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Lanchester and other models of competition or conflict as dynamical systems. I

R. K. COLEGRAVE (LONDON) and F. M. F. EL-SABAA (KUWAIT)

THE DIFFERENTIAL equations describing Lanchester and other models of competition or conflict are recast in Lagrangian or Hamiltonian form. The canonical transformation theory is invoked and it is shown that an advantage of the method is to reveal invariants which otherwise could be obscured. Some operational applications are indicated.

Równania różniczkowe opisujące model Lanchestera i inne modele współzawodnictwa i konfliktu przekształcono do postaci lagranżowskiej lub hamiltonowskiej. Zastosowano teorię transformacji kanonicznej i wykazano, że zaletą tej metody jest ujawnienie niezmienników, które w przypadku stosowania innych metod pozostałyby ukryte. Wskazano na pewne możliwości zastosowań otrzymanych wyników.

Дифференциальные уравнения, описывающие модель Ланчестера и другие модели соревнования и конфликта, преобразованы к лагранжевому или гамильтоновому виду. Применена теория канонического преобразования и показано, что достоинством этого метода является выявление инвариантов, которые в случае применения других методом остались бы неизвестными. Указаны некоторые возможности применений полученных результатов.

1. Introduction

SYSTEMS of equations that arise in modelling in the sphere of economics or operational research may often, and perhaps always, be regarded as equivalent problems in mechanics. We shall consider in detail some simple Lanchester models for armed conflict together with some other models of competition. The whole content of the original system of equations will be expressed in terms of a single Hamiltonian or Lagrangian function. A similar identification has long been applied to the coupled equations giving the currents in electrical circuits or electronic devices, and seeking suitable Lagrangians has always been a central problem in the classical and quantum field theory. One advantage of such a procedure is that the powerful and well-known methods of analytical dynamics such as the canonical transformation theory may be applied and new light shed on the model. Identifying the equivalent dynamical system via the Lagrangian constitutes the so-called inverse problem of Lagrangian dynamics which has a long history starting with the celebrated work of HELMHOLTZ [7]. The procedure has recently been reviewed and extended by SARLET [16], whose method we shall apply in the present paper.

An extensive literature exists on the models of LANCHESTSER [9]. An introduction to the subject and further references are given by CONOLLY [5]. In the simplest Lanchester model, called the square law model, the strengths x and y of the opposing forces at time t are governed by the equations

$$\dot{x} = -\lambda_1(t)y, \quad \dot{y} = -\lambda_2(t)x, \quad x \geq 0, \quad y \geq 0, \tag{1.1}$$

with ever positive attrition rates λ_1 and λ_2. The solution for constant λ_1 and λ_2 is discussed by Conolly. TAYLOR and COMSTOCK [19] and TAYLOR and BROWN [18] are amongst those who have considered time-dependent λ_1 and λ_2. With the replenishments π_1 and π_2, which may be taken as positive functions of the time, Eqs. (1.1) generalize to

$$\dot{x} = \pi_1(t) - \lambda_1(t)y, \quad \dot{y} = \pi_2(t) - \lambda_2(t)x, \quad x \geq 0, \quad y \geq 0. \tag{1.2}$$

Our discussion in the present paper will be concerned largely with the systems of Eqs. (1.1) or (1.2). In Sect. 2 we shall determine the Lagrangian and Hamiltonian functions for Eqs. (1.2) and some other models of competition or conflict. In Sect. 3 we shall illustrate the use of the canonical transformation theory and the determination and application of some time-dependent invariants. Such invariants have been extensively discussed in classical and quantum mechanics (LEWIS and RIESENFELD [12], PRINCE and ELIEZER [14], LEWIS and LEACH [11], WOLLENBERG [22, 23], RAY and REID [15], COLEGRAVE *et al.* [3], [4]). Time-independent invariants were used from the beginning by Lanchester as criteria for victory. For example, for constant λ_1 and λ_2 the famous "square law"

$$\lambda_2 x^2 - \lambda_1 y^2 = \lambda_2 x^2(0) - \lambda_1 y^2(0) \tag{1.3}$$

shows that X is the victor if and only if the right hand side is positive.

2. Lagrangian and Hamiltonian functions for some models of competition and conflict

We consider the problem of constructing Hamiltonians for some well-known Lanchester or competitive models. We shall use Sarlet's method to construct a Lagrangian. For simplicity we shall consider only one degree of freedom.

Given a second-order differential equation

$$\ddot{q} = f(q, \dot{q}, t), \tag{2.1}$$

we wish to determine a multiplier $\alpha(q, \dot{q}, t)$ to that Eq. (2.1) may be identified with a Lagrangian L according to

$$\alpha(\ddot{q} - f) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}. \tag{2.2}$$

Following Sarlet we define

$$A(q, \dot{q}, t) = -\frac{1}{2}(\partial f/\partial \dot{q}), \tag{2.3}$$

then Theorem 1 of SARLET [16] states that α is a solution of the equation

$$d\alpha/dt = 2A\alpha. \tag{2.4}$$

Thus α always exists in a case of one degree of freedom although, as Sarlet warns us, it may not always be easy to find. Some care has to be taken to assign the coordinate $q = q(x)$ in such a way that Eq. (2.4) is immediately integrable.

2.1. The square law model

Originally with the constant λ_1 and λ_2, Eqs. (1.1) were proposed by Lanchester as a model of "modern warfare", embodying the notion of concentration. Equations (1.2) were introduced more recently to incorporate the idea of replenishment.

For Eqs. (1.2) and with λ_1, λ_2, π_1, π_2 any functions of the time, we may find the Hamiltonian very easily by identifying $x = q$, $y = p$ as conjugate dynamical variables and Eqs. (1.6) as the Hamilton equations:

$$\begin{aligned} &\dot{q} = \pi_1 - \lambda_1 p = \partial H/\partial p \Rightarrow H = \pi_1(t)p - \frac{1}{2}\lambda_1(t)p^2 + f(q, t), \\ &\dot{p} = \pi_2 - \lambda_2 q = -\partial H/\partial q \Rightarrow H = -\pi_2(t)q + \frac{1}{2}\lambda_2(t)q^2 + g(p, t). \end{aligned} \tag{2.5}$$

Hence the Hamiltonian function is

$$H(q, p, t) = -\frac{1}{2}\lambda_1(t)p^2 + \frac{1}{2}\lambda_2(t)q^2 + \pi_1(t)p - \pi_2(t)q. \tag{2.6}$$

The corresponding Lagrangian function is given by

$$L(q, \dot{q}, t) = p\dot{q} - H, \quad p = (\pi_1 - \dot{q})/\lambda_1 \tag{2.7}$$

and, discarding a constant, this yields a Lagrangian L which may be found alternatively by Sarlet's method:

$$L = -\frac{1}{2}\dot{q}^2/\lambda_1(t) - \frac{1}{2}\lambda_2(t)q^2 + \pi_1(t)\dot{q}/\lambda_1(t) - \pi_2(t)q. \tag{2.8}$$

2.2. The mixed model

The model is discussed by CONOLLY ([5], Eq. (1.45)). The equations for the strengths x and y of the opposing forces are

$$\dot{x} = -\lambda_1(t)xy, \quad \dot{y} = -\lambda_2(t)x, \quad x \geq 0, \quad y \geq 0. \tag{2.9}$$

We put $x = q$ and eliminate y. This leads to

$$\ddot{q} = f \equiv \dot{q}^2/q + (\dot{\lambda}_1/\lambda_1)\dot{q} + \lambda_1\lambda_2 q^2, \tag{2.10}$$

$$A = -\frac{1}{2}(\partial f/\partial \dot{q}) = -\frac{1}{2}(2\dot{q}/q + \dot{\lambda}_1/\lambda_1). \tag{2.11}$$

Equation (1.4) integrates to give

$$\alpha = (\lambda_1 q^2)^{-1}. \tag{2.12}$$

We split up Eq. (1.2) into

$$(2.13)\qquad \begin{aligned} &\frac{1}{\lambda_1}\frac{\ddot{q}}{q^2}-\frac{\dot{q}^2}{\lambda_1 q^3}-\frac{\dot{\lambda}_1}{\lambda_1^2}\frac{\dot{q}}{q^2}+F(q,\dot{q},t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right).\\ &\lambda_2+F(q,\dot{q},t) = \frac{\partial L}{\partial q}.\end{aligned}$$

Taking $F = -\dot{q}^2/(\lambda_1 q^3)$, Eqs. (2.13) give

$$\frac{\partial L}{\partial \dot{q}} = \frac{1}{\lambda_1}\frac{\dot{q}}{q^2} \Rightarrow L = \frac{1}{2\lambda_1}\frac{\dot{q}^2}{q^2}+A(q,t),$$

$$\frac{\partial L}{\partial q} = \lambda_2-\frac{\dot{q}^2}{\lambda_1 q^3} \Rightarrow L = \lambda_2 q+\frac{\dot{q}^2}{2\lambda_1 q^2}+B(\dot{q},t),$$

from which we see that

$$(2.14)\qquad L = (1/2\lambda_1)(\dot{q}/q)^2+\lambda_2 q.$$

We can now calculate the Hamiltonian:

$$(2.15)\qquad p \equiv \partial L/\partial \dot{q} = \dot{q}/(\lambda_1 q^2) \Rightarrow \dot{q} = \lambda_1 p q^2,$$

$$(2.16)\qquad H(q,p,t) \equiv p\dot{q}-L = \frac{1}{2}\lambda_1 p^2 q^2-\lambda_2 q.$$

2.3. The linear model

The "linear" model (CONOLLY [5] Eqs. (2.60)), which Lanchester proposed as embodying the notion of "ancient warfare" is

$$(2.17)\qquad \dot{x} = -\lambda_1(t)xy, \quad \dot{y} = -\lambda_2(t)xy, \quad x \geqq 0, \quad y \geq 0.$$

We follow the same procedure as in our last example and put $x = q$, eliminating y. This leads to

$$(2.18)\qquad \begin{aligned} &\ddot{q} = f \equiv \dot{q}^2/q+(\dot{\lambda}_1/\lambda_1)\dot{q}+\lambda_2 q\dot{q},\\ &(1/\alpha)d\alpha/dt = -(2\dot{q}/q+\dot{\lambda}_1/\lambda_1+\lambda_2 q) = -(2\dot{q}/q+\dot{\lambda}_1/\lambda_1-\dot{y}/y),\end{aligned}$$

where we have used Eq. $(2.17)_2$. Thus

$$(2.18')\qquad \alpha = -(\lambda_1^2 q\dot{q})^{-1} \quad (q = x).$$

However, the integration of Eq. (2.4) leads to an extremely awkward expression for L and we try a better assignment of q. Let us put $x = e^q$, then the elimination of y leads to

$$(2.19)\qquad \begin{aligned} &\ddot{q} = f \equiv (\dot{\lambda}_1/\lambda_1)\dot{q}+\lambda_2 e^q\dot{q},\\ &(1/\alpha)d\alpha/dt = -(\dot{\lambda}_1/\lambda_1+\lambda_2 e^q) = -\ddot{q}/\dot{q}.\end{aligned}$$

Thus the integrating factor is

$$(2.19')\qquad \alpha = -1/\dot{q} \quad (q = \ln x)$$

and Eq. (2.4) is easily solved to give

$$(2.20)\qquad L(q,\dot{q},t) = \dot{q}\ln(|\dot{q}|/\lambda_1)-\dot{q}-\lambda_2 e^q,$$

where it should be noted that Eq. $(2.17)_1$ gives $\dot{q} \geq 0$ since $\lambda_1(t) > 0$. The conjugate momentum is

$$p \equiv \partial L/\partial \dot{q} = \ln(|\dot{q}|/\lambda_1) \tag{2.20'}$$

and the Hamiltonian is

$$H \equiv p\dot{q} - L = -\lambda_1(t)e^p + \lambda_2(t)e^q. \tag{2.20''}$$

A quicker way would be to use the symmetry of the model equations (2.17) and to assign (subject to consistency)

$$x = e^q, \quad y = e^p. \tag{2.21}$$

Then an identification of Eqs. (2.17) with Hamilton's equations as in Eqs. (2.5) leads directly to (2.20''), (2.20') is a Hamiltonian equation and canonical consistency is manifest.

2.4. The predator-prey model of Volterra

For our next example we consider the predator-prey model (CONOLLY [5] p. 14) with the equations

$$\dot{x} = (\lambda_1 - \mu_1 y)x, \quad \dot{y} = -(\lambda_2 - \mu_2 x)y, \quad x \geq 0, \quad y \geqq 0, \tag{2.22}$$

where λ_1, λ_2, μ_1, μ_2 are non-negative functions of the time. Following the success in our last example, we put $x = e^q$; then on eliminating y we obtain

$$\begin{aligned} \ddot{q} = f &\equiv -(\dot{\mu}_1/\mu_1)(\lambda_1 - \dot{q}) - (\lambda_2 - \mu_2 e^q)\dot{q} - \lambda_1 \mu_2 e^q + \dot{\lambda} + \lambda_1 \lambda_2, \\ (1/\alpha)\, d\alpha/dt &= -\dot{\mu}_1/\mu_1 - (\lambda_2 - \mu_2 e^q) = -\dot{\mu}_1/\mu_1 - \dot{y}/y, \end{aligned} \tag{2.23}$$

where we have used Eq. $(2.18)_2$, cf. Eq. $(2.18)_2$. Thus

$$\alpha = (\mu_1 y)^{-1} = (\lambda_1 - \dot{q})^{-1}. \tag{2.23'}$$

From Eq. $(2.22)_1$ it is clear that $\lambda_1 - \dot{q} \geq 0$.

We find that we may write Eq. (2.2) in the form

$$\frac{d}{dt}[\ln \mu_1 - \ln(\lambda_1 - \dot{q})] - (\lambda_2 - \mu_2 e^q) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

from which it follows easily that

$$\begin{aligned} L(q, \dot{q}, t) &= \dot{q} \ln \mu_1 + (\lambda_1 - \dot{q})[\ln(\lambda_1 - \dot{q}) - 1] + \lambda_2 q - \mu_2 e^q, \\ p &\equiv \partial L/\partial \dot{q} = \ln[\mu_1/(\lambda_1 - \dot{q})], \\ H(p, q, t) &= (p - \ln \mu_1)(\lambda_1 - \mu_1 e^{-p}) + \mu_1 e^{-p}[1 - \ln(\mu_1 e^{-p})] - \lambda_2 q + \mu_2 e^q. \end{aligned} \tag{2.24}$$

2.5. A model from epidemic theory

As our final example let us consider the model of KERMACK and MCKENDRICK [8] (see CONOLLY [5] p. 26) with the equations

$$\begin{aligned} \dot{x} = -\lambda xy + \nu z, \quad \dot{y} = \lambda xy - \mu y, \quad \dot{z} = \mu y - \nu z, \\ x \geq 0, \quad y \geqq 0, \quad z \geq 0, \end{aligned} \tag{2.25}$$

where x, y, z are, respectively, the numbers of susceptibles, infectives and removals at time t and λ, μ, ν are functions of the time. We use the integral

$$x+y+z = N \tag{2.25'}$$

and set $y = q$. Then the relation (2.23) may be rewritten in the form

$$\dot{x} = -\lambda xq+\nu(N-x-q), \quad \dot{q} = (\lambda x-\mu)q. \tag{2.26}$$

On eliminating x we find

$$\ddot{q} \equiv f = (\dot{q}+\mu q)[(\lambda\dot{q}+\dot{\lambda}q)/(\lambda q)-\lambda q-\nu]+\lambda\nu q(N-q)-\mu\dot{q}-\dot{\mu}q. \tag{2.27}$$

The equation for α is

$$d\alpha/dt = f\partial\alpha/\partial\dot{q}+\dot{q}\partial\alpha/\partial q+\partial\alpha/\partial t = 2A\alpha = -[2\dot{q}/q+\dot{\lambda}/\lambda+\mu+\nu+\lambda q]\alpha. \tag{2.28}$$

We note that this is the first example in which α depends on $\dot{q}$. The solution of Eq. (2.28) has the form

$$\begin{aligned} \alpha(q, \dot{q}, t) &= (\lambda q^2)^{-1}A(t)F(q, \dot{q}, t), \\ A(t) &= \exp\left[-\int^t (\mu+\nu)\,dt\right], \\ F(q, \dot{q}, t) &= \exp\left[-\int^t \lambda q\,dt\right]. \end{aligned} \tag{2.29}$$

To find F we need to solve the equation

$$(d/dt)F(q, \dot{q}, t) = -\lambda qF(q, \dot{q}, t), \tag{2.30}$$

i.e. using Eq. (2.27)

$$\dot{q}\partial F/\partial q+f(q, \dot{q}, t)\partial F/\partial\dot{q}+\partial F/\partial t+\lambda qF = 0. \tag{2.30'}$$

We cannot proceed any further at present in our search for L (or H).

We feel that this example is useful because it illustrates a difficulty that may be encountered in applying Sarlet's method. We recall the simplification in the treatment of the model (2.17) when the canonical variables q, p were used. It seems desirable to reformulate Sarlet's procedure directly in terms of Hamiltonian. We hope to report progress on this at a later date.

3. Applications

In this section we consider some applications of a Hamiltonian formulation for the models considered in Sect. 2 Our examples will mostly be concerned with the Lanchester square law model (1.1) or (1.2). The advantages of the method may be classified under two headings: a) applications of the canonical transformation theory, b) the discovery and application of invariants. Further advantages may well appear in the future, e.g. applications of the (nonlinear) stability theory, adiabatic invariants and an extension to include stochastic processes.

3.1. The square law model with exponentially growing or decaying attrition rates

We return to the example of Sect. 2 and for simplicity we take the case of no replenishment. Then the Hamiltonian is given by Eq. (2.7) with $\pi_1 = \pi_2 = 0$. We shall suppose that the attrition rates λ_1 and λ_2 are

$$\lambda_1(t) = \lambda_1(0)\exp(2\gamma_1 t), \quad \lambda_2(t) = \lambda_2(0)\exp(2\gamma_2 t) \tag{3.1}$$

with $\lambda_1(0) > 0$, $\lambda_2(0) > 0$. We seek a solution of the form

$$x = q = A_1\exp(m_1 t), \quad y = p = A_2\exp(m_2 t) \tag{3.2}$$

for the stengths of the two combatants X and Y. Substituting into Eqs. (1.1) we find that a solution with constant m_1 and m_2 is possible in the case $\gamma_1 = -\gamma_2 = \gamma$. This condition describes a situation of "demoralization" where the ability of one participant to press home his advantage is matched by his opponent's diminishing aggressive power. Then the solution is

$$\begin{aligned} x = q &= [\lambda_1(0)]^{1/2}e^{\gamma t}[A_1(\Gamma-\gamma)^{1/2}e^{\Gamma t}+A_2(\Gamma+\gamma)^{1/2}e^{-\Gamma t}], \\ y = p &= [\lambda_2(0)]^{1/2}e^{-\gamma t}[-A_1(\Gamma+\gamma)^{1/2}e^{\Gamma t}+A_2(\Gamma-\gamma)^{1/2}e^{-\Gamma t}], \end{aligned} \tag{3.3}$$

where

$$\Gamma = [\gamma^2+\lambda_1(0)\,\lambda_2(0)]^{1/2} \tag{3'3.}$$

and A_1, A_2 are arbitrary constants. We note that a solution (3.3) with $A_2 = 0$ is not possible since this would violate the constraints $x \geqslant 0$, $y \geqslant 0$. The solution with $A_1 = 0$ would require the initial force strengths $x(0)$, $y(0)$ to satisfy

$$(\Gamma-\gamma)^{1/2}\lambda_2^{1/2}(0)\,x(0) = (\Gamma+\gamma)^{1/2}\lambda_1^{1/2}(0)\,y(0). \tag{3.4}$$

Let us now return to the more general problem with attrition rates described by Eqs. (3.1) with $\gamma_1 \neq -\gamma_2$.

3.1.1. An exact solution. The Hamiltonian (1.6) with $\pi_1 = \pi_2 = 0$ may be written in the form

$$H = e^{2\alpha t}\left[-\frac{1}{2}\lambda_1(0)e^{2\beta t}p^2+\frac{1}{2}\lambda_2(0)e^{-2\beta t}q^2\right], \tag{3.5}$$

where

$$\alpha = \frac{1}{2}(\gamma_1+\gamma_2), \quad \beta = \frac{1}{2}(\gamma_1-\gamma_2). \tag{3.5'}$$

By using the method of LEWIS and LEACH [11], the Hamiltonian (3.5) can be reduced to a time-independent form, thus rendering the equations of motion trivially solvable. However, since the transformation involved is complicated and not amenable to explicit analytic expression, we prefer to reduce Eq. (3.5) to a simpler form which is not entirely time independent. This leads to an exact solution in terms of Bessel functions.

First we apply the scaling transformation $(q, p, t) \to (Q, P, t)$:

$$P = e^{\beta t}p, \quad Q = e^{-\beta t}q \tag{3.6}$$

then (cf. COLEGRAVE and ABDALLA [2])

$$H \to K = e^{2\alpha t}\left[-\frac{1}{2}\lambda_1(0)P^2 + \frac{1}{2}\lambda_2(0)Q^2 - \beta e^{-2\alpha t}QP\right]. \tag{3.7}$$

Secondly we apply the "generalized" canonical transformation (LEWIS and LEACH [11]) $(Q, P, t) \to (\bar{Q}, \bar{P}, \tau)$:

$$\bar{Q} = 0, \quad \bar{P} = P, \quad \tau(t) = \int_0^t e^{2\alpha t'}dt' = (e^{2\alpha t} - 1)/2\alpha \tag{3.8}$$

so that

$$K \to \bar{K} = -\frac{1}{2}\lambda_1(0)\bar{P}^2 + \frac{1}{2}\lambda_2(0)\bar{Q}^2 - \beta(2\alpha\tau+1)\bar{Q}\bar{P}. \tag{3.9}$$

The idea of a "reduced" time τ was introduced by LEVI CIVITA [10] and is discussed by CALDIROLA [1]. It is referred to as operational time by some workers in operational research. Denoting differentiation with respect to τ by a prime, the relation (3.9) gives the equations of motion (from now on we drop the unnecessary bars on Q, P):

$$\begin{aligned} Q' &= \partial K/\partial P = -\lambda_1(0)P - \beta Q/(2\alpha\tau+1), \\ P' &= -\partial K/\partial Q = -\lambda_2(0)Q + \beta P/(2\alpha\tau+1). \end{aligned} \tag{3.10}$$

The elimination of P gives

$$\begin{aligned} d^2Q/d\tau^2 &= [\lambda^2 + \beta(\beta+2\alpha)/(2\alpha\tau+1)^2]Q, \\ \lambda^2 &\equiv \lambda_1(0)\lambda_2(0). \end{aligned} \tag{3.11}$$

This equation is recognised as being of the Bessel type. To reduce it to standard form we write

$$Q = \sigma^{1/2}v, \quad \sigma = 2\alpha\tau+1 = e^{2\alpha t}(\Rightarrow x = e^{(\alpha+\beta)t}v) \tag{3.12}$$

when we obtain

$$\frac{d^2v}{d\sigma^2} + \frac{1}{\sigma}\frac{dv}{d\sigma} - \left[\frac{\lambda^2}{4\alpha^2} + \frac{(\alpha+\beta)^2}{4\alpha^2\sigma^2}\right]v = 0. \tag{3.13}$$

The general solution may be written

$$v = AJ_\nu(ik\sigma) + BJ_{-\nu}(ik\sigma), \tag{3.14}$$

where A, B are suitable (complex) constants and

$$\begin{aligned} k &= \lambda/2\alpha = \lambda_1^{1/2}(0)\lambda_2^{1/2}(0)/(\gamma_1+\gamma_2), \\ \nu &= (\alpha+\beta)/2\alpha = \gamma_1/(\gamma_1+\gamma_2). \end{aligned} \tag{3.14'}$$

The solution may be more conveniently expressed in terms of the modified Bessel functions $I_\nu(k\sigma)$ and $K_\nu(k\sigma)$, see for instance WATSON, [21], but we shall not discuss computational details here. We wish to emphasize that a canonical reduction of the Hamiltonian has led us to an exact solution, the existence of which was previously unknown to the authors. It is interesting to compare this with the Bessel function solution of TAYLOR and COMSTOCK [19] for power attrition rates.

3.1.2. Special solutions. We shall now examine some interesting special cases of our general solution.

a) From Eq. $(3.11)_1$ we see that, besides the case $\alpha = 0$ ($\gamma_1 = -\gamma_2$: one combatant demoralized) described by the solution (3.3), the cases $\beta = 0$ ($\gamma_1 = \gamma_2$: a flagging or exacerbated conflict), $\beta \pm 2\alpha = 0$ ($3\gamma_1 = -\gamma_2/\gamma_1 = -3\gamma_2$) and $\alpha \pm \beta = 0$ ($\gamma_1 = 0$ or $\gamma_2 = 0$) are solved simply in terms of exponential functions.

b) In the case of demoralization ($\gamma_1 = -\gamma_2 = \gamma$) Eq. (3.7) reduces to

$$K = -\frac{1}{2}\lambda_1(0)P^2 + \frac{1}{2}\lambda_2(0)Q^2 - \gamma QP. \tag{3.15}$$

Since K is the Hamiltonian we know that

$$dK/dt = \partial K/\partial t = 0 \tag{3.16}$$

and, somewhat fortuitously, we have discovered an invariant. In terms of the original variables this reads

$$\frac{1}{2}\lambda_2(0)e^{-2\gamma t}x^2 - \frac{1}{2}\lambda_1(0)e^{2\gamma t}y^2 - \gamma xy = \text{const} = K, \tag{3.17}$$

cf. CONOLLY [5] Eq. (1.29) in the case $\gamma = 0$. We do not think that Eq. (3.17) is a very obvious consequence of the model equations (1.1) with $\lambda_1(t) = [\lambda_2(t)]^{-1} = e^{2\gamma t}$. A corresponding invariant exists in the general case (cf. LEWIS and LEACH [11]). From the solution (3.3) we find that

$$K = 2A_1A_2\Gamma^2[\lambda_1(0)\lambda_2(0)]^{1/2}. \tag{3.18}$$

Let us consider the case $K = 0$; then a time $t = 0$ Eq. (3.17) gives a quadratic equation for $z = x(0)/y(0)$. In view of the constraints $x \geqslant 0$, $y \geqslant 0$ the acceptable solution is $z = (\gamma+\Gamma)/\lambda_2(0)$. After time t the model (2.17) shows that the following relation exists between $x(t)$ and $y(t)$:

$$x(t) = [(\gamma+\Gamma)/\lambda_2(0)]e^{2\gamma t}y(t). \tag{3.19}$$

From Eq. (3.18) we know that $K = 0$ corresponds to $A_1 = 0$ ($A_2 = 0$ being unacceptable). Equation (3.19) is thus the evolution of the relation (3.4) for $t < 0$. Obviously neither x nor y will go to zero at a finite time in this case. If a combatant surrenders when his force is reduced to (say) one tenth of the initial size, it is clear from Eq. (3.19) that Y will surrender first if $\gamma > 0$.

Let us suppose now that $K > 0$; then from Eq. (3.17)

$$\frac{1}{2}\lambda_2(0)e^{-2\gamma t}x^2(t) - \frac{1}{2}\lambda_1(0)e^{2\gamma t}y^2(t) - \gamma x(t)y(t)$$
$$= \frac{1}{2}\lambda_2(0)x^2(0) - \frac{1}{2}\lambda_1(0)y^2(0) - \gamma x(0)y(0) > 0. \tag{3.20}$$

Suppose $y(t_1) = 0$, then the relation (3.20) confirms that $x(t_1) > 0$, so that X is the victor. Similarly, if $K < 0$, then Y is the victor. Thus accepting the limitations of a deterministic model, the invariant K is of special importance in the planning of force levels (see for instance TAYLOR [17]). To summarize: in order to win x must ensure that his initial force satisfies the relation (3.20).

c) In a flagging conflict where there is no demoralization, but the combatants' aggressiveness wanes exponentially at the same rate, we have $\gamma_1 = \gamma_2 < 0$ so that

$$\alpha = -|\alpha| = -|\gamma_1|. \tag{3.21}$$

From Eq. (3.9) we see that $\bar{K}$ is independent of the time:

$$\bar{K} = -\frac{1}{2}\lambda_1(0)p^2 + \frac{1}{2}\lambda_2(0)q^2. \tag{3.22}$$

Equations (3.10) reduce to

$$q' = -\lambda_1(0)p, \quad p' = -\lambda_2(0)q \tag{3.23}$$

or, on eliminating p and writing $\lambda^2 = \lambda_1(0)\,\lambda_2(0)$

$$d^2q/d\tau^2 = \lambda^2 q, \quad \tau = (1-e^{-2|\alpha|t})/2|\alpha|. \tag{3.23'}$$

Reverting to the original variables ($q \to x$, $p \to y$) the combatants are seen to have strengths at time t given by

$$\begin{aligned} x(t) &= x(0)\cosh[\lambda(1-e^{-2|\alpha|t})/2|\alpha|] - y(0)\,[\lambda_1(0)/\lambda_2(0)]^{1/2}\sinh[\lambda(1-e^{-2|\alpha|t})/2|\alpha|], \\ y(t) &= y(0)\cosh[\lambda(1-e^{-2|\alpha|t})/2|\alpha|] - x(0)\,[\lambda_2(0)/\lambda_1(0)]^{1/2}\sinh[\lambda(1-e^{-2|\alpha|t})/2|\alpha|]. \end{aligned} \tag{3.24}$$

On the assumption that the conflict has not previously ended, the long-time limits are

$$\begin{aligned} x(\infty) &= x(0)\cosh[\lambda/2|\alpha|] - y(0)[\lambda_1(0)/\lambda_2(0)]^{1/2}\sinh[\lambda/2|\alpha|], \\ y(\infty) &= y(0)\cosh[\lambda/2|\alpha|] - x(0)\,[\lambda_2(0)/\lambda_1(0)]^{1/2}\sinh[\lambda/2|\alpha|]. \end{aligned} \tag{3.25}$$

The contest must finish when either x or y first becomes zero. In the case $\lambda \gg |\alpha|$ and $x(0) > y(0)[\lambda_1(0)/\lambda_2(0)]^{1/2}$ we see that X is sure to be the victor. On the other hand, if $\lambda \ll |\alpha|$ we see from Eq. $(3.25)_1$ that X survives provided $x(0) > \frac{1}{2}\lambda_1(0)\,y(0)/|\alpha|$ but wins only if $x(0) > 2|\alpha|\,y(0)/\lambda_2(0)\left(\gg \frac{1}{2}\lambda_1(0)\,y(0)/|\alpha|\right)$. These may be important operational conclusions.

3.2. The square law model with periodic attrition rates

Let us consider a situation in which the attrition rates in the square law model (1.1) show slight periodic fluctuations. Furthermore, let us suppose that each side experiences encouragement or demoralization as his opponent shows weakness or strength. This model could be of interest in psychology or in the control theory. We take

$$\lambda_1(t) = \lambda_1(0)e^{-2\mu\sin\nu t}, \quad \lambda_2(t) = \lambda_2(0)e^{2\mu\sin\nu t}, \quad (\mu \ll 1). \tag{3.26}$$

The canonical transformation (COLEGRAVE and ABDALLA [3])

$$Q = qe^{\mu\sin\nu t}, \quad P = pe^{-\mu\sin\nu t} \tag{3.27}$$

leads to the new Hamiltonian

$$K = -\frac{1}{2}\lambda_1(0)P^2 + \frac{1}{2}\lambda_2(0)Q^2 + \mu\nu\cos\nu t\,QP \tag{3.28}$$

and equations of motion of the generalized Hill type (MAGNUS and WINKLER [13]), with $\lambda^2 = \lambda_1(0)\,\lambda_2(0)$:

$$\ddot{Q} = (\lambda^2 - \mu\nu^2 \sin\nu t + \mu^2\nu^2\cos^2\nu t)Q \tag{3.29}$$

with P given by

$$P = [\lambda_1(0)]^{-1}(\mu\nu\cos\nu t Q - \dot{Q}). \tag{3.29'}$$

For $\mu \ll 1$ these equations may be solved in a straightforward manner by the variation of the parameters method. To the second order in μ we find the solution

$$Q = Q_0 + \mu Q_1 + \mu^2 Q_2 \tag{3.30}$$

where, on writing

$$\begin{aligned}
&\theta = \nu^2/(\nu^2+4\lambda^2), \qquad \delta = \lambda/\nu,\\
&Q_0 = Q(0)\cosh\lambda t - [P(0)/\lambda_2(0)]\sinh\lambda t,\\
&Q_1 = 2\theta Q(0)\cos(\nu t/2)(2\delta\sinh\lambda t\cos\nu t/2 + \cosh\lambda t\sin\nu t/2)\\
&\qquad + 2\theta[P(0)/\lambda_2(0)]\sin(\nu t/2)(2\delta\cosh\lambda t\sin\nu t/2 + \sinh\lambda t\cos\nu t/2),
\end{aligned} \tag{3.30'}$$

$$\begin{aligned}
Q_2 = {}& -\theta Q(0)\left[\left(\frac{3}{4}-\theta\right)\cosh\lambda t - \frac{1}{4}(2\delta\sinh\lambda t\sin 2\nu t - \cosh\lambda t\cos 2\nu t)\right.\\
&\left. - 2\theta\delta(2\delta\cosh\lambda t\cos\nu t + \sinh\lambda t\sin\nu t) - \lambda t\sinh\lambda t\right] + \theta[P(0)/\lambda_2(0)]\left[\left(\frac{3}{4}-\theta\right)\right.\\
&\times\sinh\lambda t - \frac{1}{4}(2\delta\cosh\lambda t\sin 2\nu t - \sinh\lambda t\cos 2\nu t) - 2\theta\delta(2\delta\sinh\lambda t\cos\nu t\\
&\left. - \cosh\lambda t\sin\nu t - \lambda t\cosh\lambda t\right],
\end{aligned}$$

P may be derived from this solution by using Eq. (3.29'). In the case $\lambda \ll \nu$ our solution becomes approximately, with Q_0 growing only slowly,

$$\begin{aligned}
Q_1 &\approx \sin\nu t Q_0(-t),\\
Q_2 &\approx \frac{1}{2}\sin^2\nu t Q_0(t) + t\dot{Q}_0(t).
\end{aligned} \tag{3.31}$$

The last term in the relation $(3.31)_2$ shows a gradual build-up or instability caused by the periodically fluctuating attrition rates. For long-term survival the combatant X must ensure that $\dot{Q}_0(t) > 0$ for $t \ll 0$. From Eq. $(3.30')_2$ this means

$$x(0) > y(0)/\lambda_2(0). \tag{3.32}$$

A similar result is found for the attrition rates

$$\lambda_1(t) = \lambda_1(0)(1+2\mu\sin\nu t)^{-1}, \qquad \lambda_2(t) = \lambda_2(0)(1+2\mu\sin\nu t) \tag{3.33}$$

with $\mu \ll 1$. This agrees with the relations (3.26) to first order and Q_1 is still given by Eqs. $(3.30')_3$. Q_2 again contains nonperiodic terms which give a long term build up. Second order aperiodicity is a feature of the response of this Lanchester model with $\lambda_1(t)$, $\lambda_2(t)$ slightly fluctuating and periodic, but connected by $\lambda_1(t)\,\lambda_2(t) = $ constant.

3.3. Time-dependent invariants

Time-dependent invariants have been studied widely in classical and quantum mechanics. In Eq. (3.17) we have already met such an invariant. They may be sought systematically by the method of LEWIS and RIESENFELD [12] and recent applications to the time-dependent harmonic oscillator have been made by WOLLENBERG [22, 23] and by COLEGRAVE *et al.* [3, 4].

Suppose that $J(q, p, t)$, with the time entering in certain coefficients is to be constant in time, then it must satisfy (GOLDSTEIN [6])

$$0 = dJ/dt = \partial J/\partial t + (\partial J/\partial q)\,\partial H/\partial p - (\partial J/\partial p)\,\partial H/\partial q. \tag{3.34}$$

To illustrate the method, we take the system (1.2) but with the constans λ_1, λ_2, π_1, π_2 as treated by CONOLLY [5] p. 23. Let us consider the possibility of an invariant of the form

$$J = \alpha(t)q + \beta(t)p + \gamma(t) \tag{3.35}$$

for the model described by the Hamiltonian (2.6) (cf. VAIDYANATHAN [20] who finds a similar invariant for a driven oscillator). Equations (3.34) and (3.55) require

$$\dot{\alpha} = \lambda_2 \beta, \quad \dot{\beta} = \lambda_1 \alpha, \quad \dot{\gamma} = -\pi_1 \alpha - \pi_2 \beta, \tag{3.36}$$

so that

$$\ddot{\alpha} = \lambda^2 \alpha, \quad \ddot{\beta} = \lambda^2 \beta \quad (\lambda^2 \equiv \lambda_1 \lambda_2). \tag{3.37}$$

The solution is

$$\begin{aligned} \alpha &= \lambda_2^{1/2}[Ae^{\lambda t} + Be^{-\lambda t}], \\ \beta &= \lambda_1^{1/2}[Ae^{\lambda t} - Be^{-\lambda t}], \\ \gamma &= -\pi_1 \lambda_1^{-1/2}(Ae^{\lambda t} - Be^{-\lambda t}) - \pi_2 \lambda_2^{-1/2}(Ae^{\lambda t} + Be^{-\lambda t}) \quad (\lambda > 0). \end{aligned} \tag{3.38}$$

Writing $J = AJ_1 + BJ_2$, we see that we have discovered two linear invariants for our model:

$$\begin{aligned} J_1(t) &= (\lambda_2^{1/2} q + \lambda_1^{1/2} p - \pi_1 \lambda_1^{-1/2} - \pi_2 \lambda_2^{-1/2}) e^{\lambda t}, \\ J_2(t) &= (\lambda_2^{1/2} q - \lambda_1^{1/2} p + \pi_1 \lambda_1^{-1/2} - \pi_2 \lambda_2^{-1/2}) e^{-\lambda t}. \end{aligned} \tag{3.39}$$

In terms of these invariants, the solution of the problem may be written in the form

$$\begin{aligned} x = q &= \pi_2/\lambda_2 + \frac{1}{2}\lambda_2^{-1/2} J_2(0) e^{\lambda t} + \frac{1}{2}\lambda_2^{-1/2} J_1(0) e^{-\lambda t}, \\ y = p &= \pi_1/\lambda_1 - \frac{1}{2}\lambda_1^{-1/2} J_2(0) e^{\lambda t} + \frac{1}{2}\lambda_1^{-1/2} J_1(0) e^{-\lambda t}, \end{aligned} \tag{3.40}$$

and J_1 (0) and J_2 (0) correspond to B and A in Conolly's notation. The condition for side X to be the victor is obviously that $J_2(0) > 0$.

Equations (3.36) still hold if λ_1, λ_2, π_1, π_2 are time dependent and their solutions give more general linear invariants of the form (3.35) which should have useful operational applications.

The Hamiltonian (2.6) is the same as for a driven harmonic oscillator (of imaginary frequency). We are very familiar with the associated dynamics and invariants in this case. We turn our attention to the more exotic Hamiltonians derived in Sect. 2.

It seems unlikely that any invariants can be found for the "mixed" model described by the Hamiltonian (2.16), or for the predator-prey model Hamiltonian (2.24). For the "linear" model with the Hamiltonian (2.20) we may seek an invariant of the form

$$J(q, p, t) = \alpha(t)e^p + \beta(t)e^q. \tag{3.41}$$

To satisfy Eq. (3.34) we need to choose α, β to satisfy

$$\dot{\alpha}e^p + \dot{\beta}e^q - (\lambda_1\beta + \lambda_2\alpha)e^{p+q} = 0 \tag{3.42}$$

i.e.

$$\dot{\alpha} = 0, \quad \dot{\beta} = 0, \quad \lambda_1\beta + \lambda_2\alpha = 0. \tag{3.43}$$

A solution exists only if λ_1, λ_2 have the same time factor, e.g. $\lambda_1(t) = \lambda_1(0)\,e^{-\gamma t}$, $\lambda_2(t) = \lambda_2(0)\,e^{-\gamma t}$. Then Eq. (3.43) is satisfied by $\alpha = \lambda_1(0)$, $\beta = -\lambda_2(0)$ and an invariant is

$$J(x, y) = \lambda_2(0)x - \lambda_1(0)y = \text{const} = J. \tag{3.44}$$

The existence of Eq. (3.44) is manifest from the model (2.17). If X arranges that $x(0) > \lambda_1(0)\,y(0)/\lambda_2(0)$, then $J > 0$ and he is sure to win the contest. The invariant (3.44) is equivalent to the form obtained by Lanchester for time-independent λ_1, λ_2.

4. Discussion

We have demonstrated the usefulness of representing systems of equations from Lanchester and other models by a Hamiltonian function which summarizes their whole content. In Sect. 2 we have shown how the Hamiltonian may be constructed by taking a series of examples of models with one degree of freedom. Some ingenuity needs to be exercised in the choice of the coordinate, as this seriously affects the complexity of the Hamiltonian. This is shown in our example in Sect. 2.3. In Sect. 3.1 we have seen that a systematic canonical reduction of the Hamiltonian leads to an unsuspected exact solution of the model equations. We have shown how invariants are revealed. Subject only to our acceptance of the limitation of deterministic models, these have operational significance, as shown in Sect. 2.1.

We conclude that the consideration of models of conflict or competition as Hamiltonian dynamical systems is a fruitful one. The model is seen from a different point of view and in many cases this leads to new insight into the solution or properties of the model.

Models in operational research are either deterministic or probabilistic. In this paper we have confined our attention to the former class and this approach to modelling is identifiable with classical mechanics. We feel that a probabilistic or stochastic model could similarly be regarded as a quantum mechanical system associated with the same Hamiltonian.

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DEPARTMENT OF MATHEMATICS, CHELSEA COLLEGE
UNIVERSITY OF LONDON, LONDON, ENGLAND
and
DEPARTMENT OF MATHEMATICS
KUWAIT UNIVERSITY, KUWAIT.

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