpha, t)$. Regions of constant λ are propagating surfaces, called wavelets, with unit normal and normal velocity in B_R given by

(2.6) $$N_\alpha(\lambda) = \frac{\psi_{,\alpha}}{|\nabla\lambda|}, \quad U(\lambda) = -\frac{\dot\psi}{|\nabla\lambda|}.$$

The equation of motion (2.4) and the compatibility condition in the region of a simple wave are

(2.7) $$\frac{\partial T_{i\alpha}}{\partial x_{j\beta}} x'_{j\beta}\psi_{,\alpha} = \varrho u'_i \dot\psi, \qquad x'_{i\beta}\dot\psi = u'_j \psi,$$

where the prime indicates differentiation with respect to λ. If $\dot\psi \neq 0$, Eqs. (2.7) may be written as

(2.8) $$(Q_{ij} - \varrho U^2\delta_{ij})u'_j = 0, \qquad Ux'_{j\beta} + u'_j N_\beta = 0,$$

where

(2.8′) $$Q_{ij} = \frac{\partial T_{i\alpha}}{\partial x_{j\beta}} N_\alpha N_\beta.$$

is the acoustic tensor. For an incompressible material, substitution of Eq. (2.3) into Eq. (2.8), and the identity $X_{\alpha i,\alpha} = 0$ lead to the equation

$$(\tilde Q_{ij} - \varrho V^2\delta_{ij})u'^j - p_{,\alpha}X_{\alpha i}U(|\nabla\psi|)^{-1} = 0$$

or, since in the region of a simple wave $p_{,\alpha} = p'|\nabla\psi|N_\alpha$, to the equation

(2.9) $$(\tilde Q_{ij} - \varrho U^2\delta_{ij})u'_j - p'UX_{\alpha i}N_\alpha = 0.$$

We denote here

(2.10) $$\tilde Q_{ij} = \varrho\sigma_{ij}^{\alpha\beta} N_\alpha N_\beta, \quad \sigma_{ij}^{\alpha\beta} = \frac{\partial^2\sigma}{\partial x_{i\alpha}\partial x_{j\beta}}.$$

From the incompressibility condition (2.2) we have

$$I_{,\beta} = X_{\alpha i}x_{i\alpha}\psi_{,\beta} = 0.$$

Using this equation, together with Eq. (2.8) and the relation (cf. [6])

(2.11) $$Un_i = uN_\alpha X_{\alpha i},$$

where n_i is a unit normal and u the speed of propagation of the wave in B, the scalar p' can be eliminated to obtain

(2.12) $$(Q^*_{ij} - \varrho U^2\delta_{ij})u'^j = 0.$$

The tensor

(2.13) $$Q^*_{ij} = \tilde Q_{ij} - Q_{kj}n_k n_i$$

is called the reduced acoustic tensor

3. Incident shock and reflection pattern

The constraint of incompressibility restrict the propagating waves to transverse waves only. In general, the reflection problem may have no solution in terms of simple waves, as there are at the most two possible families of reflected waves in such a case; this means that there are two free parameters, with three boundary conditions to be met. However, solutions may exist for some types of incompressible materials, with particular deformation and boundary conditions. In this paper we examine such special cases.

Suppose that a plane transverse shock wave propagates through the half-space $X_2 > 0$ filled with an incompressible elastic material in a direction perpendicular to the boundary. Thus this travelling discontinuity surface belongs to a one-parameter family of parallel

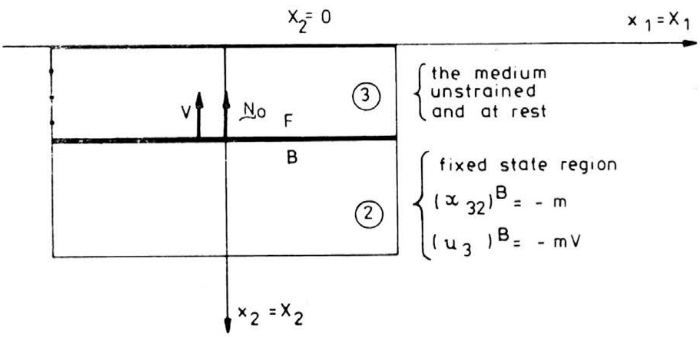

FIG. 1. Incident shock.

planes with normals $\mathbf{N}_0 = (0, -1, 0)$ (Fig. 1). It is assumed that the material is isotropic and it is defined by the constitutive equation

$$W(I_1, I_2) = \varrho\sigma(I_1, I_2) = C_1(I_1-3)+C_2(I_2-3)+C_3(I_1^2-9) \tag{3.1}$$

proposed by ZAHORSKI [8] where

$$I_1 = B_{ii}, \quad I_2 = \frac{1}{2}(B_{ii}B_{jj}-B_{ij}B_{ji})$$

are the invariants of the left Cauchy–Green strain tensor B_{ij}.

It is convenient to choose, with no loss of generality, the coordinate axes so that the X_3-axis is parallel to the amplitude vector of the shock. Since the medium ahead of the shock wave is unstrained and at rest, the nonzero jumps in the relations (2.5b) are

$$[\![u_3]\!] = (u_3)^B = -mV, \quad [\![x_{32}]\!] = (x_{32})^B = -m, \tag{3.2}$$

where $m = |a|$ is the shock strength. Substituting the relations (3.2) into Eqs. (2.5), we obtain the equation for the shock speed [3]

$$V^2 = C(1+\eta m^2), \tag{3.3}$$

where

$$C = \frac{2}{\varrho}(C_1+C_2+6C_3), \quad \eta = \frac{4C_3}{\varrho C}. \tag{3.4}$$

The shock wave may be specified completely by a single parameter, its strength m, thus fixing the state behind the wave. Equations (3.2) and (3.3) determine the deformation gradient

$$(x_{i\alpha}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & v & 1 \end{bmatrix} \tag{3.5}$$

and particle velocity

$$\mathbf{u} = (0, 0, u) \tag{3.6}$$

in this state. Here $v = (x_{32})^B$, $u = (u_3)^B$.

Only some components of $T_{i\alpha}$ and $\sigma_{ij}^{\alpha\beta}$, evaluated in region 1 (Fig. 1) for the material (3.1) are required in this paper. These are

$$\begin{gathered} T_{12} = 0, \quad T_{32} = 2\varrho\left(\frac{\partial\sigma}{\partial I_1} + \frac{\partial\sigma}{\partial I_2}\right)v, \\ T_{22} = 2\varrho\left(\frac{\partial\sigma}{\partial I_1} + 2\frac{\partial\sigma}{\partial I_2}\right) + p; \end{gathered} \tag{3.7}$$

$$\begin{gathered} \sigma_{33}^{22} = 2\left(\frac{\partial\sigma}{\partial I_1} + \frac{\partial\sigma}{\partial I_2}\right) + 4\frac{\partial^2\sigma}{\partial I_1^2}v^2, \\ \sigma_{23}^{22} = 4\frac{\partial^2\sigma}{\partial I_1^2}v, \quad \sigma_{ij}^{22} = 0 \quad \text{for} \quad i \neq j \end{gathered} \tag{3.8}$$

and

$$\frac{\partial\sigma}{\partial I_1} = \frac{1}{\varrho}(C_1 + 2C_3 I_1), \quad \frac{\partial\sigma}{\partial I_2} = \frac{C_2}{\varrho}, \quad \frac{\partial^2\sigma}{\partial I_1^2} = \frac{2C_3}{\varrho},$$
$$I_1 = I_2 = 3 + v^2.$$

The shock wave is reflected from the boundary $X_2 = 0$ and the reflected wave propagates into the state region just fixed (Fig. 2). The problem now is to fit this reflected wave so as to connect the state at the boundary that is compatible with the boundary conditions (region 0), with the state in front of the wave (region 2). Mathematically, the situation is similar to the problem of waves initiated by a impulsive load on the boundary [1, 2].

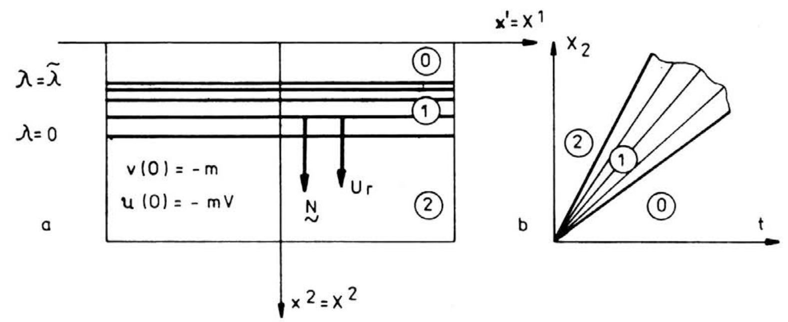

FIG. 2. Reflected wave. (a) in (X_2, x_2)-plane, (b) in (X_2, t)-plane.

We assume that the reflected wave is a simple wave (region 1, Fig. 2), propagating in the direction of the positive X_2-axis. Its wavelets are parallel planes, with normals $\mathbf{N} = (0, 1, 0)$. Since the components n_i of the normals refered to the present configuration B remain the same: $n_i = N_i$, the acoustic tensor (2.13) assumes a simpler form:

$$Q_{ij}^* = \tilde{Q}_{ij} = \varrho\sigma_{ij}^{22} \quad \text{for} \quad i \neq 2, \quad Q_{2j}^* = 0$$

which, together with Eqs. (3.8), leads to the propagation condition for simple waves (2.12) being reduced to a linear algebraic equation in U^2:

$$\sigma_{33}^{22} - U^2 = 0.$$

The eigenvalue $U_r^2 = \sigma_{33}^{22} = C(1+3\eta v^2)$, (ref. (3.8)), is a function of v^2 only. The requirement that U_r^2 must be a single-valued real function imposes a restriction on the internal energy function σ that the inequality $C(1+3\eta v^2) > 0$ holds in the whole region of a simple wave; it is necessary then that $C > 0$. We denote $c^2 = C$ and the equation for the speed of a simple wave, the propagation condition, is

$$U_r^2 = c^2(1+3\eta v^2). \tag{3.9}$$

The corresponding eigenfunction $\mathbf{u}'$ is given by

$$\mathbf{u}' = (0, 0, f), \tag{3.10}$$

where f is an arbitrary scalar function of λ. The differential equation relating the particle velocity and the deformation gradient is obtained from the compatibility conditions $(2.8)_2$:

$$U_r v' + u' = 0; \tag{3.11}$$

here U_r is the positive root of Eq. (3.9).

The geometrical significance of the above relations is evident. Differentiating $\psi(X_2, t) = \lambda$ along the line of the constant λ we obtain (Ref. (2.6))

$$\frac{dx_2}{dt} = U_r. \tag{3.12}$$

The curves given by Eq. (3.12) are the characteristics of the differential system (2.12). The trajectories of the wavelets in the (X_2, t) plane are given by the characteristics of the equations of motion in the region of a simple wave. The changes of the field of quantities in this region are governed by the ordinary differential equation (3.10) and Eq. (3.11).

It is convenient to assume $f = -U_r$. From Eqs. (3.10) and (3.11) it follows then that

$$v' = 1, \quad u' = -U_r. \tag{3.13}$$

Since the deformation gradient and velocity are continuous throughout the regions 0, 1 and 2, the initial values for the differential equations (3.13) that describe region 1 are the constant values of region 2. Thus from Eqs. (3.2), (3.5) and (3.6)

$$v(0) = -m, \quad u(0) = -mV, \tag{3.14}$$

where the shock speed V is given by Eq. (3.3), and m is the shock strength.

4. Boundary value problems

4.1. Frictionless-rigid boundary

Let us consider the case of "mixed" boundary conditions on the plane $X_2 = 0$, when the normal displacement and the shearing stresses T_{12} and T_{32} are zero. Since the motion is in the direction of the X_3-axis, the displacement condition and the first stress condition (ref. (3.7)) are satisfied identically. The second stress condition: $T_{32} = 0$ is met when

$$v = 0 \quad \text{on} \quad X_2 = 0. \tag{4.1}$$

Integrating the relations (3.13) with the conditions (3.14) and (4.1), we obtain now

$$u(\lambda) = -\int_0^\lambda U_r(\lambda)d\lambda - mV, \tag{4.2}$$

$$v(\lambda) = \lambda - m, \quad 0 \leqslant \lambda \leqslant m. \tag{4.3}$$

Both the velocity and the deformation gradient are constant along the wavelets $\lambda = $ = const., $0 \leqslant \lambda \leqslant m$. Substitution of the relations (4.3) into Eq. (3.9) completes the solution.

Whether the solution $U^2 = c^2(1+3\eta v^2)$, $-m \leqslant v \leqslant 0$, represents a simple wave depends on the sign of the constant which, by the relations (3.4), is clearly a property of the material (3.1). Each of the forward-propagating wavelets is identified by assigning to it a fixed value of v equal to $\lambda - m$. The leading wavelet is assigned here $v = -m$ and the following wavelets are given by the increasing values of the deformation gradient v. Thus a wavelet $\{v\}$ precedes the wavelet $\{v+\varepsilon\}$, $\varepsilon > 0$. The trailing wavelet corresponds to $v = 0$. If η is positive, that is if U_r is decreasing across the wave as v varies from $-m$ to 0, then the wavelets given by higher values of v travel at lower speed. In the (X_2, t) plane (Fig. 2b) the corresponding characteristics form a fan of diverging rays issuing from the origin. In this case the solution (3.9) represents a simple wave propagating into a deformed material and unloading it to a state of zero strain. The region of a simple wave is given by

$$c \cdot t \leqslant X_2 \leqslant c\sqrt{1+3\eta m_2} t. \tag{4.4}$$

If η is negative, the speed of the reflected wave is an increasing function of v. Consequently, the wavelet $\{v+\varepsilon\}$ propagates at a higher speed than the preceding wavelet $\{v\}$ and a solution (3.9) no longer represents a simple wave.

Suppose that $\eta < 0$ and that region 1 (Fig. 2) connecting regions 0 and 2 is a shock wave propagating in the positive direction of the X_2-axis; the wave front is parallel to the boundary $X_2 = 0$. The equations of motion (2.4) are now replaced by the jump conditions (2.5) connecting the corresponding quantities in regions 0 and 2 across the wave.

The constant state ahead of the wave (region 2) is given by Eqs. (3.5), (3.6) and (3.14). The constant state behind the wave (region 0) has zero strain. Thus $[\![v]\!] = -(v)^F = m$. Eliminating the velocity jump $[\![u]\!]$ from Eqs. (2.5) we obtain the equation for the reflected shock speed: $[\![T_{32}]\!] = \varrho V_r^2 [\![v]\!]$, or

$$V_r^2 = c^2(1+\eta m^2) \tag{4.5}$$

which is exactly the equation for the speed of the incident wave.

Equation (4.5) represents a weak solution of the reflection problem considered in this section. As such, this solution does not possess the uniqueness property of the smooth solutions. According to LAX [4], for Eq. (4.5) to represent an admissible shock, it must also satisfy a stability criterion:

$$U_r^B > V_r > U_r^F, \tag{4.6}$$

where U_r^B and U_r^F is the characteristic (acoustic) speed (3.9) in the material region behind and in front of the shock, respectively.

Since $(U_r^B)^2 = c^2$, $(U_r^F)^2 = c^2(1+3\eta m^2)$ and the $\eta < 0$ condition (4.7) is satisfied and Eq. (4.6) represents a shock $X_2 = V_r t$ (Fig. 3), the shock unloading the material.

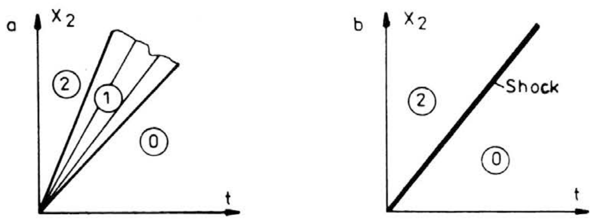

FIG. 3. Reflected wave, mixed boundary. (a) $\eta > 0$, (b) $\eta < 0$.

4.2. Clamped boundary

Let us assume that the incident shock is reflected from a rigidly constrained boundary; this means that

$$u = 0 \quad \text{on} \quad X_2 = 0. \tag{4.7}$$

In this case it is convenient to choose u as a parameter of the reflected wave. From Eqs. (3.14), (4.7) and (3.11) it follows that $-mV \leqslant u \leqslant 0$ and that v is a decreasing function of u, $v(0) < -m$. As v has a fixed negative sign throughout the wave, the sign derivative of the wave speed (3.9)

$$\frac{dU_r}{du} = -3c^2\eta U_r^{-2}v \tag{4.8}$$

is the same as the sign of η. If $\eta < 0$, then U_r is decreasing as u varies from $u = -mV$ to $u = 0$ and the solution $U_r^2 = c^2(1+3\eta v^2)$, $v = v(u)$, $-mV \leqslant u \leqslant 0$ represents a simple wave. The gradient v increases in absolute value across the wave, thus the wave is of the loading type. The region of the reflected wave is given by

$$c\sqrt{1+3\eta v_0^2}\,t \leqslant X_2 \leqslant c\sqrt{1+3\eta m^2}\,t, \qquad \eta < 0, \tag{4.9}$$

where $v_0 = v(0)$ is a solution of Eq. (4.2) for $u = 0$.

If η is positive, the speed U_r is an increasing function of u and the solution (3.9) does not represent a simple wave. Let us suppose now that the reflected wave is a shock (region 1). The constant state behind the shock (region 0) has nonzero strain, the deformation given by v_0. The jump of the deformation gradient across region 1 is $[\![v]\!] = v_0+m$. Since

$[\![u]\!] = -[\![v]\!]V_r$ and $[\![u]\!] = mV$, it must be $v_0 < -m$. From Eqs. (2.5) and (3.7) we obtain the equation for the reflected shock speed V_r:

$$\begin{aligned} [\![T_{32}]\!] &= \varrho[\![v]\!]V_r^2, \\ V_r^2 &= c^2(1+\eta(v_0^2-v_0m+m^2)). \end{aligned} \tag{4.10}$$

The characteristic speed U_r behind and ahead of the wave can be found from Eq. (3.9) by substituting for v the value v_0 or m, respectively. Thus $(U_r^B)^2 = c^2(1+3\eta v_0^2)$, $(U_r^F)^2 = c^2(1+3\eta m^2)$. Also, since $v_0 < -m$ and m is positive,

$$3v_0^2 > v_0^2-v_0m+m^2 > 3m^2,$$

and the stability condition (4.6) for the solution (4.10) is satisfied for an arbitrary value of the incident shock strength m. Thus the reflection solution is a shock: $X_2 = V_r t$, the wave loading the material.

4.3. Free boundary

Consider a case in which the stress vector $t_i = T_{i\alpha}K_\alpha$, $K_\alpha = (0, -1, 0)$ vanishes on the plane $X^2 = 0$. The first stress condition $(3.7)_1$ is satisfied identically. The second stress condition $(3.7)_2$ leads to the condition (4.1). The third stress condition which must be satisfied on $X^2 = 0$ in both regions 3 and 0, (Fig. 2a) determines the hydrostatic pressure p in region 3:

$$p_3 = -2(C_1+2C_2+6C_3) \tag{4.11}$$

and in region 0:

$$p_0 = p_3. \tag{4.12}$$

The function p is continuous throughout regions $0-1-2$ but it suffers a jump across the shock surface that separates regions 3 and 2. To find p_2 in region 2, we use the jump conditions $(2.5)_1$. The condition $[\![T_{12}]\!] = 0$ satisfied iden'ically the second condition $[\![T_{22}]\!] = 0$ together with Eqs. $(3.7)_3$ and $(3.14)_1$, give the jump of p. The function p_2 in the region of constant state is

$$p_2 = p_3-4C_3m^2. \tag{4.13}$$

In region 1 the deformation gradient and velocity are completely determined by Eq. (4.2) as continuous functions of the wave parameter in the interval $0 \leqslant \lambda \leqslant m$. The derivative p' of the scalar function can be computed from Eqs. (2.9) and (2.11) [3].

$$p' = uU^{-2}\tilde{Q}_{ij}u'^j n^i. \tag{4.14}$$

These equation was derived for an arbitrary homogeneous incompressible elastic material and arbitrary deformation. In the special case considered here this equation is reduced to the form

$$p' = \frac{\varrho u'}{U_r}\sigma_{23}^{22} = -8C_3v(\lambda). \tag{4.15}$$

Direct integration with the aid of the expression $(3.13)_1$ gives

$$p(\lambda) = -4C_3(v(\lambda))^2+c. \tag{4.16}$$

Due to continuity throughout regions $0-1-2$, the function $p(\lambda)$ must satisfy two conditions $p(0) = p_2$ and $p(\tilde{\lambda}) = p_0$ where p_2 and p_0 are given by the relations (4.11) and (4.12). Hence

$$p(\lambda) = -4C_3\big(v(\lambda)\big)^2 + p_0 \tag{4.17}$$

and this equation satisfies the second condition. This implies that the reflection problem for the case of the free boundary coincides with the case of the frictionless-rigid boundary and has a solution in the form (4.2), (4.3) with the additional pressure distribution given by Eq. (4.17).

5. Numerical solutions

The reflection solutions for $\eta > 0$ discussed in Sect. 4 are examined here numerically for the material (3.1) with the constants: $C_1 = 0.64$, $C_2 = 0.09$, $C_3 = 0.07$ (in kG/cm²) and for some values of the incident shock wave strength: $m = 0{,}1$, 0.6, 1.1, 1.6, 2.1, 2.6. For the boundary conditions considered here the differential equation which govern the variation of the component $v(\lambda)$ of the deformation gradient and the equation of the particle velocity $u(\lambda)$ can be integrated directly with the initial conditions (3.14).

5.1. Frictionless-rigid boundary

As is seen from the relations (4.3), the component $v(\lambda)$ of the deformation gradient is a linear function of the wave parameter λ. The equation for the speed of propagation of the simple wave (3.9) with the relations (4.3) is

$$U_r(\lambda) = \sqrt{c^2(1+3\eta(\lambda-m)^2}, \quad 0 \leqslant \lambda \leqslant m \tag{5.1}$$

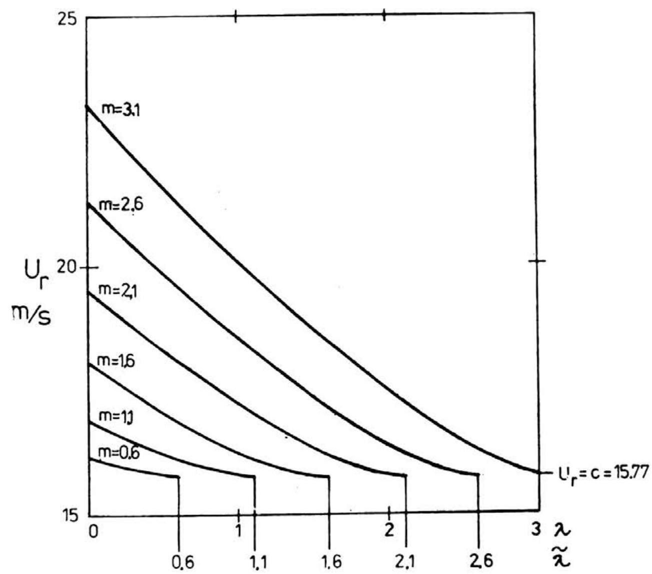

FIG. 4. Speed of the reflected simple wave $U_r(\lambda)$ as function of wave parameter and some values of m.

and represent one-parameter family of hyperbolical segments. The minimum value $U_r = c$ occurs for $\lambda = m$. Figure 4 shows that the graphs of $U_r(\lambda)$ for various values of the parameter m, become more curved with increasing m.

The form of $u(\lambda)$ Eq. (4.2) is more complex. Direct integration of Eq. (4.2) with the aid of the relations (4.3) gives

$$(5.2) \qquad u(\lambda) = \frac{c}{2}\sqrt{3\eta}\left[m(\psi-\phi)-\lambda\psi+\frac{1}{3\eta}\ln\frac{|\phi-m|}{|\psi+\lambda-m|}\right]-mV, \quad 0\leqslant\lambda\leqslant m,$$

where

$$\phi(m) = \sqrt{\frac{1}{3\eta}+m^2}, \quad \psi(\lambda, m) = \sqrt{\frac{1}{3\eta}+(\lambda-m)^2}.$$

The numerical results show that the graphs $u(\lambda)$ become less "curved" with the decreasing values of the incident shock wave strength (Fig. 5). For a small values of m the curve

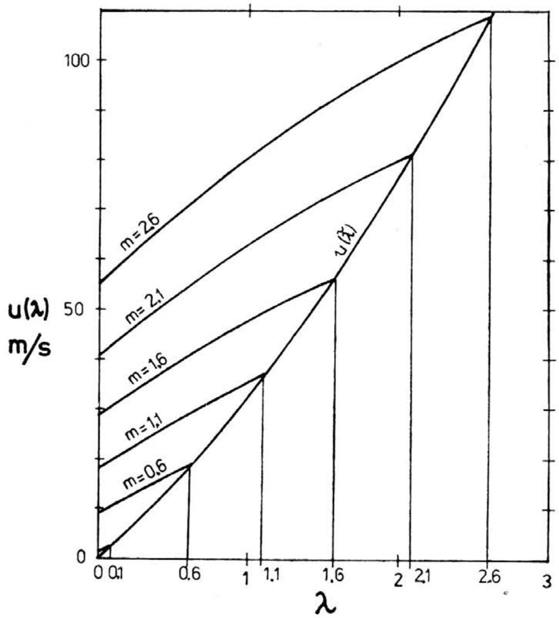

FIG. 5. Relation between particle velocity $u(\lambda)$ and wave parameter.

$u(\lambda)$ is almost a straight line. The function $u(\lambda)$ is an increasing function of the wave parametr λ, when λ ranges from 0 to its extreme value $\tilde{\lambda} > 0$. In the case of oblique reflection the component $x_2^3 = v(\lambda)$ vanishes on a frictionless rigid boundary plane $X^2 = 0$. The other component $x_1^3(\lambda)$ and particle velocity $u(\lambda)$ are decreasing with λ changing from 0 to $\tilde{\lambda} < 0$.

In the case considered here only the functions $u(\lambda)$ and $v(\lambda)$ can change in the region of the simple wave. The values of the function $v(\lambda)$ on the plane $X^2 = 0$ change rapidly from $v = -m$ to $v(\tilde{\lambda}) = 0$ and the work of elastic forces causes the increment of the kinetic energy which can be realized probably at the growth of the particle velocity $u(\lambda)$.

5.2. Free boundary

The numerical solutions for $v(\lambda)$ and $u(\lambda)$ are such as for a frictionless-rigid boundary. The pressure $p(\lambda)$ in the region of the simple wave is given by Eq. (4.17). Using the relations (4.3), this equation can be written as

$$p(\lambda) = -4C_3(\lambda - m)^2 + p_0. \tag{5.3}$$

The graphs in Fig. 6 form a one-parameter family of parabolical segments. For $\lambda = m$ the function $p(\lambda)$ reaches its minimum value $p(\tilde{\lambda}) = p_0 = -2.48$ kG/cm².

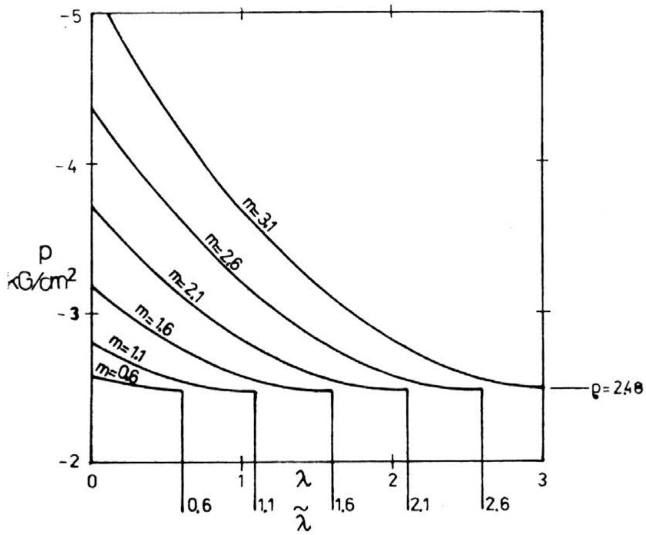

FIG. 6. The pressure distribution in region of simple wave.

5.3. Clamped boundary

According to the discussion given in Sect. 4, for $\eta > 0$ the solution in the form of the simple wave fails and the reflection solution is a shock wave. The jump of the deformation gradient across region 1 is $[\![v]\!] = v_0 + m$. Since $[\![u]\!] = -[\![v]\!]V_r$ and $[\![u]\!] = mV$, we obtain the equation for the component v_0 of the deformation gradient behind the reflected shock wave

$$-(v_0 + m)V_r = mV. \tag{5.4}$$

Now, using Eqs. (4.10), the above equation can be written as the algebraic cubic equation for v_0, with the coefficients depending on η and m. This equation is solved numerically by the Newton–Raphson procedure for some values of the incident shock wave strength. Some results are presented graphically in Fig. 7. The inequality $v_0 < -m$ selects uniquely the real roots. Figure 8 refers to a stability criterion (4.6) and presents the reflected shock wave speed V_r, the acoustic speed $(U_r)^B$, $(U_s)^F$ in the material region behind and in front of the reflected shock wave as functions of the parameter m.

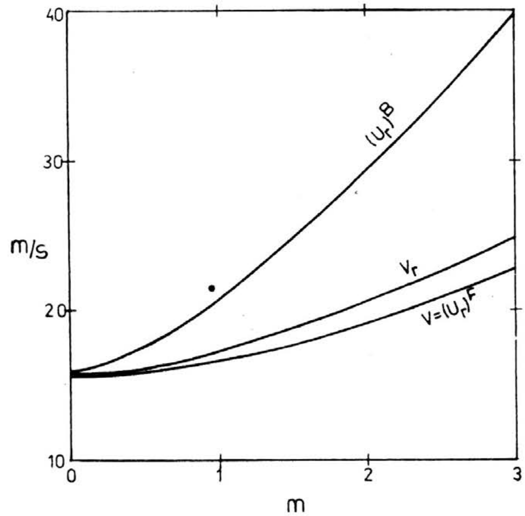

FIG. 7. The component of the deformation gradient v_0 behind the reflected shock wave.

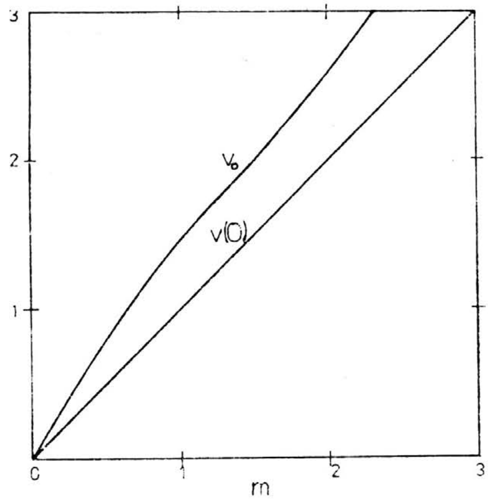

FIG. 8. Wave speeds $(U_r)^B$, V_r, $(U_r)^F$ as functions of m.

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