31.

REPLY TO PROFESSOR BOOLE'S OBSERVATIONS ON A THEOREM CONTAINED IN THE LAST NOVEMBER NUMBER OF THE JOURNAL.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 171—174.]

THE restricted space that can be spared for discussion in these pages, necessitates me to compress within the narrowest limit the remarks which I feel bound to make on Mr Boole's extraordinary observations* in the February number of this *Journal*, on my theorem contained in the antecedent number thereof†, which statements I cannot, in the interests of truth and honesty, suffer to pass unchallenged. The object of that theorem was to show how the determinant of the quadratic function resulting from the elimination of any set of the variables between a given quadratic function and a number of linear functions of the same variables, could be represented *without performing* the actual elimination by a fraction, of which the numerator would be constant whichever set of the variables might be selected for elimination, and the denominator the square of the determinant corresponding to the coefficients of the variables so eliminated. The numerator itself is a determinant, obtained by forming the square corresponding to the determinant of the given quadratic function, and bordering it horizontally and vertically with the lines and columns corresponding to the coefficients of all the variables in the given linear equations. An *immediate corollary* from this theorem leads to Mr Boole's. Conversely upon the principle that "tout est dans tout" Mr Boole devotes a page and a half of close print merely to indicate the steps of a method by which from his theorem mine is capable of being deduced, ending with the announcement, that the numerator in question is equal to the quantity

$$\phi_1\phi_2 \dots \phi_r\theta(Q),$$

(the symbols above employed being Mr Boole's own), and concludes with assuring his readers that "he has ascertained that Mr Sylvester's result is reducible to the above form." Mr Sylvester would be very sorry to put his

[* *Cambr. and Dublin Math. Jour.* VI. (1851), pp. 90, 284.] [† p. 135 above.]

result under any such form. Mr Boole could scarcely have reflected upon the effect of his words when he indulged in the remark which follows—"*there cannot be a doubt* that for the discovery of the actual relation in question, the above theorem is far more convenient than Mr Sylvester's." Of the value to be attached to this assertion the annexed comparison of results is submitted as a specimen.

Let the quadratic function be

$$ax^2 + by^2 + cz^2 + dt^2 + 2exy + 2\epsilon zt + 2gxz + 2\gamma yt + 2hyz + 2\eta xt,$$

and the linear functions (taken two in number)

$$lx + my + nz + pt,$$
$$l'x + m'y + n'z + p't.$$

My numerator will be the determinant (hereinafter cited as the *extended* determinant),

$$\begin{vmatrix} a & e & g & \eta & l & l' \\ e & b & h & \gamma & m & m' \\ g & h & c & \epsilon & n & n' \\ \eta & \gamma & \epsilon & d & p & p' \\ l & m & n & p & 0 & 0 \\ l' & m' & n' & p' & 0 & 0 \end{vmatrix}.$$

To find the numerator of Mr Boole's fraction, we must form the symbolical operator

$$\left\{\begin{aligned} & l^2\frac{d}{da} + m^2\frac{d}{db} + n^2\frac{d}{dc} + p^2\frac{d}{dd} \\ & + 2lm\frac{d}{de} + 2np\frac{d}{d\epsilon} + 2ln\frac{d}{dg} + 2mp\frac{d}{d\gamma} + 2lp\frac{d}{dh} + 2mn\frac{d}{d\eta} \end{aligned}\right\}$$

$$\times\left\{\begin{aligned} & l'^2\frac{d}{da} + m'^2\frac{d}{db} + n'^2\frac{d}{dc} + p'^2\frac{d}{dd} \\ & + 2l'm'\frac{d}{de} + 2n'p'\frac{d}{d\epsilon} + 2l'n'\frac{d}{dg} + 2m'p'\frac{d}{d\gamma} + 2l'p'\frac{d}{dh} + 2m'n'\frac{d}{d\eta} \end{aligned}\right\}$$

and after expanding the determinant hereunder written

$$\begin{vmatrix} a & e & g & \eta \\ e & b & h & \gamma \\ g & h & c & \epsilon \\ \eta & \gamma & \epsilon & d \end{vmatrix},$$

perform the operations above indicated upon the result so obtained.

These are the operations and processes which, on Professor Boole's authority, we are to accept "*as without doubt far more convenient*" than the one simple process of forming, and when necessary, calculating the extended determinant above given. Here for the present I leave the case between Mr Boole and myself to the judgment of the readers of this *Journal.*

In the April Number of the *Philosophical Magazine**, I have shown that the extended determinant serves, not only to represent the full and complete determinant of the reduced quadratic function, but likewise all the minor determinants thereof; the last *set* of which will be evidently no other than the coefficients themselves. For instance, in the example above given, if we wish to find the coefficient of x^2 after z and t have been eliminated, we have only to strike out the line and column $e\ b\ h\ \gamma\ m\ m'$ from the extended determinant; if we wish to find the coefficient of y^2, we must strike out the line and column $a\ e\ g\ \eta\ l\ l'$; to find the coefficient of xy, we must strike out the line $a\ e\ g\ \eta\ l\ l'$ and the column $e\ b\ h\ \gamma\ m\ m'$, or *vice versâ.*

In each of these cases the determinant so obtained is the numerator of the equivalent fraction; the denominator remaining always the same function of the coefficients of transformation as in the original theorem.

Again, if there be taken only one linear equation, and by aid of it x is supposed to be eliminated; and if the reduced quadratic function be called

$$Ly^2 + Mz^2 + Nt^2 + 2Pzt + 2Qyt + 2Rzy,$$

the same extended determinant as before given will serve, when stripped of its outer border, consisting of the line and column $l'\ m'\ n'\ p'$, to produce the various equivalent fractions: thus form the square

$$\begin{matrix} L & R & Q \\ R & M & P \\ Q & P & N. \end{matrix}$$

The numerator of the fraction equivalent to $\begin{vmatrix} L & R \\ R & M \end{vmatrix}$, that is, to $LM - R^2$, may be found by striking out from the form of the extended determinant the line and column $\eta\ \gamma\ \epsilon\ d\ p$; that corresponding to $\begin{vmatrix} L & Q \\ R & P \end{vmatrix}$, that is, $LP - RQ$, will be found by striking out the line $g\ h\ c\ \epsilon\ n$ and the column $\eta\ \gamma\ \epsilon\ d\ p$, or *vice versâ*; and so forth for all the first minor determinants; and similarly the second minors, that is, L, M, N, P, Q, R, may be obtained by striking out in each case a correspondent pair of lines and pair of columns. Thus, to find the numerator of L the same pair of lines and columns, namely, $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, must be elided. To find the numerator of R, the pair of lines $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, and the pair of columns $(e\ b\ h\ \gamma\ m)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, or *vice versâ*, will have to be elided; and so forth for the remaining second minors. I may conclude with observing, that the theorem contested by Mr Boole is an immediate corollary from the general Theory of Relative Determinants alluded† to in the "Sketch" inserted in the present number of the *Journal.*

[* p. 241 below.] [† p. 188 below.]