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Nonlinear microstructural continuous model of a laminated composite
I. Quasi-static phenomenological model

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A NONLINEAR continuous mathematical model of a discrete laminated composite is considered. From the assumption that the elastic energy is a function of macro-deformation and the curvature variations of a family of initially parallel material surfaces, equations of quasi-static equilibrium and constitutive relations at finite strains are derived. The set of equations derived describes a particular case of the general Cosserat medium and relates the moment stresses with the process of bending of the reinforcement layers. The boundary conditions written in the moment form have a clearly defined mechanical and geometrical meaning.

Rozpatrzono nieliniowy ciągły model matematyczny kompozytu o budowie dyskretno-warstwowej. Z założenia o zależności energii sprężystej od makroodkształcenia i od zmiany krzywizn wybranej rodziny materialnych, początkowo równoległych, powierzchni wyprowadzono równania równowagi quasi-statycznej i zależności konstytutywne przy skończonych odkształceniach. Otrzymany układ równań opisuje konkretną realizację ogólnego schematu materiału Cosseratów wiążącą naprężenia momentowe ze zginaniem warstw zbrojenia. Momentowe warunki brzegowe mają jasno określony sens mechaniczny i geometryczny.

Рассмотрена нелинейная непрерывная математическая модель композита с дискретно-слоистым строением. Из предположения о зависимости упругой энергии от макродеформации и от изменения кривизн избранного семейства материальных, вначале параллельных, поверхностей, выведены уравнения квазистатического равновесия и определяющие зависимости при конечных деформациях. Полученная система уравнений описывает конкретную реализацию общей схемы материала Коссера, связывающую моментные напряжения с изгибом слоев армирования. Моментные граничные условия имеют ясно определенный механический и геометрический смысл.

1. Introduction

THE MODERN state of development of engineering research makes it unnecessary to justify the necessity of constructing higher order models simulating the behaviour of composite materials in the cases when the characteristic dimensions of the problem (curvature radii, periodic disturbance wave length etc.) are of the same order of magnitude as the structural parameters of the material. Such an approach has led to interesting results in linear mechanics of composite materials consisting of two kinds of plane layers (matrix and reinforcement) [1–4]. Let us mention, in particular, the paper [5] in which the application of the microstructural theory of mixtures made it possible to reproduce with high accuracy the dynamic properties (dispersion curves) of real materials of a discrete-laminar structure by means of a continuous model. However, in spite of their considerable complexity such models cannot embrace all the problems of composite mechanics; from the point of view of practical applications it does not seem to be reasonable to construct such models for materials consisting of more than two components, and the problems of establishing proper boundary conditions are not always clear (in cases of periodic wave propagation it is not of primary importance, though). Almost untouched remain the problems of modelling the structural composites, both fibrous and laminated, of complex geometry, for instance those consisting of curved layers.

This author is unable to quote any references dealing with simple nonlinear structural models; the paper [11] could probably be mentioned here as an example although its authors do not propose any general model and confine their considerations to the application of the model of mixtures to a single specific nonlinear problem: propagation of a finite amplitude pulse along the layers under the plane strain conditions. Nonlinear models which would enable us to distinguish the materials with "thin" and "thick" reinforcement seem to lack.

In the papers [6, 7] the present author proposed a phenomenological model which made it possible for us to take into account the bending rigidity of reinforcement in fibre-reinforced composites. In view of the fact that rotary inertia was disregarded in that model, this author is inclined to consider the conclusions concerning the dispersive properties of the medium analyzed as unreliable. However, general scheme of the description and, for example, applications of the model to the analysis of internal stability of the material remains valid in the author's opinion. The proposed model should thus be treated as quasi-static since it takes into account the additional elastic energy following from the superposition of the local and global energies, but it disregards the similar effects connected with the kinetic energy.

In this paper a similar model will be proposed for laminated composites. Consider a continuous model which represents an ordered layered conglomerate of a simple elastic material (matrix) and a set of equidistant reinforcement layers kinematically connected with the matrix and modelled by the Cosserat surfaces (in the sense of [8]); our considerations will be confined to the model with constrained rotations, the field of directors being identified with the field of unit normal vectors. Such a model, similarly to the case of a fibrous material [6], enables immediate interpretation of the boundary conditions.

Our considerations will be limited to quasi-statical phenomena: dynamic terms will be disregarded, squares of velocities of their spatial derivatives and terms of the type $\mathbf{v} \cdot \nabla \mathbf{v}$ will be assumed to be small as compared with their first powers.

2. Geometrical foundations

Let us here recall some and derive other geometrical and kinematical relationships necessary for further considerations, and let us propose a certain tensorial measure used in the description of layer bending.

Let in a continuous medium be given a single-parameter continuous family of equidistant (along the normals) material surfaces $\{S(p)\}$ $(p \in \langle a, b\rangle \subset R)$. The field of unit vectors normal to the surfaces in their undeformed state $\overset{0}{\mathbf{N}}$ is then a well-defined smooth vector field $\overset{0}{\mathbf{N}}(x^i)$. Under such conditions, the trajectories of this field form a two-parameter family of straight lines what, however, has no major consequences for our further consider ations.

The deformation gradient tensor $\mathbf{F}$ written in material convected coordinates $\{\xi^i\}$ has the form

$$\mathbf{F} = \delta^i{}_j(\mathbf{g}_i \otimes \mathbf{G}^j), \tag{2.1}$$

Here $\mathbf{g}_i$, $\mathbf{G}^j$ denote the covariant base vectors in the current state and the contravariant base vectors in the undeformed state, respectively. The base vectors are thus connected by the following relations:

$$\begin{aligned}
\mathbf{g}_i &= \mathbf{F}\mathbf{G}_i = \mathbf{G}_i\mathbf{F}^T,\\
\mathbf{G}_i &= \overset{-1}{\mathbf{F}}\mathbf{g}_i = \mathbf{g}_i\overset{-1}{\mathbf{F}}{}^T,\\
\mathbf{g}^i &= \mathbf{G}^i\overset{-1}{\mathbf{F}} = \overset{-1}{\mathbf{F}}{}^T\mathbf{G}^i,\\
\mathbf{G}^i &= \mathbf{g}^i\mathbf{F} = \mathbf{F}^T\mathbf{g}^i.
\end{aligned} \tag{2.2}$$

It is easily seen that if a certain vector $\overset{0}{\mathbf{A}} = A^i\mathbf{G}_i$ is tangent at point (ξ^i) to the surface $S(p)$ passing through that point, then the vector $\mathbf{A}$

$$\mathbf{A} = A^i\mathbf{g}_i = \mathbf{F}\overset{0}{\mathbf{A}} \tag{2.3}$$

remains tangent to the same material surface $S(p)$ in the deformed body. This fact is particularly obvious if the material coordinate system $\{\eta_i\}$ is such that $\eta_3 = \text{const}$ over $S(p)$.

From the definition it follows that each vector $\overset{0}{\mathbf{A}}$ is normal to $\overset{0}{\mathbf{N}}$, thus

$$\overset{0}{\mathbf{N}}\cdot\overset{0}{\mathbf{A}} = \overset{0}{\mathbf{N}}\overset{-1}{\mathbf{F}}\mathbf{F}\overset{0}{\mathbf{A}} = (\overset{0}{\mathbf{N}}\overset{-1}{\mathbf{F}})\cdot\mathbf{A} = 0. \tag{2.4}$$

It follows that if the vector $\overset{0}{\mathbf{N}} = N_i\mathbf{G}^i$ is orthogonal to $\overset{0}{\mathbf{A}}$, the vector $\mathbf{N} = \overset{0}{\mathbf{N}}\overset{-1}{\mathbf{F}} = N_i\mathbf{g}^i$ must be orthogonal to $\mathbf{A}$; but the same is true for each vector tangent to $S(p)$, so that the vector $\mathbf{N}$ must be normal to $S(p)$, and the vector

$$\mathbf{n} = \frac{\mathbf{N}}{\sqrt{\mathbf{N}\cdot\mathbf{N}}} = \frac{N_i}{\sqrt{N_kN_lg^{kl}}}\,\mathbf{g}^i \tag{2.5}$$

is a unit vector normal to $S(p)$ in the current state; the field $\mathbf{n} = \mathbf{n}(\xi^i)$ represents a field of unit vectors normal to the family of surfaces $\{S_{(i)}\}$ in the current state. An analogous vector field

$$\overset{0}{\mathbf{n}}(\xi^i) = n_i\mathbf{G}^i = \mathbf{n}\mathbf{F} \tag{2.6}$$

is a field of normal (but not necessarily unit) vectors in the undeformed body.

Using the known formula

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \tag{2.7}$$

in which the dot denotes the material derivative, and

$$\mathbf{L} \equiv \operatorname{grad}\mathbf{v} = v^i_{,j}(\mathbf{g}_i \otimes \mathbf{g}^j), \tag{2.8}$$

where $\mathbf{v}$ is the velocity vector and the comma denotes the covariant derivative in the current state, the relations (2.2) yield the formula

$$\dot{\mathbf{g}}_k = \dot{\mathbf{F}}\mathbf{G}_k = \mathbf{L}\mathbf{F}\mathbf{G}_k = \mathbf{L}\mathbf{g}_k = v^i_{,k}\mathbf{g}_i. \tag{2.9}$$

From the relation $\overline{\mathbf{F}\overset{-1}{\dot{\mathbf{F}}}} = \mathbf{0}$ we obtain

$$\overset{\dot{-1}}{\mathbf{F}} = -\overset{-1}{\mathbf{F}}\mathbf{L}, \tag{2.10}$$

whence it follows

$$\dot{\mathbf{g}}^k = \mathbf{G}^k \overset{\dot{-1}}{\mathbf{F}} = -\mathbf{G}\overset{-1}{\mathbf{F}}\mathbf{L} = -\mathbf{g}^k\mathbf{L} = -v^k_{,j}g^j \tag{2.11}$$

and hence

$$\overline{\dot{g}^{ij}} = \overline{\dot{\mathbf{g}^i \cdot \mathbf{g}^j}} = -(v_{l,k}+v_{k,l})g^{ki}g^{lj} \equiv -2D^{ij}. \tag{2.12}$$

The relations (2.9), (2.11) and (2.12) yield directly the formulae

$$\dot{\mathbf{n}} = \overline{\left(\frac{\mathbf{N}\overset{0\ -1}{\mathbf{F}}}{\sqrt{M_i N_j g^{ij}}}\right)}^{\cdot} = -[\mathbf{n}\mathbf{L}-(\mathbf{n}\mathbf{L}\mathbf{n})\mathbf{n}] = -(n_i v^i_{,j} - n^i n^k v_{i,k} n_j)\mathbf{g}^j, \tag{2.13}$$

$$\overline{\dot{n}_k} = \overline{\left(\frac{N_k}{\sqrt{N_i N_j g^{ij}}}\right)}^{\cdot} = n_k(n_i v^i_{,j} n^j). \tag{2.14}$$

Bars over the symbols differentiated with respect to time are used to stress the fact that the derivatives should be referred not to the vectors or tensors but to their components in the convective material coordinate system.

Let us observe that such magnitudes represent the components of the Lie derivatives taken with respect to the velocity field, so that if the vector $\mathbf{s}$ is an objective magnitude, the vector $\bar{\dot{s}}_i g^i$ is also objective in spite of the fact that $\bar{\dot{\mathbf{s}}} = \bar{\dot{s}}_i g^i + s_i \bar{\dot{g}}^i$ is not objective in general; the same holds true with respect to tensors of arbitrary rank.

In further considerations in which the elastic energy will be related to bending of the surface $S(p)$, the assumption of a convenient and physically grounded measure of bending becomes of primary importance. In the shell theories such a measure is usually assumed to be the variation of components of the second quadratic form in the convective coordinate system; this is justified by the fact that the moment theory of shells is rarely applied to the cases when large deformations appear in the tangent plane. In such cases the bending moments are usually disregarded and the membrane theory is used, what makes it possible to avoid the paradox that in absence of bending, e.g. in the case of an inflated thin surface, the second metric form is varied in spite of the that fact the strains do not change sign across the thickness, what is typical for bending. Let us now define the bending measure which will be insensitive to affine deformation of the body.

Construct the following tensor:

$$\boldsymbol{\gamma} \equiv -[\nabla\mathbf{n} - \overset{-1}{\mathbf{F}}(\overset{0\,0}{\nabla\mathbf{n}})\overset{-1}{\mathbf{F}}], \tag{2.15}$$

where $\overset{0}{\nabla}$ denotes the gradient taken in the undeformed state,

$$\overset{0\,0}{\nabla\mathbf{n}} = \left(\frac{\partial n_i}{\partial \xi^j} - \overset{0}{\Gamma}{}^k_{ij} n_k\right)(\mathbf{G}^i \otimes \mathbf{G}^j). \tag{2.16}$$

Let us recall that the vector components of $\overset{0}{\mathbf{n}}$ in the base $\{\mathbf{G}^i\}$ are the same as the components of the vector $\mathbf{n}$ in the base $\{\mathbf{g}^i\}$. A similar procedure yields the tensor $\nabla\mathbf{n}$; moreover, the formulae (2.1) are used to derive the relation

$$\boldsymbol{\gamma} = \mathbf{K}\mathbf{n}, \tag{2.17}$$

where (in convective cordinates)

$$\mathbf{K} = (\Gamma^i_{jk} - \overset{0}{\Gamma}{}^i_{jk})\,(\mathbf{g}^i \otimes \mathbf{g}^k \otimes \mathbf{g}_i). \tag{2.18}$$

Let us define the tensor $\boldsymbol{\beta}$ as follows:

$$\boldsymbol{\beta} = (\mathbf{1} - \mathbf{n}\otimes\mathbf{n})\boldsymbol{\gamma}(\mathbf{1} - \mathbf{n}\otimes\mathbf{n}); \tag{2.19}$$

this means that

$$\beta^{kl} = (g^{ki} - n^k n^i)(\overset{0}{\Gamma}{}^m_{ij} - \Gamma^m_{ij})n_m(g^{jl} - n^j n^l). \tag{2.20}$$

The tensor $\mathbf{K}$ and, hence, also the tensor $\boldsymbol{\beta}$ vanish at affine deformation; the material coordinate system may be selected so as to satisfy the condition $\overset{0}{\Gamma}{}^m_{ij} = 0$ and it is clear that in each affine deformation this property will be preserved. On the other hand, if the family $\{S(p)\}$ in the undeformed state is a family of parallel planes and the assumed coordinate system is such that $\mathbf{n} = \mathbf{g}_3 = \mathbf{g}^3$, then

$$\overset{0}{\mathbf{n}} = \mathbf{n}\mathbf{F} = \mathbf{g}^3\mathbf{F} = \mathbf{G}^3, \tag{2.21}$$

$$\Gamma^3_{ij} = -\frac{\partial \mathbf{G}^3}{\partial \xi^i}\cdot\mathbf{G}_j. \tag{2.22}$$

The vector $\mathbf{G}^3$ is perpendicular to the planes $\{S(p)\}$ so that for $i, j = 1, 2$ $\overset{0}{\Gamma}{}^3_{ij} = 0$, but $n_k = 1$ for $k = 3$ and $n_k = 0$ for $k \neq 3$, and thus

$$\overset{0}{\Gamma}{}^k_{ij} n_k = \overset{0}{\Gamma}{}^3_{ij} = 0 \quad \text{for} \quad i, j \neq 3. \tag{2.23}$$

Expressions of the type of Γ^k_{3j} do not appear in the expression for β^{kl} since $n^3 = 1$, $g^{33} = 1$ and $g^{3i} = 0$ for $i \neq 3$. Hence for the surfaces $S(p)$ which were plane in the undeformed configuration

$$\boldsymbol{\beta} = \Gamma^3_{\alpha\beta}(g^\alpha \otimes g^\beta) \equiv b_{\alpha\beta}(g^\alpha \otimes g^\beta), \tag{2.24}$$

where $\alpha, \beta = 1, 2$, and the vectors $\mathbf{g}^\alpha$, $\mathbf{g}^\beta$ are tangent to the surface $S(p)$, while $b_{\alpha\beta}$ are coefficients of the second quadratic form. In this case the bending measure coincides with that used in the plate and shell theories.

Let us now pass to the determination of the time derivative of this measure. First of all, observe that, due to the commutativity of differentiation with respect to time and to convective coordinates, we have

$$\dot{\overline{\Gamma^i_{jk}}} = \dot{\overline{\frac{\partial \mathbf{g}_j}{\partial \xi^k}\cdot\mathbf{g}^i}} = \frac{\partial \dot{\mathbf{g}}^j}{\partial \xi^k}\cdot\mathbf{g}^i + \frac{\partial \mathbf{g}^j}{\partial \xi^k}\cdot\dot{\mathbf{g}}^i. \tag{2.25}$$

Application of the formulae (2.9) and (2.11) leads to the simple result

$$\dot{\overline{\Gamma^i_{jk}}} = v^i{}_{,jk}. \tag{2.26}$$

Using the relation

$$\dot{n}^i = \overline{n_k g^{ki}}^{\bullet} = \dot{n}_k g^{ki} + n_k \dot{g}^{ki} = \dot{n}_k g^{ki} - 2n_k D^{ki} \tag{2.27}$$

and Eqs. (2.12), (2.14) and (2.26), we obtain, after lengthy but elementary transformations, the formula

$$\overline{\beta^{pq}}^{\bullet} = -2D_{ij}\left[(g^{jp} - n^j n^p)\beta^{iq} + (g^{iq} - n^i n^q)\beta^{jp} - \frac{1}{2} n^i n^j \beta^{pq}\right] + v^i_{,kl} n_i (g^{kp} - n^k n^p)(g^{lq} - n^l n^q). \tag{2.28}$$

This result will be used in the following section.

3. Elastic energy

In this section a formula for elastic energy will be proposed expressing the energy in terms of the assumed state variables (strain measures) in accordance with the symmetry of the system.

In a purely mechanical theory corresponding to the description of isothermal or adiabatic processes, the energy of an elastic body is identified with the Gibbs or Helmholtz free energy, respectively. Hence it represents a function of the state parameters and of certain parametric tensors expressing the material symmetries. According to a recently proved theorem by J. RYCHLEWSKI [9] (which is a generalization of an earlier result by LIU [10]) the elastic energy will be sought in the form of an isotropic function of the parametric tensors and the strain measures (state variables). Our considerations will be confined to composites consisting of isotropic components. According to the previously assumed symmetry of the layers, the local symmetry of the composite will be described completely by prescribing the vector $\overset{0}{\mathbf{N}}$ and tensor $\overset{0}{\mathbf{b}}$ which are the coefficients of the second quadratic form in the undeformed state. Observe that not only the direction but also the sense of the vector $\overset{0}{\mathbf{N}}$ is important since, by changing the sense of vector $\overset{0}{\mathbf{N}}$, the sign of coefficients of the second quadratic form is also changed.

The symmetry group itself may also contain or not contain such elements as the symmetry with respect to the planes tangent to $S(p)$, or the rotation by angle $\pi/2$ about the axes lying in that plane since the cases a and b in Fig. 1 or Fig. 2 can hardly be considered as referring to the same material.

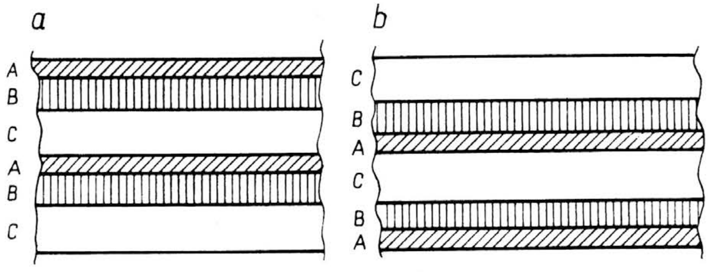

FIG. 1. Three-component material asymmetric with respect to its mirror reflection.

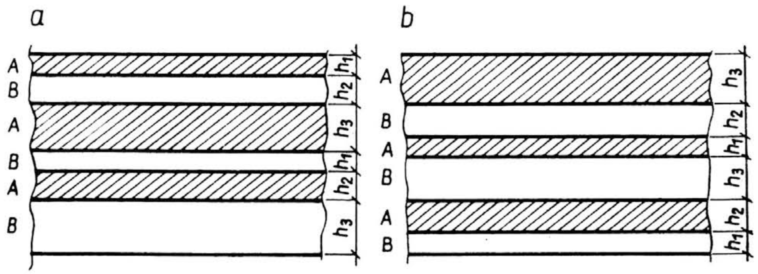

FIG. 2. Two-component material asymmetric with respect to its mirror reflection.

Thus, even in the case of the initially plane layers, we have to deal with a polar material in the sense of [8]. The state variables may be assumed to consist of an arbitrary objective strain measure and the tensor $\boldsymbol{\beta}$ (which also depends on the sense of $\overset{0}{\mathbf{N}}$). The tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ will be assumed as the strain measure.

In the convective, material coordinate system assumed we have

$$\mathbf{B} = G^{ij}(\mathbf{g}_i \otimes \mathbf{g}_j) \tag{3.1}$$

what means that in the convective base the representation of the tensor $\mathbf{B}$ is identical with the contravariant representation of the metric tensor of the initial Lagrange system. The mass density of the elastic energy may now be represented in the form of an isotropic (i.e. containing no parametric matrices) function of the matrices

$$w = w(N_i, \overset{0}{b}{}^{ij}, G^{ij}, \beta^{ij}, g_{ij}). \tag{3.2}$$

Out of all the arguments of the function w, only two: β^{ij} and g_{ij} are time-dependent; moreover, $\dot{\overline{g_{ij}}} = 2D_{ij}$ and the material derivative of β^{ij} is defined by Eq. (2.28). We then obtain

$$\begin{aligned} \dot{w} &= \frac{\partial w}{\partial g_{ij}} \dot{\overline{g_{ij}}} + \frac{\partial w}{\partial \beta^{ij}} \dot{\overline{\beta^{ij}}} \\ &= 2\left\{ \frac{\partial w}{\partial g_{ij}} - \frac{\partial w}{\partial \beta^{mn}} \left[(g^{jm} - n^j n^m)\beta^{in} + (g^{in} - n^i n^n)\beta^{jm} - \frac{1}{2} n^i n^j \beta^{mn} \right] \right\} D_{ij} \\ &\quad + \frac{\partial w}{\partial \beta^{mn}} (g^{km} - n^k n^m)(g^{ln} - n^l n^n) n^i v_{i,kl}. \end{aligned} \tag{3.3}$$

The representation (3.2) of the energy density function, although satisfactory from the formal point of view, may prove to be inconvenient in practical applications; it is sometimes useful to increase the number of variables by introducing the dependent variables, for instance g^{ij} or the tensors ε^{ijk}, ε_{ijk}. It does not change the character of the first term of the expression (3.2) representing a contraction of D_{ij} with a certain tensorial expression, since from the relation (2.12) we obtain

$$\dot{\overline{g_{ij}}} = -2D^{ij}$$

and

$$\begin{aligned} \varepsilon_{ijk} &= \varepsilon\langle ijk \rangle \sqrt{g}, \\ \varepsilon^{ijk} &= \varepsilon\langle ijk \rangle / \sqrt{g}. \end{aligned} \tag{3.4}$$

Here $\varepsilon\langle ijk\rangle$ is the permutation symbol and g — the determinant of the matrix g_{ij} i.e. $g = |g_{ij}|$. It follows that

$$\overline{\varepsilon_{ijk}}^{\,\bullet} = \varepsilon_{ijk} D^m{}_m,$$
$$\overline{\varepsilon^{ijk}}^{\,\bullet} = -\varepsilon^{ijk} D^m{}_m. \tag{3.5}$$

It is not clear, however, what is the minimal base of polynomial invariants which makes it possible to express w as a function of invariants only.

The situation is not typical in the sense that both tensors $\overset{0}{\mathbf{b}}$ and $\boldsymbol{\beta}$ are defined as a plane, i.e. at least one of their eigenvalues must vanish; moreover, their position with respect to the vectors $\overset{0}{\mathbf{N}}$ and $\mathbf{N}$ is not arbitrary since

$$N_i \overset{0}{b}{}^{ij} = 0,$$
$$N_i \beta^{ij} = 0. \tag{3.6}$$

Finally, their signs depend on the sense of $\overset{0}{\mathbf{N}}$.

The question will not be answered in this paper; it should be mentioned that, in addition, precise knowledge of the minimal base is of no great value for practical purposes.

4. Equations of quasi-static equilibrium and constitutive relations

Using the results of Sects. 2 and 3 and the general conservation principles, let us derive the complete set of quasi-static equations of the medium considered.

Under the conditions of a quasi-static process, let us postulate the following form of the energy balance to be satisfied for each material region Ω with a picewise smooth boundary $\partial\Omega$:

$$\overline{\int_{\Omega} \varrho w dV}^{\,\bullet} = \int_{\partial\Omega} \mathbf{t}\cdot\mathbf{v} dS + \int_{\partial\Omega} \mathbf{m}\cdot\boldsymbol{\omega} dS + \int_{\Omega} \varrho\mathbf{f}\cdot\mathbf{v} dV, \tag{4.1}$$

where $\mathbf{f}$ is the body force density, $\mathbf{t}$ is the stress vector, $\mathbf{m}$ the surface moment vector, and $\boldsymbol{\omega}$ denotes the rotation rate vector of $\mathbf{n}$

$$\boldsymbol{\omega} = \mathbf{n}\times\dot{\mathbf{n}}. \tag{4.2}$$

Following the Cauchy postulate which implies the existence of such a tensor $\mathbf{T}$ that

$$\mathbf{t} = \boldsymbol{\nu}\mathbf{T}, \tag{4.3}$$

where $\boldsymbol{\nu}$ is the outer unit normal vector to the surface, let us postulate the existence of such a tensor $\mathbf{M}$ that

$$\mathbf{m} = \boldsymbol{\nu}\mathbf{M}. \tag{4.4}$$

On substituting the expressions (4.3) and (4.2) into the relation (4.1), changing the surface integrals into volume integrals and taking into account that the relation (4.1) holds true for each material region (i.e. the integration signs may be disregarded) we obtain the local form of the energy balance law

$$\varrho\dot{w} = \operatorname{div}(\mathbf{T}\mathbf{v}) + \operatorname{div}[\mathbf{M}(\mathbf{n}\times\dot{\mathbf{n}})] + \varrho\mathbf{f}\cdot\mathbf{v}. \tag{4.5}$$

Use has been made of the fact that Ω is a material region, and hence

$$\overline{\int_\Omega \varrho w dV}^{\bullet} = \int_\Omega \varrho\dot{w} dV. \tag{4.6}$$

Assuming for $\dot{\overline{\beta^{pq}}}$ and $\dot{\mathbf{n}}$ the respective values determined by Eqs. (2.28) and (2.13), we obtain, after differentiation and certain rearrangements, the following form of Eq. (4.5)

$$\begin{aligned}\{T^{ji}{}_{,j}+\varrho f^i\}v_i - \Big\{2\varrho\frac{\partial w}{\partial g_{ik}} - \Big[(g^{km}-n^kn^m)\beta^{in}+(g^{im}-n^in^m)\beta^{kn} \\ -\frac{1}{2}n^in^k\beta^{mn}\Big]2\varrho\frac{\partial w}{\partial\beta^{mn}} + T^{(ki)} + \frac{1}{2}[M^{ls}g_{sj}n_r(\varepsilon^{rjk}n^i+\varepsilon^{rji}n^k)]_{,l}\Big\}D_{ik} \\ +\Big\{T^{\langle ki\rangle}+\frac{1}{2}[M^{ls}g_{sj}n_r(\varepsilon^{rjk}n^i-\varepsilon^{rji}n^k)]_{,l}\Big\}W_{ik} \\ +\Big\{\frac{1}{2}(M^{ls}\varepsilon^{jkr}+M^{ks}\varepsilon^{jlr})g_{sj}n_r-(g^{km}-n^kn^m)(g^{ln}-n^ln^n)\varrho\frac{\partial w}{\partial\beta^{mn}}\Big\}n^iv_{i,kl} = 0.\end{aligned} \tag{4.7}$$

Here $T^{(ki)}$ and $T^{\langle ki\rangle}$ are the symmetric and skew-symmetric parts of the stress tensor, respectively, and W_{ik} is the skew-symmetric part of the velocity gradient.

The values of the fields v_i, D_{ik}, W_{ik} and $n^iv_{i,kl}$ are mutually independent at the point, what implies that each of the expressions in the braces of (4.7) must identically vanish.

Introducing the notation

$$\Pi^{kl} = \varrho(g^{km}+n^kn^m)\frac{\partial w}{\partial\beta^{mn}}(g^{ln}-n^ln^n), \tag{4.8}$$

we may write the condition of vanishing of the last expression in the braces of Eq. (4.7) in the form

$$\frac{1}{2}(M^{ls}\varepsilon^{jkr}-M^{ks}\varepsilon^{jlr})g_{sj}n_r = \Pi^{kl}. \tag{4.9}$$

Multiplication of Eq. (4.9) by n_kp_l and then by p_kn_l (where p_k — components of an arbitrary vector normal to $\mathbf{n}$) yields for each vector $\mathbf{q}$ normal to $\mathbf{n}$ the relation

$$n_iM^{ik}q_k = q_iM^{ik}n_k = 0. \tag{4.10}$$

The tensor $\mathbf{M}$ must then have the form

$$M^{ij} = Qn^in^j + R(g^{ij}-n^in^j) + \hat{M}^{ij}, \tag{4.11}$$

where

$$\begin{aligned} n_i\hat{M}^{ij} &= \hat{M}^{ij}n_j = 0,\\ (g_{ij}-n_in_j)\hat{M}^{ij} &= 0.\end{aligned} \tag{4.12}$$

The functions Q and R do not contribute to the expression (4.9) and remain indefinite for the time being. It will be shown that they may be assumed to vanish.

It is easily verified by substitution that $\mathbf{M}$ has then the form

$$M^{ls} = \hat{M}^{ls} = -\Pi^{lm}n^r\varepsilon_{rmp}g^{ps}. \tag{4.13}$$

Multiplying the third expression in braces by $\varepsilon_{pik}\varepsilon^{prq}$ and setting it equal to zero, we obtain

$$(4.14)\qquad T^{\langle rq\rangle} = \frac{1}{2}\,[M^{ls}(n_s n_p - g_{sp})]_{,l}\,\varepsilon^{pqr}.$$

If $Q = 0$ or, at least, $n^l Q_{,l} = -Qn^l_{,l}$, then Eq. (4.14) may be rewritten in the form

$$(4.15)\qquad T^{\langle rq\rangle} = \frac{1}{2}\,M^l{}_{p,l}\,\varepsilon^{pqr}.$$

Vanishing of the first term in braces leading to the equation of quasi-static equilibrium

$$(4.16)\qquad T^{ji}{}_{,j} + \varrho f^i = 0,$$

makes the expression (4.15) satisfy the integral equation of quasi-static moment equilibrium

$$(4.17)\qquad \int_{\partial\Omega} (\mathbf{r}\times\mathbf{t})\,dS + \int_{\partial\Omega} \mathbf{m}\,dS + \int_{\Omega} (\mathbf{r}\times\varrho\mathbf{f})\,dV = 0.$$

Let us finally pass to the second expression in braces. Observe that

$$(4.18)\qquad \beta^{ij} = \beta^{ik}(g^{lj} - n^l n^j)g_{kl},$$

and so, equating this expression to zero we obtain

$$(4.19)\qquad T^{(ki)} = 2\left[\varrho\frac{\partial w}{\partial g_{ik}} - \Pi^i{}_m\beta^{mk} - \Pi^k{}_m\beta^{mi} + \frac{1}{2}\,n^i n^k \beta^{mn}\Pi_{mn}\right] - \frac{1}{2}\,[M^{ls}g_{sj}n_r(\varepsilon^{rjk}n^i + \varepsilon^{rji}n^k)]_{,l}.$$

Summing up the left and right-hand sides of Eqs. (4.19) and (4.14) we are led to the relation

$$(4.20)\qquad T^{ki} = 2\left(\varrho\frac{\partial w}{\partial g_{ik}} - \Pi^i{}_m\beta^{mk} - \Pi^k{}_m\beta^{mi} + \frac{1}{2}\,n^k n^i \beta^{mn}\Pi_{mn}\right) - (M^{ls}g_{sj}n_r\varepsilon^{rjk}n^i)_{,l}.$$

It is easily seen that the function Q (4.11) does not contribute to the stress tensor or to the energy balance, while the function R in the function (4.11) does not contribute to the equilibrium equation since

$$(4.21)\qquad [R(g^{ls} - n^l n^s)g_{sj}n_r\varepsilon^{rjk}n^i]_{,lk} = [Rn_r\varepsilon^{rlk}n^i]_{,lk} = 0.$$

Let us now consider the boundary conditions. As long as $\mathbf{M}$ and $\mathbf{m}$ are interpreted as the moment tensor and moment vector, respectively (as suggested by Eq. (4.17)), the term containing Q will describe the component of the moment perpendicular to the surface $S(p)$; this means that if it had to have a physical meaning, the surface would have to be intrinsically (in the sense of torsion within the tangent plane) a Cosserat surface what, in the case of composites consisting of layers of simple materials, has no physical justification. The term containing R would indicate, for example in the case of cylindrical bending, the presence of a bending moment component perpendicular to the generators of the cylinder; the latter result has no direct physical interpretation, either. Hence we should assume $Q = 0$ and $R = 0$ and then, substitution of Eq. (4.13) into Eqs. (4.15) and (4.20) yields the following set of equations:

$$(4.22)\qquad T^{ki}{}_{,k} + \varrho f^i = 0,$$

(4.22) [cont.]
$$T^{ki} = 2\left(\varrho \frac{\partial w}{\partial g_{ik}} - \Pi^i_m \beta^{mk} - \Pi^k_m \beta^{mi} + \frac{1}{2} n^k n^i \beta^{mn} \Pi_{mn}\right) - (\Pi^{li} n^k)_{,l},$$
$$M^{ls} = -\Pi^{lm} n^r \varepsilon_{rmp} g^{ps}.$$

Here

$$\Pi^{ij} = (g^{ik} - n^i n^k) \frac{\partial w}{\partial \beta_{kl}} (g^{lj} - n^l n^j).$$

The boundary conditions at the boundary characterized by the unit outer normal $\mathbf{v}$ have the form

$$(4.23) \qquad t^j = \nu_i T^{ij}, \qquad m^j = \nu_i M^{ij}.$$

Discussion of the range of applicability of the model and effective evaluation of the material constants of the linearized theory will be dealt with in another paper. Finally, let us observe that Eqs. (4.22) treated as the static equilibrium equations describe the class of materials wider than the class of hyper-elastic materials for which they constitute the quasi-static equilibrium equations.

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