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On the accuracy of classical linear shell theory

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DANIELSON'S [2] error bound for solutions of classical linear shell theory is shown to be inadequate if the boundary conditions involve the rotations of normals to the midsurface. This is corrected in this report.

W pracy pokazano, że otrzymane przez DANIELSONA [2] oszacowanie błędu rozwiązań klasycznej liniowej teorii powłok jest nieadekwatne, jeśli warunki brzegowe zawierają obroty normalnych do powierzchni środkowej. W niniejszej pracy zostało to skorygowane.

В работе показано, что полученная Данельсоном [2] оценка ошибки решений классической линейной теории оболочек является неадекватной, если граничные условия содержат вращения нормалей к срединной поверхности. В настоящей работе это исправлено.

1. Introduction

KOITER [1] has found that the relative mean-square error of solutions of the classical linear theory of elastic shells is of order $\sqrt{\varepsilon} = \sqrt{h/R+h/L}$, h being the shell thickness, R a typical radius of curvature of the middle surface, and L a characteristic wave length of the deformation pattern. To this end he constructed two three-dimensional solutions, a statically admissible solution and a kinematically admissible solution from the supposedly known two-dimensional shell theory solution. They were then compared with the exact elasticity theory solution with the help of energy inequalities. Later on DANIELSON [2] has obtained a more elaborate three-dimensional displacement field, better describing the true distribution of the transverse shearing stresses over the thickness, and found that the error is of order ε. This refined result, however, is verified to be of limited validity, which fact was not noted in [2]. Namely, the argument of [1] and [2] assumes that the boundary conditions of the three-dimensional problem are "regular" (in the KOITER's [1] terminology), that is, on the respective parts of the bounding surface the actual displacements $\mathbf{u}$ coincide with the kinematically admissible ones $\hat{\mathbf{u}}$ and the actual stresses conform to the statically admissible ones. This implies, in particular, that along the edge of the midsurface the rotations of normals, obtained by standard methods from $\mathbf{u}$ and $\hat{\mathbf{u}}$, should also coincide. In addition, their expressions in terms of the displacements of the midsurface should have the form characteristic of classical shell theory. Unfortunately, the edge rotation resulting from Danielson's $\hat{\mathbf{u}}$-field is not that of classical theory, and the argument in [2] becomes defective when the boundary conditions restrict the edge rotation. More precisely, the error bound in [2] is formally correct, but only for a non-classical shell theory using a different formula for the edge rotation. Since this formula turns out to be more complex than its classical counterpart, the use of that non-classical theory would aggravate computational difficulties.

This report presents a modified $\hat{\mathbf{u}}$-field, which agrees with the classical expression for the edge rotation and leads to the error of order ε. At the same time, however, this field gives a complicated, non-classical expression for the lateral deflection of the middle surface. In this connection, our result seems to complete that of Danielson: the former proves that, within the error of order ε, classical theory is applicable to shell problems whose boundary conditions do not restrict the lateral deflection, while the latter proves the same for boundary conditions which do not restrict the rotations of normals. When both quantities are prescribed (e.g. clamped edge) ε-error solutions require the use of non-classical boundary conditions, either for the deflection (our case) or for the rotations (Danielson's case).

It is pertinent to note that the term "classical" is meant here to denote all consistent variants of the linear shell theory based on the Kirchhoff–Love hypothesis, and equivalent in the sense defined by KOITER [3].

2. Formulation of the problem

Our analysis will be based on KOITER's [1] version of shell theory. The shell space is parametrized by a normal coordinate system $(x^i) = (x^\alpha, x^3 = z)$, $(i = 1, 2, 3; \alpha = 1, 2)$ where x^α are midsurface coordinates and z denotes the distance from the midsurface. The strain-displacement relations are

$$\gamma_{\alpha\beta} = \frac{1}{2}(w_{\alpha|\beta}+w_{\beta|\alpha})-b_{\alpha\beta}w_3, \tag{2.1}$$
$$\varrho_{\alpha\beta} = w_{3|\alpha\beta}-b^\lambda_\alpha b_{\lambda\beta}w_3+b^\lambda_\alpha w_{\lambda|\beta}+b^\lambda_\beta w_{\lambda|\alpha}+b^\lambda_{\alpha|\beta}w_\lambda,$$

where $w_i(x^\lambda)$ is the displacement vector, $\gamma_{\alpha\beta}(x^\lambda)$ and $\varrho_{\alpha\beta}(x^\lambda)$ are the symmetric tensors of the membrane and bending strains, $b_{\alpha\beta}(x^\lambda)$ is the curvature tensor, and a vertical stroke denotes surface covariant differentiation based on the natural metric of the middle surface. We also introduce two auxiliary geometric quantities

$$\eta_{\alpha\beta} = w_{\alpha|\beta}-b_{\alpha\beta}w_3, \qquad \Phi_\alpha = w_{3,\alpha}+b^\lambda_\alpha w_\lambda, \tag{2.2}$$

where $\Phi_\alpha(x^\lambda)$ describes the rotations of normals to the midsurface under the Kirchhoff–Love hypothesis, whereas $\gamma_{\alpha\beta}$ is verified to be the symmetric part of $\eta_{\alpha\beta}(x^\lambda)$; commas denote the partial differentiation with respect to x^i.

The quantities (2.1) and (2.2) are related by

$$\varrho_{\alpha\beta} = \frac{1}{2}(\Phi_{\alpha|\beta}+\Phi_{\beta|\alpha}+b^\lambda_\alpha\eta_{\lambda\beta}+b^\lambda_\beta\eta_{\lambda\alpha}), \tag{2.3}$$
$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[\varrho_{\beta\lambda|\mu}-b^\varkappa_\lambda(\gamma_{\varkappa\beta|\mu}+\gamma_{\varkappa\mu|\beta}-\gamma_{\beta\mu|\varkappa})] = 0,$$

where $(2.3)_2$ express two compatibility equations for the deformed middle surface (see [4], Eq. (5.10)); $\varepsilon^{\alpha\beta}(x^\lambda)$ are components of the permutation (Ricci) tensor ascribed to the midsurface.

The material is homogeneous, linearly elastic and isotropic so that the constitutive equations read

$$n_{\alpha\beta} = \frac{Eh}{1-\nu^2}[(1-\nu)\gamma_{\alpha\beta}+\nu a_{\alpha\beta}\gamma^{\lambda}_{\lambda}],$$
(2.4)
$$m_{\alpha\beta} = \frac{Eh^3}{12(1-\nu^2)}[(1-\nu)\varrho_{\alpha\beta}+\nu a_{\alpha\beta}\varrho^{\lambda}_{\lambda}],$$

where $n_{\alpha\beta}(x^\lambda)$ and $m_{\alpha\beta}(x^\lambda)$ are membrane forces and moments, E is Young's modulus, ν is Poisson's ratio, h denotes the constant thickness, and $a_{\alpha\beta}(x^\lambda)$ are components of the midsurface metric tensor.

Given a shell theory solution, we shall concisely characterize its properties by means of certain numbers, defined by

$$|\eta_{\alpha\beta}, \Phi_\alpha, h(\Phi_{\alpha|\beta}+\Phi_{\beta|\alpha}), \gamma_{\alpha\beta}, h\varrho_{\alpha\beta}| \leqslant \gamma \ll 1,$$
$$|n_{\alpha\beta}, m_{\alpha\beta}/h| \leqslant n,$$
(2.5)
$$|n_{\alpha\beta|\lambda}, m_{\alpha\beta|\lambda}/h| \leqslant n/L, \qquad |n_{\alpha\beta|\lambda\eta}, m_{\alpha\beta|\lambda\eta}/h| \leqslant n/L^2,$$
$$|b_{\alpha\beta}| \leqslant 1/R, \qquad |b_{\alpha\beta|\lambda}| \leqslant 1/R^2,$$
$$\varepsilon = h/R+h^2/L^2,$$

where the coordinates are assumed to have the dimension of length. Here γ represents an upper bound for the displacement gradients, rotations of normals and the strains, n is the magnitude of the internal forces, L is a characteristic wave length of the deformation pattern, R represents a typical radius of curvature of the midsurface, and ε is a small parameter.

The following relation will be of later use:

$$(1+\nu)m^{\beta}_{\alpha|\beta} = m^{\beta}_{\beta,\alpha}+O\left(\frac{nh^2}{RL}\right). \tag{2.6}$$

To prove this, we differentiate $(2.4)_2$, rearrange terms by means of the well-known equality $\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu} = a^{\alpha\lambda}a^{\beta\mu}-a^{\alpha\mu}a^{\beta\lambda}$, and multiply both sides of the resulting equations by $(1+\nu)$, thus finding

$$(1+\nu)m^{\beta}_{\alpha|\beta} = \frac{Eh^3}{12(1-\nu)}\varrho^{\beta}_{\beta,\alpha}+\frac{Eh^3}{12}\varepsilon_{\alpha\beta}\varepsilon^{\lambda\mu}\varrho^{\beta}_{\lambda|\mu}. \tag{2.7}$$

By $(2.4)_2$, $(2.3)_2$, the equation (2.7) is verified to give (2.6) if we remember that from $(2.4)_1$ there follows $n = O(Eh\gamma)$.

For future use we record the simplified strain-displacement relations of threedimensional elasticity (see [2])

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-b_{\alpha\beta}u_3+O(\varepsilon\gamma),$$
(2.8)
$$e_{\alpha 3} = \frac{1}{2}(u_{\alpha,3}+u_{3,\alpha}+b^{\lambda}_{\alpha}u_\lambda)+O(\varepsilon\gamma),$$
$$e_{33} = u_{3,3},$$

where $e_{ij}(x^k)$ are the strains, and $u_i(x^k)$ are the displacement components. The $O()$ — terms result from the replacement of the spatial covariant differentiation at an arbitrary point away from the midsurface by the midsurface covariant differentiation; it has been also assumed that the magnitude u of the displacements is small, namely $u/R = O(\gamma)$.

The spatial stress-strain relations may be written in the form

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}[(1-\nu)e_{\alpha\beta}+\nu a_{\alpha\beta}e^{\lambda}_{\lambda}]+\frac{\nu}{1-\nu}a_{\alpha\beta}\sigma_{33}+O(\varepsilon\sigma_{\alpha\beta}),$$

(2.9)
$$\sigma_{\alpha 3} = \frac{E}{1+\nu}e_{\alpha 3},$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{33}+\nu e^{\lambda}_{\lambda}],$$

where the error term in $(2.9)_1$ is due to the use of the midsurface metric tensor in place of the metric tensor ascribed to a parallel surface.

The previously introduced shell theory quantities are defined through the elasticity theory variables by

$$w_\alpha = u_\alpha(z=0), \quad w_3 = u_3(z=0), \quad \Phi_\alpha = -\frac{12}{h^3}\int_{-h/2}^{h/2} u_\alpha z dz,$$

(2.10)
$$n_{\alpha\beta} = \int_{-h/2}^{h/2}\sigma_{\alpha\beta}dz[1+O(\varepsilon)], \quad m^{\beta}_{\alpha|\beta} = -\int_{-h/2}^{h/2}\sigma_{\alpha 3}dz[1+O(\varepsilon)],$$

$$m_{\alpha\beta} = -\int_{-h/2}^{h/2}\sigma_{\alpha\beta}zdz[1+O(\varepsilon)].$$

The stress energy functional of an arbitrary stress field $\boldsymbol{\sigma}(x^i)$ reads

(2.11)
$$C[\boldsymbol{\sigma}] = \frac{1}{2E}\int_V [(1+\nu)\sigma^{ij}\sigma_{ij}-\nu\sigma^i_i\sigma^j_j]dV,$$

where V is the volume of the shell. Being quadratic, homogeneous and positive definite, this functional may be used as a norm squared for the stresses.

The following energy inequality [2] will be of fundamental importance in our analysis

(2.12)
$$C[\hat{\boldsymbol{\sigma}}-\boldsymbol{\sigma}] \leqslant C[\hat{\boldsymbol{\sigma}}-\tilde{\boldsymbol{\sigma}}]$$

which is true if

(2.13)
$$\sigma_{ij} = \tilde{\sigma}_{ij} \text{ on } S_\sigma, \quad u_i = \hat{u}_i \quad \text{on} \quad S_u,$$

where $\boldsymbol{\sigma}(x^i)$ is the actual stress field solving the three-dimensional problem of elasticity theory, $\tilde{\boldsymbol{\sigma}}(x^i)$ is a statically admissible stress field, $\hat{\mathbf{u}}(x^i)$ and $\hat{\boldsymbol{\sigma}}(x^i)$ constitute kinematically admissible displacement and stress fields connected through the constitutive equations and S is the bounding surface. According to (2.12), $\hat{\boldsymbol{\sigma}}$ may be regarded as an approximation to $\boldsymbol{\sigma}$, and the error involved is computable on the basis of $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}$ alone, without knowing $\boldsymbol{\sigma}$.

Since our aim is to construct $\hat{\mathbf{u}}$, $\hat{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}$ from the classical shell theory solution, equations (2.13) known as "regular" boundary conditions [1], together with (2.10) give

$$(2.14)\qquad \hat{u}_\alpha(z=0) = w_\alpha, \quad \hat{u}_3(z=0) = w_3, \quad -\frac{12}{h^3}\int_{-h/2}^{h/2} \hat{u}_\alpha z dz = \Phi_\alpha \quad \text{on} \quad C_u,$$

and

$$(2.15)\qquad \int_{-h/2}^{h/2} \tilde{\sigma}_{\alpha\beta} dz = n_{\alpha\beta}[1+O(\varepsilon)], \qquad -\int_{-h/2}^{h/2} \tilde{\sigma}_{\alpha 3} dz = m^{\beta}_{\alpha|\beta}[1+O(\varepsilon)],$$

$$-\int_{-h/2}^{h/2} \tilde{\sigma}_{\alpha\beta} z dz = m_{\alpha\beta}[1+O(\varepsilon)] \quad \text{on} \quad C_\sigma,$$

C being the edge curve of the midsurface.

To minimize the error of solutions of the classical theory, we should minimize the difference $\hat{\boldsymbol{\sigma}}-\tilde{\boldsymbol{\sigma}}$ among such fields $\hat{\mathbf{u}}$ and $\tilde{\boldsymbol{\sigma}}$ which satisfy (2.14) and (2.15). Obviously, it may happen that the error (2.12) will be smaller if we take $\hat{\mathbf{u}}$ and/or $\tilde{\boldsymbol{\sigma}}$ not conforming to (2.14), (2.15), that is, for a non-classical theory. This turns out to be the case with DANIELSON's result [2] which violates $(2.14)_3$, giving a complicated formula for the rotations of normals. We are able to prove that this defect may be overcome, but only partially. To this end we construct a modified $\hat{\mathbf{u}}$-field satisfying $(2.14)_3$, but violating $(2.14)_2$.

3. Derivation of $\tilde{\sigma}$, $\hat{u}$, $\hat{\sigma}$ and error estimates

The statically admissible stress field $\tilde{\boldsymbol{\sigma}}$ is taken here in the usual form [1, 2]

$$(3.1)\qquad \begin{aligned} \tilde{\sigma}_{\alpha\beta} &= \frac{n_{\alpha\beta}}{h} - \frac{12 z m_{\alpha\beta}}{h^3} + O\left(\frac{\varepsilon n}{h}\right), \\ \tilde{\sigma}_{\alpha 3} &= -\frac{z}{h} n^{\beta}_{\alpha|\beta} + \frac{3}{2h}\left(\frac{4z^2}{h^2}-1\right) m^{\beta}_{\alpha|\beta} + O\left(\frac{\varepsilon n}{h}\right), \\ \tilde{\sigma}_{33} &= O\left(\frac{\varepsilon n}{h}\right). \end{aligned}$$

The above expressions satisfy the three-dimensional equilibrium equations with zero body forces, suitable stress boundary conditions on the faces, and also the requirements (2.15) along the edge of the midsurface.

A close to $\tilde{\boldsymbol{\sigma}}$ kinematically admissible stress field $\hat{\boldsymbol{\sigma}}$ will result if we choose the displacement field $\hat{\mathbf{u}}$ in the form

$$(3.2)\qquad \begin{aligned} \hat{u}_\alpha &= w_\alpha - z\Phi_\alpha + \frac{z^2}{2Eh}\left[\nu n^{\beta}_{\beta,\alpha} - 2(1+\nu) n^{\beta}_{\alpha|\beta}\right] + \frac{2-\nu}{Eh}\left(\frac{2z^3}{h^2} - \frac{3z}{10}\right) m^{\beta}_{\beta,\alpha}, \\ \hat{u}_3 &= w_3 - \frac{\nu z}{Eh} n^{\beta}_{\beta} + \frac{3}{Eh}\left(\frac{2\nu z^2}{h^2} - \frac{8+\nu}{10}\right) m^{\beta}_{\beta} + b, \end{aligned}$$

b being an arbitrary constant. Indeed, substituting (3.2) into (2.8) and using (2.1), $(2.2)_2$, $(2.3)_1$, (2.6), we find the strains

$$\hat{e}_{\alpha\beta} = \gamma_{\alpha\beta} - z\varrho_{\alpha\beta} + O(\varepsilon\gamma),$$

$$\hat{e}_{\alpha 3} = \frac{1+\nu}{Eh}\left[-zn^{\beta}_{\alpha|\beta} + \frac{3}{2}\left(\frac{4z^2}{h^2} - 1\right) m^{\beta}_{\alpha|\beta}\right] + O(\varepsilon\gamma), \tag{3.3}$$

$$\hat{e}_{33} = \frac{\nu}{Eh}\left(-n^{\beta}_{\beta} + \frac{12z}{h^2} m^{\beta}_{\beta}\right).$$

Now from (2.9) with (3.3), (3.1), (2.4) we have

$$\hat{\sigma}_{ij} - \tilde{\sigma}_{ij} = O(\varepsilon n/h). \tag{3.4}$$

Since by (3.1), $\tilde{\boldsymbol{\sigma}} = O(n/h)$, it follows from (2.11), (2.12) and (3.4) that

$$C[\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}]/C[\tilde{\boldsymbol{\sigma}}] \leqslant O(\varepsilon). \tag{3.5}$$

This signifies that our three-dimensional solution $\hat{\boldsymbol{\sigma}}$ approximates the actual (unknown) solution with the relative error of order ε, provided that the boundary conditions of the three-dimensional problem are regular, that is, agree with the stresses (3.1) and the displacements (3.2). These regular boundary conditions are consistent with the boundary conditions of the classical shell theory if the equations (2.14) and (2.15) are satisfied. We have already verified that the stresses (3.1) meet equations (2.15). As for the $\hat{\mathbf{u}}$-field (3.2), it satisfies $(2.14)_{1,3}$ but instead of $(2.14)_2$ gives

$$\hat{u}_3(z = 0) = w_3 \underline{- \frac{3(8+\nu)}{10Eh} m^{\beta}_{\beta} + b} \quad \text{on} \quad C_u. \tag{3.6}$$

Within the framework of the two-dimensional shell theory, imposing this boundary condition for the lateral deflection of the midsurface is necessary to obtain approximate three-dimensional solutions with the error equal to ε. Neglecting the underlined terms leads to the classical conditions, but it increases the error (3.5) from ε to $\sqrt{\varepsilon}$. Exceptionally, (3.6) may be reduced to the classical form without loss of accuracy. This happens when m^{β}_{β}, or equivalently ϱ^{β}_{β}, is constant along C_u since then the arbitrary constant b, representing rigid body displacements, may be selected so as to make $\hat{u}_3(z = 0) = w_3$ along C_u. The DANIELSON's [2] $\hat{\mathbf{u}}$-field always satisfies the latter relation, but it may be preferable only if the boundary conditions do not restrict the rotations of normals, since his $\hat{u}_\alpha$ — distribution violates $(2.14)_3$ adding some terms to Φ_α. Finally, it may be concluded from [2] and our analysis that for boundary conditions restricting both the rotations of normals and the lateral deflection of the midsurface, the use of classical boundary conditions leads to a larger error $\sqrt{\varepsilon}$, which may be reduced to ε only at the cost of employing the complex condition (3.6) for $\hat{u}_3$ or that implied by $\hat{u}_\alpha$ of [2] for the rotations of normals.

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