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On the low and high-frequency behaviour of generalized thermoelastic waves

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THE PROPAGATION of plane harmonic waves in a homogeneous transversely isotropic generalized thermoelastic medium has been investigated with the help of the theory of algebraic functions. The low and high-frequency approximations for the propagation speeds and attenuation coefficients have been obtained for quasi-longitudinal (QL), quasi-transverse (QT) and quasi-thermal (T-mode). The limiting cases of the frequency equation have also been discussed.

Posługując się teorią funkcji algebraicznych, przeanalizowano problem rozprzestrzeniania się płaskich fal harmonicznych w jednorodnym, poprzecznie izotropowym, uogólnionym ośrodku termosprężystym. Otrzymano przybliżenia niskich i wysokich częstotliwości dla prędkości propagacji i współczynników tłumienia w przypadku fal quasi-podłużnych (QL), quasi-poprzecznych (QT) i quasi-termicznych (T). Omówiono również przypadki graniczne równania częstości.

Послуживаясь теорией алгебраических функций, проанализирована проблема распространения плоских гармонических волн в однородной, поперечно изотропной, обобщенной термоупругой среде. Получены приближения низких и высоких частот для скорости распространения и коэффициентов затухания в случае квазипродольных (QL), квазипоперечных (QT) и квазитермических (T) волн. Обсуждены тоже предедельные случаи уравнения частот.

1. Introduction

IN COUPLED thermoelasticity, the propagation of plane harmonic waves in homogeneous transversely isotropic heat conducting elastic materials has been investigated by CHADWICK and SEET [1]. CHADWICK [2] studied the basic properties of plane harmonic waves in an homogeneous anisotropic heat conducting materials. Recently, the generalized theory of thermoelasticity advanced by LORD and SHULMAN [3] has been extended to anisotropic elastic bodies by DHALIWAL and SHERIEF [4]. SINGH and SHARMA [5] discussed the propagation of plane harmonic waves in a transversely isotropic thermoelastic medium in the context of the theory developed in [4]. Three dispersive waves, namely QL, QT and T-mode apart from the SH-wave, are found to exist in such materials. SHARMA and SINGH [6] investigated the propagation of plane waves in a homogeneous anisotropic thermoelastic medium. Four dispersive waves are found to exist. The results have been verified numerically.

The aim of the present article is to give a detailed account of the low and high-frequency behaviour of generalized thermoelastic waves in transversely isotropic materials in the context of a theory developed in [4] and the theory developed by GREEN and LINDSAY [7].

2. The problem and the secular equation

We consider an infinite homogeneous transversely isotropic, thermoelastic medium at uniform temperature T_0. We take the x_3-axis as the axis of symmetry. Then the displacement vector $\mathbf{u}(x_1, x_2, x_3, t) = (u_1, u_2, u_3)$ and the temperature $T(x_1, x_2, x_3, t)$ in the context of generalized thermoelasticity satisfy the basic field equations of motion and heat conduction equation in the absence of body forces and heat sources as [5]:

$$\frac{1}{2}(c_{11}-c_{12})u_{i,jj}+\frac{1}{2}(c_{11}+c_{12})u_{j,ij}+c_{44}u_{i,33}+(c_{13}+c_{44})u_{3,i3}-\beta_1 T_{,i} = \varrho\ddot{u}_i, \tag{2.1}$$

$$c_{44}u_{3,jj}+c_{33}u_{3,33}+(c_{13}+c_{44})u_{j,3j}-\beta_3 T_{,3} = \varrho\ddot{u}_3, \qquad i,j = 1,2, \tag{2.2}$$

$$K_1 T_{,jj}+K_3 T_{,33}-\varrho C_e(\dot{T}+\tau_0\ddot{T}) = T_0[\beta_1(\dot{u}_{j,j}+\tau_0\ddot{u}_{j,j})+\beta_3(\dot{u}_{3,3}+\tau_0\ddot{u}_{3,3})], \tag{2.3}$$

where

$$\beta_1 = (c_{11}+c_{12})\alpha_1+c_{13}\alpha_3, \qquad \beta_3 = 2c_{13}\alpha_1+c_{33}\alpha_3$$

and all other symbols have their usual meanings as in [5]. The comma notation is used for spatial derivatives and the superposed dot denotes time differentiation.

We assume that

$$\begin{gathered} K_1>0, \quad K_3>0, \quad \varrho>0, \quad T_0>0, \quad C_e>0, \quad c_{11}>0, \quad c_{11}>c_{12}, \\ c_{11}^2>c_{12}^2, \quad c_{44}>0, \quad c_{33}(c_{11}+c_{12})>2c_{13}^2. \end{gathered} \tag{2.4}$$

To solve Eqs. (2.1)–(2.3) we take [5]

$$u_1 = \phi_{,1}+\chi_{,2}, \qquad u_2 = \phi_{,2}-\chi_{,1}, \qquad u_3 = \psi_{,3}. \tag{2.5}$$

Substituting Eq. (2.5) in Eq. (2.1)–(2.3), we obtain

$$\phi_{,ii}+c_2\phi_{,33}+c_3\psi_{,33}-\beta_1 T/c_{11} = \varrho\ddot{\phi}/c_{11}, \tag{2.6}$$

$$c_3\phi_{,ii}+c_2\psi_{,ii}+c_1\psi_{,33}-\beta_3 T/c_{11} = \varrho\ddot{\psi}/c_{11}, \tag{2.7}$$

$$T_{,ii}+\bar{K}T_{,33}-(\varrho C_e/K_1)(\dot{T}+\tau_0\ddot{T}) = (T_0\beta_1/K_1)[\dot{\phi}_{,ii}+\tau_0\ddot{\phi}_{,ii}+\bar{\beta}(\dot{\psi}_{,33}+\tau_0\psi_{,33})], \tag{2.8}$$

$$(c_{11}-c_{12})\chi_{,ii}+2c_{44}\chi_{,33}-2\varrho\ddot{\chi} = 0, \qquad i = 1,2, \tag{2.9}$$

where

$$c_1 = c_{33}/c_{11}, \quad c_2 = c_{44}/c_{11}, \quad c_3 = (c_{13}+c_{44})/c_{11}, \quad \bar{K} = K_3/K_1, \quad \bar{\beta} = \beta_3/\beta_1. \tag{2.10}$$

Equation (2.9) gives a purely transverse wave which is not affected by the temperature and is polarized in planes perpendicular to the x_3- axis and may be referred to the SH-wave. This wave propagates without dispersion or damping with the speed

$$V_3 = [\{(c_{11}-c_{12})\sin^2\theta+2c_{44}\cos^2\theta\}/2\varrho]^{1/2} \tag{2.11}$$

where θ is the inclination of the wave normal to the x_3-axis.

Since Eqs. (2.6)–(2.8) are independent of χ and *vice-versa*, we exclude Eq. (2.9) from further discussion.

For plane harmonic waves we take

$$\begin{gathered} \phi = A\exp\{i\omega(v^{-1}x_p n_p-t)\}, \qquad \psi = B\exp\{i\omega(v^{-1}x_p n_p-t)\}, \\ T = C\exp[i\omega(v^{-1}x_p n_p-t)], \end{gathered} \tag{2.12}$$

where $i = \sqrt{(-1)}$, A, B, C are the amplitudes, v is the phase velocity (in general complex), ω the angular frequency (assumed real) and $\mathbf{n} = (n_1, n_2, n_3)$ is the wave normal which specifies the direction of wave propagation.

Using Eqs. (2.12) in Eqs. (2.6) to (2.8), we obtain the secular equation as

$$(1-\tau_0\omega_1^* z)\zeta(\zeta-\lambda_1^*)(\zeta-\lambda_2^*)+zD(\zeta-\lambda_1)(\zeta-\lambda_2) = 0, \tag{2.13}$$

where

$$\zeta = \varrho v^2/c_{11}, \quad D = \sin^2\theta+\overline{K}\cos^2\theta, \quad \omega_1^* = C_e c_{11}/K_1, \quad \varepsilon_1 = \beta_1^2 T_0/\varrho C_e c_{11}, \quad \chi^* = \omega/\omega_1^*,$$
$$z = i\chi^*, \quad \lambda_1, \lambda_2 = \left(a_2 \pm \sqrt{(a_3^2-4a_1)}\right)/2, \quad \lambda_1^*, \lambda_2^* = (1+\varepsilon_1)\left(A_2 \pm \sqrt{(A_2^2-4A_1)}\right)/2, \tag{2.14}$$

$$a_1 = c_2\sin^4\theta+(c_1+c_2^2-c_3^2)\sin^2\theta\cos^2\theta+c_1c_2\cos^4\theta, \tag{2.15}$$

$$a_2 = (1+c_2)\sin^2\theta+(c_1+c_2)\cos^2\theta, \tag{2.16}$$

$$A_1 = C_2\sin^4\theta+(C_1+C_2^2-C_3^2)\sin^2\theta\cos^2\theta+C_1C_2\cos^4\theta, \tag{2.17}$$

$$A_2 = (1+C_2)\sin^2\theta+(C_1+C_2)\cos^2\theta, \tag{2.18}$$

$$C_1 = (c_1+\varepsilon_1\overline{\beta}^2)/(1+\varepsilon_1), \quad C_2 = c_2/(1+\varepsilon_1), \quad C_3 = (c_3+\varepsilon_1\beta)/(1+\varepsilon_1) \tag{2.19}$$

and θ is the inclination of the wave normal to the axis of symmetry.

3. Limiting cases of the secular equation

When $\omega \to 0$. i.e., $z \to 0$ Eq. (2.13) reduces to

$$\zeta(\zeta-\lambda_1^*)(\zeta-\lambda_2^*) = 0. \tag{3.1}$$

The roots of this equation are

$$\zeta_i(0) = \lambda_i^*, \quad \zeta_3(0) = 0, \quad i = 1, 2. \tag{3.2}$$

It follows from Eq. (2.14) that the velocities associated with these roots have real values $v_i = (c_{11}\lambda_i^*/\varrho)^{1/2}$. The modes associated with the first two roots ζ_i are quasi-longitudinal (QL), quasi-transverse (QT) and the third associated with ζ_3 is the thermal (T-mode). Obviously, there is no damping in either of these modes in this limiting case.

The secular equation (2.13) may also be written as

$$(\hat{z}-\tau_0\omega_1^*)\zeta(\zeta-\lambda_1^*)(\zeta-\lambda_2^*)+D(\zeta-\lambda_1)(\zeta-\lambda_2) = 0, \tag{3.3}$$

where $\hat{z} = 1/z$.

When $\omega \to \infty$, i.e., $\hat{z} \to 0$, Eq. (3.3) reduces to

$$-\tau_0\omega_1^*\zeta(\zeta-\lambda_1^*)(\zeta-\lambda_2^*)+D(\zeta-\lambda_1)(\zeta-\lambda_2) = 0. \tag{3.4}$$

Approximate values of the three roots of this equation will be obtained in the next section as a special case of high frequency approximation. However, when $\tau_0 \to 0$ this equation may be written as

$$\zeta^{-1}(\zeta-\lambda_1)(\zeta-\lambda_2) = 0. \tag{3.5}$$

The three roots of this equation are

(3.6) $$\zeta_i = \lambda_i, \quad \zeta_3 = \infty, \quad i = 1, 2.$$

The velocities of the first two modes associated with ζ_i have real values $(\lambda_i c_{11}/\varrho)^{1/2}$. The third mode (T) has infinite velocity of propagation and it thus diffusive in nature.

4. Discussion of the secular equation

The secular equation (2.13) is, in general, an irreducible cubic equation in the unknown ζ which determines the complex phase velocity. When the numerical values of all the parameters involved are given, this equation may be solved exactly or numerically by standard methods. However, it is of theoretical as well as of practical interest to investigate, as far as possible, the three roots of Eq. (2.13) as it stands. We discuss the roots of Eqs. (2.13) or (3.3) with the help of the theory of algebraic functions taking z or $\hat{z}$ as a complex variable.

4.1. Low-frequency approximations

Let

(4.1) $$F(\zeta, z) = (1-\tau_0\omega_1^* z)\zeta(\zeta-\lambda_1^*)(\zeta-\lambda_2^*)+z(\zeta-\lambda_1)(\zeta-\lambda_2) = 0.$$

The critical points of the algebraic function F defined by Eq. (4.1) are the zeros of the discriminant obtained by eliminating ζ from

(4.2) $$F(\zeta, z) = 0, \quad \partial F/\partial\zeta = 0$$

and the zeros of the coefficient of highest power in ζ, which is a sextic equation in z. Let the roots of this equation be z_j ($j = 1, 2, \ldots, 6$). The seventh critical point is at $z = 1/\tau_0\omega_1^*$. By a basic theorem in the theory of algebraic functions there are three distinct roots $\zeta_j(z)$, ($j = 1, 2, 3$) which are analytic functions of z in a domain D of the complex z-plane which excludes these critical points [8]. It may be shown that z_j are the branch points and $1/\tau_0\omega_1^*$ is a simple pole of these roots. None of z_j could be zero, for otherwise at least one of $\zeta = 0$, λ_1^*, λ_2^* would be a repeated root of $F(\zeta, 0) = 0$. Thus these roots admit Taylor series expansions in the neighbourhood of $z = 0$. We may, therefore, write

(4.3) $$\zeta_i(z) = \lambda_i^*\left[1+\sum_{n=1}^{\infty} c_n^{(i)}(-z)^n\right], \quad i = 1, 2,$$

(4.4) $$\zeta_3(z) = \sum_{n=1}^{\infty} d_n(-z)^n$$

where the first two coefficients in these series are given by

(4.5) $$c_1^{(i)} = Dg(\lambda_i^*)/f'(\lambda_i^*)\lambda_i^*, \quad d_1 = Dg(0)/f'(0),$$

(4.6) $$c_2^{(i)} = c_1^{(i)}[Dg'(\lambda_i^*)-\tau_0\omega_1^* f'(\lambda_i^*)-c_1^{(i)}\lambda_i^* f''(\lambda_i^*)/2]/f'(\lambda_i^*),$$

(4.7) $$d_2 = d_1[Dg'(0)-\tau_0\omega_1^* f'(0)-d_1 f''(0)/2]/f'(0),$$

(4.8) $$f(\zeta) = \zeta(\zeta-\lambda_1^*)(\zeta-\lambda_2^*), \quad g(\zeta) = (\zeta-\lambda_1)(\zeta-\lambda_2).$$

These series converge for $|z| < R$ where

(4.9) $$R = \mathrm{Min}\{1/\tau_0\omega_1^*, |z_j|\}, \quad j = 1, 2, 3, \ldots$$

is the distance of the nearest singularity (critical point) of ζ_i from the origin.

If we write

$$v^{-1} = V^{-1} + i\omega^{-1}q,$$

where V and q are real, the exponent in the plane wave in Eq. (2.12) becomes

(4.10) $$-qx_pn_p + i\omega(V^{-1}x_pn_p - t).$$

This shows that V is the speed of propagation and q the attenuation coefficients of the wave. Using Eqs. (4.3)–(4.4), we obtain the values of V and q for different modes:

(4.11) $$V_i = V_i^* \sqrt{R_i}/\cos(\phi_i/2), \quad q_i = \omega \sin(\phi_i/2)/V_i^* \sqrt{R_i}, \quad i = 1, 2, 3,$$

where for elastic waves

(4.12) $$R_i = \sqrt{(A_i^2 + B_i^2)}, \quad \phi_i = \tan^{-1}(\pm |B_i/A_i|),$$
$$A_i = 1 - c_2^{(i)}\chi^{*2}, \quad B_i = c_1^{(i)}\chi^*, \quad V_i^* = (c_{11}\lambda_i^*/\varrho)^{1/2}$$

and for the T-mode

(4.13) $$A_3 = -\chi^{*2}d_2, \quad B_3 = d_1\chi^*, \quad V_3^* = (c_{11}/\varrho)^{1/2}.$$

The signs + or − in the determination of ϕ_i are taken accordingly as $x_pn_p > 0$ or < 0 in the expression (4.10)

4.2. High-frequency approximation

We now obtain approximations to the roots of the secular equation (3.3) when $\hat{z}$ is in the neighbourhood of $\tau_0\omega_1^*$. The critical points of the algebraic function defined by Eq. (3.3) are at $\hat{z}_i = 1/z_i$, $(i = 1, 2, \ldots, 6)$ and $\hat{z} = \tau_0\omega_1^*$. Due to the roots (3.6), it may be shown that $\hat{z} = \tau_0\omega_1^*$ is a removable singularity for the roots $\zeta_i(\hat{z})$ and a simple pole for $\zeta_3(\hat{z})$. $\hat{z}_i$ $(i = 1, 2, \ldots, 6)$ are branch points of all the three roots $\zeta_i(\hat{z})$, $(i = 1, 2, 3)$. Thus ζ_1, ζ_2 admit Taylor's expansions and $\zeta_3(\hat{z})$ admits Laurent's expansion in the neighbourhood of $\hat{z} = \tau_0\omega_1^*$. We can, therefore, write

(4.14) $$\zeta_i(\hat{Z}) = \lambda_i \left[1 + \sum_{n=1}^{\infty} c_n^{(i)}(-\hat{Z})^n\right],$$

(4.15) $$\zeta_3(\hat{Z}) = \eta(\hat{Z})/\hat{Z},$$

where

(4.16) $$\hat{Z} = \hat{z} - \tau_0\omega_1^*, \quad \eta(\hat{Z}) = \sum_{n=0}^{\infty} d_n(\hat{Z})^n.$$

The expansions (4.14) and (4.16) are valid for

(4.17) $$|\hat{Z}| < R,$$

where

(4.18) $$R = \mathrm{Min}\{|\hat{z}_i - \tau_0\omega_1^*|\}.$$

If the critical point $\hat{z}_i$ which defines R in the relation (4.17) is labelled $\hat{z}_1$ and we write

$$\hat{z}_1 = \hat{x}_1 + i\hat{y}_1$$

then the relation (4.17) leads to

(4.19) $$\omega^2 > \omega_1^* / \{\hat{x}_1^2 + \hat{y}_1^2 - 2\hat{x}_1 \tau_0 \omega_1^*\}.$$

This gives the lower limit on the frequencies above which the expansions (4.14) and (4.15) hold.

If the expansions (4.14), (4.15) and (4.16), are introduced in Eq. (3.3), then by using the roots (3.6) we obtain the following expansions for the first two coefficients as

$$c_1^{(i)} = f(\lambda_i)/D\lambda_i g'(\lambda_i), \quad c_2^{(i)} = c_1^{(i)} \left[f'(\lambda_i) - \frac{1}{2} D\lambda_i c_1^{(i)} g''(\lambda_i) \right] / Dg'(\lambda_i),$$

(4.20) $$d_0 = -D, \quad d_1 = \sum_{i=1}^{2} (\lambda_i^* - \lambda_i),$$

$$d_2 = 2D^{-1}[\lambda_1^* \lambda_2^* - \lambda_1 \lambda_2 + d_1 \{2\lambda_1^* + \lambda_2^*) - \lambda_1 - \lambda_2 + 2d_1\}],$$

where $f(\zeta)$ and $g(\zeta)$ have been defined in Eq. (4.8). To obtain the propagation velocities and attenuation coefficients, we write

(4.21) $$\hat{Z} = \hat{z} - \tau_0 \omega_1^* = re^{i\psi},$$

where

(4.22) $$r = (\chi^{*-2} + \tau_0^2 \omega_1^{*2})^{1/2}, \quad \psi = \tan^{-1}(1/\tau_0 \omega).$$

We then obtain

(4.23) $$V_i = c_i \sqrt{r_i}/\cos(\psi_i/2), \quad q_i = \omega \sin(\psi_i/2)/c_i \sqrt{r_i}, \quad i = 1, 2, 3,$$

where

$$r_i = \sqrt{(a_i^2 + b_i^2)}, \quad \psi_i = \tan^{-1}(\pm |b_i/a_i|),$$

(4.24) $$a_i = 1 - r\cos\psi c_1^{(i)} + r^2 \cos 2\psi c_2^{(i)},$$

$$b_i = -r\sin\psi c_1^{(i)} + r^2 \sin 2\psi c_2^{(i)}, \quad c_i = (c_{11} \lambda_i/\varrho)^{1/2},$$

for elastic waves

$$a_3 = -D\cos(\psi)/r + r\cos\psi d_1 + r^2 \cos 2\psi d_2,$$

(4.25) $$b_3 = D\sin(\psi)/r + r\sin\psi d_1 + r^2 \sin 2\psi d_2, \quad c_3 = \sqrt{(c_{11}/\varrho)} \quad \text{for T-mode.}$$

The + or − signs in the determination of ψ are to be taken accordingly as $x_p n_p < 0$ or > 0.

The approximate values of the roots of Eq. (3.4) can be obtained from Eqs. (4.14) and (4.15) on letting $\chi^* \to \infty$. We obtain

$$\zeta_i = \lambda_i [1 + \tau_0 \omega_1^* c_1^{(i)} + \tau_0^2 \omega_1^{*2} c_2^{(i)} + \ldots],$$

$$\zeta_3 = 1/\tau_0 \omega_1^* + d_1 - d_2 \tau_0 \omega_1^* + \ldots, \quad i = 1, 2.$$

The values of τ_0 for which these expansions hold follow from the relation (4.19), and are to satisfy the inequality

$$\hat{x}_1^2 + \hat{y}_1^2 - 2\hat{x}_1 \tau_0 \omega_1^* > 0.$$

The real values of propagation velocities follow from the above expansions directly. We obtain

$$V_i = c_i\{1+\tau_0\omega_1^* c_1^{(i)} + \tau_0^2 c_2^{(i)}\omega_1^{*2} + \ldots\} \quad \text{for elastic waves}$$

$$V_3 = c_3\left\{\frac{1}{\tau_0\omega_1^*} + d_1 - \tau_0\omega_1^* d_2 + \ldots\right\} \quad \text{for thermal wave.}$$

This last result shows that the T-mode has now a finite velocity of propagation whereas in the coupled thermoelasticity ($\tau_0 \to 0$) this mode is evidently diffusive.

5. Analysis on the basis of Green–Lindsay theory

In this section we shall discuss the propagation of plane harmonic waves in a transversely isotropic thermoelastic medium in the context of the Green and Lindsay [7] theory of thermoelasticity. The basic equations of motion and heat conduction in the absence of body forces and heat sources are

$$(5.1) \quad \frac{1}{2}(c_{11}-c_{12})u_{i,jj} + \frac{1}{2}(c_{11}+c_{12})u_{j,ij} + c_{44}u_{i,33} + (c_{13}+c_{44})u_{3,i3} - \varrho\ddot{u}_i = \\ = \beta_1(T+\alpha_0\dot{T})_{,i},$$

$$(5.2) \quad c_{44}u_{3,jj} + c_{33}u_{3,33} + (c_{13}+c_{44})u_{j,3j} - \varrho\ddot{u}_3 = \beta_3(T+\alpha_0\dot{T})_{,3},$$

$$(5.3) \quad K_1 T_{,jj} + K_3 T_{,33} - \varrho C_e(\dot{T}+\alpha_0^*\ddot{T}) = T_0(\beta_1\dot{u}_{j,j} + \beta_3\dot{u}_{3,3}),$$

where α_0 and α_0^* are the thermal relaxation times and all other symbols have their usual meanings as defined in [5]. The parameters α_0 and α_0^* satisfy the inequality

$$(5.4) \quad \alpha_0 \geqslant \alpha_0^* \geqslant 0.$$

If $\alpha_0 \neq 0$, the stresses depend on the temperature velocity and if $\alpha^* \neq 0$, the heat propagates with a finite speed. Since $\alpha_0^* \neq 0$ implies that $\alpha_0 \neq 0$, it follows that the heat cannot propagate with finite speed, unless the stresses depend on the temperature velocity.

Using Eqs. (2.5) and (2.12) in Eqs. (5.3), we see that the purely transverse (SH) wave again gets decoupled from the rest of the motion and propagates without damping or dispersion with speed given by Eq. (2.11). The secular equation in this case for the rest of the motion is given by

$$(5.5) \quad (1-\alpha_0^*\omega_1^* z)\zeta(\zeta-\lambda_1^*)(\zeta-\lambda_2^*) + zD(\zeta-\lambda_1')(\zeta-\lambda_2') = 0,$$

where $\zeta, \omega_1^*, z, \lambda_1^*, \lambda_2^*, D$ are given by Eqs. (2.14) and

$$(5.6) \quad \lambda_1' = [a_2+b_1+\{(a_2+b_1)^2-4a_1(1+b_2)\}^{1/2}]/2(1+b_2),$$

$$(5.7) \quad \lambda_2' = [a_2+b_1-\{(a_2+b_1)^2-4a_1(1+b_2)\}^{1/2}]/2(1+b_2),$$

$$(5.8) \quad b_1 = \varepsilon_1(\alpha_0-\alpha_0^*)\omega_1^* D^{-1}[c_2\sin^4\theta + (c_1-2c_3\bar{\beta}+\bar{\beta}^2)\sin^2\theta\cos^2\theta + c_2\bar{\beta}\cos^4\theta],$$

$$(5.9) \quad b_2 = \varepsilon_1\omega_1^*(\alpha_0-\alpha_0^*)D^{-1}(\sin^2\theta+\bar{\beta}^2\cos^2\theta).$$

Proceeding as in the previous section we again obtain the following approximations for the speed of propagation and the attenuation coefficients of the QL, QT and thermal (T-mode) waves:

i) Low-frequency approximations

(5.10) $V_i = V_i^* \sqrt{R_i^*}/\cos(\phi_i^*/2), \quad q_i = \omega \sin(\phi_i^*/2)/V_i^* \sqrt{R_i^*}, \quad i = 1, 2, 3,$

where

(5.11) $R_i^* = \sqrt{(L_i^2 + M_i^2)}, \quad \phi_i^* = \tan^{-1}(\pm |M_i/L_i|),$

(5.12) $L_i = 1 - c_2^{(i)} \chi^{*2}, \quad M_i = c_1^{(i)} \chi^*, \quad V_i^* = (c_{11} \lambda_i^*/\varrho)^{1/2}$ for elastic waves,

(5.13) $L_3 = -\chi^{*2} d_2, \quad M_3 = d_1 \chi^*, \quad V_3^* = (c_{11}/\varrho)^{1/2}$ for T-mode.

The quantities $c_1^{(i)}$, $c_2^{(i)}$, d_1 and d_2 in this case are

(5.14) $c_1^{(i)} = Dg^*(\lambda_i^*)/\lambda_i^* f'(\lambda_i^*), \quad c_2^{(i)} = c_1^{(i)}[Dg^{*\prime}(\lambda_i^*) - \alpha_0^* \omega_1^* f'(\lambda_i^*) - c_1^{(i)} \lambda_i^* f''(\lambda_i^*)/2]/f'(\lambda_i^*),$

(5.15) $d_1 = Dg^*(0)/f'(0), \quad d_2 = d_1[Dg^{*\prime}(0) - \alpha_0^* \omega_1^* f'(0) - d_1 f''(0)/2]/f'(0),$

where $f(\zeta)$ is given by the relations (4.8) and

(5.16) $g^*(\zeta) = (\zeta - \lambda_1')(\zeta - \lambda_2').$

ii) High-frequency approximations

(5.17) $V_i = c_i' \sqrt{r_i^*}/\cos(\psi_i^*/2), \quad q_i = \omega \sin(\psi_i^*/2)/c_i' \sqrt{r_i^*}, \quad i = 1, 2, 3,$

where

(5.18) $r_i^* = \sqrt{(l_i^2 + m_i^2)}, \quad \psi_i^* = \tan^{-1}(\pm |l_i/m_i|),$

(5.19) $l_i = 1 - r^* \sin\psi' c_1^{(i)} + r^{*2} \cos 2\psi' c_2^{(i)}$ for elastic waves,

(5.20) $m_i = -r^* \sin\psi' c_1^{(i)} + r^{*2} \sin 2\psi' c_2^{(i)}, \quad c_i' = (c_{11} \lambda_i'/\varrho)^{1/2},$

(5.21) $l_3 = -D\cos(\psi')/r^* + r^* \cos\psi' d_1 + r^{*2} \cos 2\psi' d_2,$

(5.22) $m_3 = D\sin(\psi')/r^* + r^* \sin\psi' d_1 + r^{*2} \sin 2\psi' d_2, \quad c_3 = (c_{11}/\varrho)$ for T-mode.

(5.23) $r^* = (\chi^{*-2} + \alpha_0^{*2} \omega_1^{*2})^{1/2}, \quad \psi' = \tan^{-1}(1/\alpha_0^* \omega),$

(5.24) $c_1^{(i)} = f(\lambda_i')/D\lambda_i' g^{*\prime}(\lambda_i'), \quad c_2^{(i)} = c_1^{(i)}[f'(\lambda_i') - Dc_1^{(i)} \lambda_i' g^{*\prime\prime}(\lambda_i')]/Dg^{*\prime}(\lambda_i'),$

(5.25)
$$d_1 = \sum_{i=1}^{2} (\lambda_i^* - \lambda_i'),$$
$$d_2 = 2D^{-1}[\lambda_1^* \lambda_2^* - \lambda_1' \lambda_2' + d_1 \{2(\lambda_1^* + \lambda_2^*) - \lambda_1' - \lambda_2' + 2d_1\}].$$

If we take $\alpha_0 = \alpha_0^* = \tau_0$, then all these results reduce to the corresponding results obtained in the previous section, for Eq. (5.5) reduces to Eq. (2.13) and b_1, b_2 tends to zero. Taking $\alpha_0 = \alpha_0^* = \tau_0 = 0$, all the results reduce to the corresponding ones in the context of coupled thermoelasticity. Thus we conclude that the results obtained in this section are more general ones.

6. Conclusion

The SH-wave gets decoupled from the rest of the motion in the case of both theories and propagates without dispersion or damping. The resulting motion is represented by three types of waves: QL, QT and T-mode, which are affected by the thermomechanical coupling and relaxation times. At sufficiently low frequencies the waves are found to be independent of relaxation times where as at high frequencies these are affected by the thermal relaxations, which supports the conclusion that "second sound" effects are short-lived. Though the waves in the context of the Green–Lindsay theory are subject to stronger modifications than those in the Dhaliwal–Sherief theory, in general, both theories lead to similar types of conclusions and results.

References

1. P. Chadwick and L.T.C. Seet, *Wave-propagation in transversely isotropic heat conducting elastic material,* Mathematica, **17**, 255–274, 1970.
2. P. Chadwick, *Basic properties of plane harmonic waves in a prestressed heat-conducting elastic material,* J. Thermal Stresses, **2**, 193–214, 1979.
3. H. W. Lord and Y. Shulman, *The generalized dynamical theory of thermoelasticity,* J. Mech. Phys. Solids, **15**, 299–2309, 1967.
4. R. S. Dhaliwal and H. H. Sherief, *Generalized thermoelasticity for anisotropic media,* Quart. Appl. Maths., **38**, 1–8, 1980.
5. H. Singh and J. N. Sharma, *Generalized thermoelastic waves in transversely isotropic media,* J. Acoust. Soc., Amer., **77**, 1046–1053, 1985.
6. J. N. Sharma and H. Singh, *Generalized thermoelastic waves in anisotropic media,* J. Acoust. Soc., Amer. [in press].
7. A. E. Green and K. A. Lindsay, *Thermoelasticity,* J. Elasticity, **2**, 1–7, 1972.
8. L. V. Alfors, *Complex analysis* (2nd Ed.), pp. 287–297, McGraw-Hill Kogakusha, Ltd., Tokyo 1966.

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