30.

ON CERTAIN GENERAL PROPERTIES OF HOMOGENEOUS FUNCTIONS.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 1—17.]

LET χ denote the operation

$$x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \ldots + x_n \frac{d}{da_n},$$

and A the operation

$$a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} + \ldots + a_n \frac{d}{dx_n}:$$

and now suppose that ω, a homogeneous function of ι dimensions of $a_1, a_2 \ldots a_n$, and not of any of the quantities $x_1, x_2 \ldots x_n$, is subjected to the successive operations indicated by $A^s \chi^r$.

We have

$$A^s \chi^r \omega = A^{s-1} A \chi^r \omega,$$

$$A\chi^r\omega = \left(a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} + \ldots + a_n \frac{d}{dx_n}\right)\left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \ldots + x_n \frac{d}{da_n}\right)^r \omega$$

$$= r\left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \ldots + a_n \frac{d}{da_n}\right)\chi^{r-1}\omega$$

$$= r(\iota - r + 1)\chi^{r-1}\omega,$$

for $\chi^{r-1}\omega$ is of $(r-1)$ dimensions, lower than ω (which is of ι dimensions) in $a_1, a_2 \ldots a_n$.

Hence

$$A^s \chi^r \omega = r(\iota - r + 1) A^{s-1} \chi^{r-1} \omega$$

$$= \&\text{c.} = \{r(r-1)\ldots(r-s+1)\}$$
$$\{(\iota - r + 1)(\iota - r + 2)\ldots(\iota - r + s)\}\,\chi^{r-s}\omega. \qquad (1)$$

Now in the expression

$$\chi^r \omega (a_1, a_2 \dots a_n),$$

suppose that we write

$$x_1 = u_1 + a_1 \epsilon,$$

$$x_2 = u_2 + a_2 \epsilon,$$

$$\dots\dots\dots\dots\dots$$

$$x_n = u_n + a_n \epsilon,$$

we have, by Taylor's theorem,

$$\chi^r \omega = U^r \omega + A U^r \omega \, \epsilon + A^2 U^r \omega \frac{\epsilon^2}{1.2} + \dots + A^r U^r \omega \frac{\epsilon^r}{1.2.3\dots r},$$

where $U^r \omega$ denotes what $\chi^r \omega$ becomes, on substituting u's for x's, and A now represents

$$u_1 \frac{d}{da_1} + u_2 \frac{d}{da_2} + \dots + u_n \frac{d}{da_n}.$$

This expansion stops spontaneously at the $(r+1)$th term, because $\chi^r \omega$ is only of r dimensions in $x_1, x_2 \dots x_n$.

Applying now theorem (1), we obtain

$$\chi^r \omega = U^r \omega + r(\iota - r + 1) U^{r-1} \omega \epsilon + \tfrac{1}{2} r(r-1) \{(\iota - r + 1)(\iota - r + 2)\} U^{r-2} \omega \epsilon^2$$
$$+ \dots + \{(\iota - r + 1)(\iota - r + 2) \dots \iota\} \omega \epsilon^r. \qquad (2)$$

In using this theorem in the course of the ensuing pages, it will be found convenient to assign to ϵ a specific value, and I shall suppose it equal to $\frac{x_n}{a_n}$; this gives

$$u_1 = x_1 - \frac{a_1}{a_n} x_n,$$

$$u_2 = x_2 - \frac{a_2}{a_n} x_n,$$

$$\dots\dots\dots\dots\dots$$

$$u_n = x_n - \frac{a_n}{a_n} x_n$$
$$= 0.$$

And inasmuch as the U symbol now contains $a_1, a_2, \dots a_n$, so that UU^r no longer equals U^{r+1}, I shall write U_r for U^r. Theorem (2) will thus assume the form

$$\chi^r \omega = U_r \omega + r(\iota - r + 1) U_{r-1} \omega \frac{x_n}{a_n} + \tfrac{1}{2} r(r-1)(\iota - r + 1)(\iota - r + 2) U_{r-2} \omega \left(\frac{x_n}{a_n}\right)^2$$
$$+ \dots + \{(\iota - r + 1) \dots \iota\} \omega \left(\frac{x_n}{a_n}\right)^r, \qquad (3)$$

where U_r for all values of r denotes what

$$\left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_{n-1} \frac{d}{da_{n-1}}\right)^r \omega$$

becomes, on substituting $u_1, u_2, \dots u_{n-1}$ for $x_1, x_2, \dots x_{n-1}$, *after* the processes of derivation have been completed: this it is essential to observe, because $u_1, u_2, \dots u_{n-1}$ now involve $a_1, a_2, \dots a_{n-1}, a_n$. The term $x_n \dfrac{d}{da_n}$ is omitted from the symbol of linear derivation, because in the substitutions x_n will be replaced by zero.

As an example of this last theorem, take

$$\omega = a^3 + b^3 + c^3 + kabc;$$

then

$$\chi\omega = 3a^2x + 3b^2y + 3c^2z + kbcx + kcay + kabz,$$

$$\chi^2\omega = 6ax^2 + 6by^2 + 6cz^2 + 2kcxy + 2kayz + 2kbzx,$$

$$\chi^3\omega = 6x^3 + 6y^3 + 6z^3 + 6kxyz.$$

$$U_1\omega = 3a^2\left(x - \frac{az}{c}\right) + 3b^2\left(y - \frac{bz}{c}\right) + kbc\left(x - \frac{az}{c}\right) + kca\left(y - \frac{bz}{c}\right),$$

$$U_2\omega = 6a\left(x - \frac{az}{c}\right)^2 + 6b\left(y - \frac{bz}{c}\right)^2 + 2kc\left(x - a\frac{z}{c}\right)\left(y - \frac{bz}{c}\right),$$

$$U_3\omega = 6\left(x - \frac{az}{c}\right)^3 + 6\left(y - \frac{bz}{c}\right)^3,$$

and it will be found that the equations given by theorem (3) are satisfied, namely

$$\chi\omega = U\omega + 3\frac{z}{c}\omega,$$

$$\chi^2\omega = U_2\omega + 2.2\frac{z}{c}U\omega + 2.3\frac{z^2}{c^2}\omega,$$

$$\chi^3\omega = U_3\omega + 3\frac{z}{c}U_2\omega + 3.1.2\frac{z^2}{c^2}U\omega + 1.2.3\frac{z^3}{c^3}\omega.$$

Probably, as this theorem is of rather a novel character, the annexed sketch of a somewhat different course of demonstration may be not unacceptable to my readers.

We have

$$\chi\omega = \left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_n \frac{d}{da_n}\right)\omega;$$

and by the well-known law for homogeneous functions,

$$\iota\omega = \left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \dots + a_n \frac{d}{da_n}\right)\omega.$$

Hence

$$\left(\chi - \iota \frac{x_n}{a_n}\right) \omega = \left(u_1 \frac{d}{da_1} + u_2 \frac{d}{da_2} + \ldots + u_{n-1} \frac{d}{da_{n-1}}\right) \omega$$

$$= U \omega.$$

Hence

$$\chi \omega = \left(U + \iota \frac{x_n}{a_n}\right) \omega,$$

$$\chi^2 \omega = \left\{U + (\iota - 1) \frac{x_n}{a_n}\right\} \left(U + \iota \frac{x_n}{a_n}\right) \omega,$$

$$\chi^3 \omega = \left\{U + (\iota - 2) \frac{x_n}{a_n}\right\} \left\{U + (\iota - 1) \frac{x_n}{a_n}\right\} \left(U + \iota \frac{x_n}{a_n}\right) \omega,$$

&c. = &c.

But in performing the process indicated by the several factors it must be carefully borne in mind that UU_r is not $= U_{r+1}$; this would be the case were it not for the terms $-\frac{a_1}{a_n} x_n$, $-\frac{a_2}{a_n} x_n$, &c., which enter into $u_1, u_2 \ldots u_{n-1}$. But on account of these terms, we have

$$UU_r \omega = \left(u_1 \frac{d}{da_1} + u_2 \frac{d}{da_2} + \ldots + u_{n-1} \frac{d}{da_{n-1}}\right) \left(u_1 \frac{d}{da_1} + u_2 \frac{d}{da_2} + \ldots + u_{n-1} \frac{d}{da_{n-1}}\right)^r$$

$$= U_{r+1} \omega - r \frac{x_n}{a_n} \left\{u_1 \frac{d}{da_1} + u_2 \frac{d}{da_2} + \ldots + u_{n-1} \frac{d}{da_{n-1}}\right\}^{r-1} \omega,$$

for $$\frac{d}{da_1} u_1 = \frac{d}{da_2} u_2 = \ldots = \frac{d}{da_{n-1}} u_{n-1} = -\frac{x_n}{a_n}.$$

Hence

$$UU_r \omega = U_{r+1} \omega - r \frac{x_n}{a_n} U_r \omega.$$

Let $\frac{x_n}{a_n}$ be called ϵ; we find

$$\chi = U + \iota\epsilon,$$

$$\chi^2 = \{U + (\iota - 1) \epsilon\} (U + \iota\epsilon)$$

$$= UU + (2\iota - 1) \epsilon U + (\iota - 1) \iota\epsilon^2$$

$$= U_2 + 2 (\iota - 1) \epsilon U + (\iota - 1) \iota\epsilon^2;$$

$$\chi^3 = \{U + (\iota - 2) \epsilon\} \chi^2$$

$$= UU_2 + 2 (\iota - 1) \epsilon UU + (\iota - 1) \iota\epsilon^2 U$$

$$+ (\iota - 2) \epsilon U_2 + 2 (\iota - 2) (\iota - 1) \epsilon^2 U + (\iota - 2) (\iota - 1) \iota\epsilon^3$$

$$= U_3 + 3 (\iota - 2) \epsilon U_2 + 3 (\iota - 2) (\iota - 1) \epsilon^2 U + (\iota - 2) (\iota - 1) \iota\epsilon^3.$$

The same process being continued will lead to results identical with those previously obtained and expressed in theorem (3).

The expansion of χ^r, treated according to this second method, appears to require the solution of the partial equation in differences

$$a_{r+1,\,s+1} = a_{r,\,s+1} + (\iota - 2r)\,a_{r,\,s},$$

$a_{0,s}$ being given as unity for $s=1$ and as zero for all other values of s.

It is probable however that the solution of this equation might be evaded by some artifice peculiar to the particular case to be dealt with. I do not propose to dwell upon this inquiry, which would be foreign to the object of my present research. It may however not be out of place to make the passing remark, that the equations expressing χ^r in terms of powers of U admit easily of being reverted, as indeed may the more general form

$$\chi_r = u_r + \epsilon_r u_{r-1} + \frac{1}{1\,.\,2}\,\epsilon_r\epsilon_{r-1}u_{r-2} + \&c.$$

which becomes the equation of formula (3), on making

$$\epsilon_r = r(\iota + 1 - r)\frac{x_n}{a_n}, \quad \chi_r = \chi^r\omega, \text{ and } u_r = U_r\omega;$$

for let

$$u_r = \epsilon_1\epsilon_2 \ldots \epsilon_r v_r,$$

$$\chi_r = \epsilon_1\epsilon_2 \ldots \epsilon_r y_r,$$

then

$$y_r = v_r + v_{r-1} + \frac{v_{r-2}}{1\,.\,2} + \frac{v_{r-3}}{1\,.\,2\,.\,3} + \&c.;$$

whence

$$v_r = e^{-\frac{d}{dr}}y_r$$

$$= y_r - y_{r-1} + \frac{y_{r-2}}{1\,.\,2} - \frac{y_{r-3}}{1\,.\,2\,.\,3} + \&c.:$$

and therefore

$$u_r = \chi_r - \epsilon_r\chi_{r-1} + \tfrac{1}{2}\epsilon_r\epsilon_{r-1}\chi_{r-2} + \&c.$$

Thus we obtain, from equation (3),

$$U_r\omega = \chi^r\omega - r(\iota - r + 1)\,\chi^{r-1}\omega\,\frac{x_n}{a_n} + \&c.$$

As a first application of theorem (3), I shall proceed to show how Joachimsthal's equation to the surface drawn from a given point $(\alpha, \beta, \gamma, \delta)$ through the intersection of two surfaces $\phi(x, y, z, t) = 0$, $\theta(x, y, z, t) = 0$, may be expressed under the *explicit* form of the equation to a cone.

The equation in question is obtained by eliminating λ between

$$\phi\lambda^m + \chi\phi\lambda^{m-1} + \frac{1}{1\,.\,2}\chi^2\phi\lambda^{m-2} + \&c. = 0,$$

$$\theta\lambda^m + \chi\theta\lambda^{m-1} + \frac{1}{1\,.\,2}\chi^2\theta\lambda^{m-2} + \frac{1}{1\,.\,2\,.\,3}\chi^3\theta\lambda^{m-3} + \&c. = 0,$$

where

$$\phi = \phi(\alpha, \beta, \gamma, \delta), \quad \theta = \theta(\alpha, \beta, \gamma, \delta), \quad \chi = x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + t\frac{d}{d\delta}.$$

By theorem (3), these two equations, on writing $\frac{x_n}{a_n} = \epsilon$, become

$$\phi\lambda^m + \{U\phi + m\phi\epsilon\}\lambda^{m-1} + \{U^2\phi + 2(m-1)U\phi\epsilon + (m-1)m\phi\epsilon^2\}\frac{\lambda^{m-2}}{1.2} + \&c. = 0,$$

$$\theta\lambda^n + \{U\theta + n\theta\epsilon\}\lambda^{n-1} + (U^2\theta + \&c.)\frac{\lambda^{n-2}}{1.2} + \{U^3\theta + 3(n-2)U^2\theta\epsilon + 3(n-2)(n-1)U\theta\epsilon^2 + (n-2)(n-1)n\epsilon^3\}\frac{\lambda^{n-3}}{1.2.3} + \&c.$$

Now on writing $\lambda = \mu - \epsilon$, these equations take the forms

$$\phi\mu^m + U\phi\mu^{m-1} + U^2\phi\frac{\mu^{m-2}}{1.2} + \&c. = 0,$$

$$\theta\mu^n + U\theta\mu^{n-1} + U^2\theta\frac{\mu^{n-2}}{1.2} + \&c. = 0,$$

as is easily seen by substituting back $\lambda + \epsilon$ in place of μ. Consequently ϵ no longer appears in the coefficients of the terms of the equations between which the elimination is to be performed, and the resultant will accordingly come out as a function only of ϕ, $U\phi$, $U^2\phi$, &c., that is, of α, β, γ, δ, and of

$$x - \frac{\alpha}{\delta}t, \quad y - \frac{\beta}{\delta}t, \quad z - \frac{\gamma}{\delta}t,$$

showing that the equation in x, y, z, t, is of the form of that to a cone, as we know *à priori* it ought to be. Precisely a similar method may be applied to the elucidation of the corresponding theorem for a system of rays drawn from a given point through the locus of the intersection of two curves.

Before entering upon some further and more interesting applications of theorem (3), it will be convenient to explain a nomenclature which has been employed by me on another occasion, and which is almost indispensable in inquiries of the nature we are now engaged upon. Homogeneous functions may be characterized by their degree, by the number of letters which enter into them, and lastly, by the lowest number of linear functions of the letters which may be introduced in place of the letters to represent such functions. Any such linear function I designate as an order, and am now able to discriminate between the number of letters and the number of orders which enter into a given function. The latter number, *generally* speaking, is the same as the former; it can never exceed it, but *may* be any number of units less than it.

I need scarcely observe that a pair of points becoming coincident, a conic becoming a pair of lines, a conoid becoming a cone, and so forth, for the higher realms of space, will be expressed by the homogeneous function of the second order which characterizes such loci*, losing one order, that is, having an order less than the number of letters entering therein. Calling such characteristic $\phi(x, y, z \ldots t)$, it is well known that the condition of such loss of an order is the vanishing of the determinant

$$\begin{vmatrix} \dfrac{d^2\phi}{dx^2}, & \dfrac{d^2\phi}{dxdy} & \cdots & \dfrac{d^2\phi}{dxdt} \\ \dfrac{d^2\phi}{dydx}, & \dfrac{d^2\phi}{dy^2} & \cdots & \dfrac{d^2\phi}{dydt} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{d^2\phi}{dtdx}, & \dfrac{d^2\phi}{dtdy} & \cdots & \dfrac{d^2\phi}{dt^2} \end{vmatrix}.$$

A conoid becoming a pair of planes, a cone becoming a pair of coincident lines, a pair of points becoming indeterminate, will, in like manner, be denoted by their characteristic losing two orders, and so forth, for the higher degrees of degradation. In like manner, in general, a homogeneous function of three letters of any degree losing an order, typifies that the locus to which it is the characteristic will break up into a system of right lines.

Now let ω be a homogeneous function of $\alpha, \beta, \gamma \ldots \delta$, and suppose that we have the equations $\omega = 0$, $\chi\omega = 0$, $\chi^2\omega = 0$, where χ as above

$$= x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + \ldots + t\frac{d}{d\delta}.$$

I say that on eliminating any of the variables $x, y, z \ldots t$ between the second and third of the above equations, the resulting equation will be of one order less than the number of letters, that is, the expulsion of one letter will be attended by the expulsion of *two* orders.

For we have, by theorem (3),

$$\chi\omega = U\omega + 2\frac{x_n}{a_n}\omega = 0,$$

$$\chi^2\omega = U_2\omega + 2\frac{x_n}{a_n}U\omega + 2\left(\frac{x_n}{a_n}\right)^2\omega = 0,$$

and by hypothesis

$$\omega = 0.$$

Hence we have also

$$U\omega = 0,$$
$$U_2\omega = 0;$$

and since $U\omega$, $U_2\omega$ contain one order less than the number of letters in

* If $U=0$ is the equation to any locus, U may be said to *characterize* the same, or to be its characteristic.

ω, the resultant of the elimination between them will contain two orders less than the number of letters in ω; and consequently, whichever of the letters $x, y, z \dots t$ we eliminate between $\chi\omega = 0$ and $\chi^2\omega = 0$, provided that $\omega = 0$, the resultant equation will contain one order less than the number of letters remaining.

Thus we see how it is that the tangent line to a conic meets it in two coincident points, the tangent plane to a conoid in two intersecting lines, and so forth, for the higher regions of space*. For instance, if we take $\omega(x, y, z, t) = 0$, the equation to a conoid, and $\alpha, \beta, \gamma, \delta$, the coordinates to any point therein, we shall have $\omega(\alpha, \beta, \gamma, \delta) = 0$,

$$\left(x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + t\frac{d}{d\delta}\right)\omega, \text{ that is, } \chi\omega = 0,$$

and $$\omega(x, y, z, t), \text{ that is, } \chi^2\omega = 0,$$

x, y, z, t representing the coordinates of any point in the intersection of the conoid by the tangent plane.

Consequently, by what has been shown above, on eliminating any one of the four letters x, y, z, t, the resultant function of three letters will contain only two orders, and will thus represent a pair of lines, real or imaginary, intersecting one another at $\alpha, \beta, \gamma, \delta$.

The fact which has just been demonstrated (that the resultant of $\chi\omega = 0$, $\chi^2\omega = 0$, loses an order if $\omega = 0$), indicates that on expressing one of the quantities $x, y, z \dots t$ in terms of the others, by means of the first equation, and then substituting this value in the second, the determinant of the equation so obtained must be zero.

Now by virtue of a theorem which was given by me in a note† to my paper in the preceding number of this *Journal*, this determinant will be equal to the squared reciprocal of the coefficient in the equation $\chi\omega = 0$ of the letter eliminated multiplied by the determinant in respect to $x, y, z \dots t, \lambda$ of

$$\chi^2\omega + \chi\omega\lambda.$$

This latter determinant is therefore zero; but this determinant is the resultant of the equations

$$\left.\begin{array}{l}
\dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \text{\&c.}\right)^2\omega + \dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \dots\right)\omega = 0, \\
\dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \text{\&c.}\right)^2\omega + \dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \dots\right)\omega = 0, \\
\qquad \text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.} \\
\qquad\qquad \chi\omega = 0, \text{ that is, } \left(x\dfrac{d}{da} + y\dfrac{d}{db} + \dots\right)\omega = 0,
\end{array}\right\}.$$

* Thus a tangential section of a hyperlocus of the second degree at any point cuts it in two cones.

[† p. 135 above.]

Thus we obtain the singular law, that the symmetrical determinant

$$\left|\begin{matrix} \frac{d}{da}\frac{d}{da}\omega, & \frac{d}{da}\frac{d}{db}\omega, \cdots & \frac{d}{da}\frac{d}{dl}\omega, & \frac{d}{da}\omega \\ \frac{d}{db}\frac{d}{da}\omega, & \frac{d}{db}\frac{d}{db}\omega, \cdots & \frac{d}{db}\frac{d}{dl}\omega, & \frac{d}{db}\omega \\ \frac{d}{dc}\frac{d}{da}\omega, & \frac{d}{dc}\frac{d}{db}\omega, \cdots & \frac{d}{dc}\frac{d}{dl}\omega, & \frac{d}{dc}\omega \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d}{dl}\frac{d}{da}\omega, & \frac{d}{dl}\frac{d}{db}\omega, \cdots & \frac{d}{dl}\frac{d}{dl}\omega, & \frac{d}{dl}\omega \\ \frac{d}{da}\omega, & \frac{d}{db}\omega, \cdots & \frac{d}{dl}\omega, & 0 \end{matrix}\right|$$

is zero when ω is zero.

This is easily shown independently by means of a remarkable and I believe novel theorem, relative to homogeneous functions.

If ω be any homogeneous function of ι dimensions of $a, b, c \ldots l$, we have (by Euler's theorem already repeatedly applied), remembering that $\frac{d\omega}{da}, \frac{d\omega}{db} \cdots \frac{d\omega}{dl}$ are all homogeneous,

$$-\iota\omega + \left(a\frac{d}{da} + b\frac{d}{db} + \ldots\ldots\ldots\ldots\ldots + l\frac{d}{dl}\right)\omega = 0,$$

$$-(\iota-1)\frac{d\omega}{da} + \left(a\frac{d}{da}\frac{d}{da} + b\frac{d}{da}\frac{d}{db} + \ldots + l\frac{d}{da}\frac{d}{dl}\right)\omega = 0,$$

$$-(\iota-1)\frac{d\omega}{db} + \left(a\frac{d}{db}\frac{d}{da} + \ldots\ldots\ldots\ldots\ldots + l\frac{d}{db}\frac{d}{dl}\right)\omega = 0,$$

&c. &c. &c.

$$-(\iota-1)\frac{d\omega}{dl} + \left(a\frac{d}{dl}\frac{d}{da} + \ldots\ldots\ldots\ldots\ldots\ldots + l\frac{d}{dl}\frac{d}{dl}\right)\omega = 0.$$

Between these equations we may eliminate *all the letters*, $a, b, c \ldots l$, and we obtain the equation

$$\left|\begin{matrix} \frac{d}{da}\frac{d}{da}\omega, & \frac{d}{da}\frac{d}{db}\omega, \cdots & \frac{d}{da}\frac{d}{dl}\omega, & \frac{d}{da}\omega \\ \frac{d}{db}\frac{d}{da}\omega, & \frac{d}{db}\frac{d}{db}\omega, \cdots & \frac{d}{db}\frac{d}{dl}\omega, & \frac{d}{db}\omega \\ \frac{d}{dc}\frac{d}{da}\omega, & \frac{d}{dc}\frac{d}{db}\omega, \cdots & \frac{d}{dc}\frac{d}{dl}\omega, & \frac{d}{dc}\omega \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d}{dl}\frac{d}{da}\omega, & \frac{d}{dl}\frac{d}{db}\omega, \cdots & \frac{d}{dl}\frac{d}{dl}\omega, & \frac{d}{dl}\omega \\ \frac{d}{da}\omega, & \frac{d}{db}\omega, \cdots & \frac{d}{dl}\omega, & \frac{\iota}{\iota-1}\omega \end{matrix}\right| = 0.$$

As a corollary to this theorem, we see that if $\omega = 0$ the determinant obtained in the previous investigation becomes zero, agreeing with what has been already shown; in fact the last-named determinant is always equal to

$$\frac{\iota - 1}{\iota}\,\omega \times \begin{vmatrix} \frac{d}{da}\frac{d}{da}\omega, & \frac{d}{da}\frac{d}{db}\omega, \ldots & \frac{d}{da}\frac{d}{dl}\omega \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{d}{dl}\frac{d}{da}\omega, & \frac{d}{dl}\frac{d}{db}\omega, \ldots & \frac{d}{dl}\frac{d}{dl}\omega \end{vmatrix}.$$

This remarkable theorem, which I have communicated to friends nearly a twelvemonth back, is here, I believe, published for the first time*.

Suppose next that $\omega(x, y, z)$ is the characteristic of a line of any degree, to which a tangent is drawn at the point α, β, γ, using U in a manner correspondent to its previous signification to denote

$$\left(x - \frac{\alpha}{\gamma}z\right)\frac{d}{d\alpha} + \left(y - \frac{\beta}{\gamma}z\right)\frac{d}{d\beta},$$

and understanding $\omega(\alpha, \beta, \gamma)$ by ω, we have for determining the point of intersection, $\omega = 0$, $\chi\omega = 0$, $\chi^n\omega = 0$; and consequently, by aid of our theorem (3), we shall obtain

$$\omega = 0,$$

$$U\omega = 0,$$

$$U_n\omega + n\frac{z}{\gamma}U_{n-1}\omega + \ldots = 0.$$

By means of the two latter equations, we obtain

$$\left(x - \frac{\alpha z}{\gamma}\right)^2 F\left(x - \frac{\alpha z}{\gamma}\right) = 0,$$

$$\left(y - \frac{\beta z}{\gamma}\right)^2 G\left(y - \frac{\beta z}{\gamma}\right) = 0,$$

* Thus let z be a homogeneous function in x and y of ι dimensions, and let

$$\frac{dz}{dx}, \quad \frac{dz}{dy}, \quad \frac{d^2z}{dx^2}, \quad \frac{d^2z}{dx\,dy}, \quad \frac{d^2z}{dy^2},$$

be called p, q, r, s, t; we shall find

$$\begin{vmatrix} r, & s, & p \\ s, & t, & q \\ p, & q, & \frac{\iota}{\iota - 1}\omega \end{vmatrix} = 0,$$

that is, $$\omega = \frac{\iota - 1}{\iota}\,\frac{rq^2 - 2pqs + tp^2}{rt - s^2}.$$

where F and G are each of only $(n-2)$ dimensions, and serve to determine the intersections of the tangent with the curve, extraneous to the two coincident ones at the point of contact.

Again, suppose that ω is a function of any degree of any number of letters α, β, γ, &c., and that we have given $\omega = 0$, $\chi\omega = 0$, $\chi^2\omega = 0, \ldots \chi^m\omega = 0$; it is evident from our fundamental theorem that these equations may be replaced by

$$\omega = 0, \quad U_1\omega = 0, \quad U_2\omega = 0, \ldots \; U_m\omega = 0;$$

and consequently that the expulsion of $(m-1)$ letters, by aid of the last m of the given equations, will be attended by the disappearance of m orders, or, in other words, the resultant will be minus an order, that is, will have one order less than the number of letters remaining in it.

In applying to space conceptions the preceding theorem, it will be convenient to use a general nomenclature for geometrical species of various dimensions.

Thus we may call a line a monotheme, a surface a ditheme, the species beyond a tritheme, and so on, *ad infinitum.*

A system of points according to the same system of nomenclature would be called a kenotheme.

An n-theme has for its characteristic a homogeneous function of $(n+2)$ letters.

Again, it will be convenient to give a general name to all themes expressed by equations of the first degree. Right lines and planes agree in conveying an idea of levelness and uniformity; they may both be said to be homalous. I shall therefore employ the word homaloid to signify in general any theme of the first degree.

Now let $\omega(x, y, z \ldots t)$ be the characteristic to an n-theme of the nth degree.

The number of letters $x, y, z \ldots t$ is $(n+2)$.

As usual, let ω represent $\omega(\alpha, \beta, \gamma \ldots \delta)$, and suppose

$$\omega = 0, \quad \chi\omega = 0, \quad \chi^2\omega = 0 \ldots \chi^n\omega = 0,$$

and consequently

$$U_1\omega = 0, \quad U_2\omega = 0 \ldots U_n\omega = 0.$$

On eliminating $(n-1)$ letters between the n last equations, the resulting function will be of three letters but of only two orders, and of the $1 \cdot 2 \cdot 3 \ldots n$ degree. Hence we see that at every point of an n-theme of the nth degree, and lying in the tangent homaloid thereto, $1 . 2 \ldots . n$ right lines may be drawn coinciding throughout with the n-theme.

Thus one right line can be drawn at each point of a line of the first order lying on the line; two right lines at each point of a surface of the second order lying on the surface; six right lines at each point of a hyperlocus of the third degree, and so forth.

It is obvious that a surface may be treated as the homaloidal section of a tritheme, just as a plane curve may be regarded as a section of a surface. I shall proceed to show upon this view, how we may obtain a theorem given by Mr Salmon for surfaces of the third degree of a particular character from the law just laid down, according to which a tritheme of the third degree admits of six right lines being drawn upon it at every point*.

Let $\omega(x, y, z, t, u)$ be the characteristic of any tritheme of the third degree; $\alpha, \beta, \gamma, \delta, \epsilon$, coordinates to any point in the same. Then $\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, and the equation to the tangent homaloid will be $\chi\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, and the equation to the polar of the second degree to the given tritheme in relation to the assumed point as origin, (that is, the infinite system of homaloids that may be drawn from the point to touch the tritheme), will be $\chi^2\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$.

But the section of any polar through its origin is the polar of the section to the same origin; hence the polar to the intersection of $\omega(x, y, z, t, u) = 0$, with $\chi\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, is the intersection of $\chi\omega = 0$ with $\chi^2\omega = 0$.

The projections of these intersections upon the space x, y, z, t will be found by eliminating u, and getting the correspondent two equations between x, y, z, t. Hence we see that the projection of the latter intersection upon any space x, y, z, t is a cone; or, in other words, this intersection itself, that is, the polar to the intersection of the tritheme with its tangent homaloid, is a cone; that is to say, the surface of the third degree formed by cutting a tritheme of the third degree by any tangent homaloid has a conical point at the point of contact; so that every surface of the third degree with a conical point may be considered as the intersection of a tritheme of the third degree with any tangent homaloid thereto†.

* The reduction of any equation of the sixth degree to depend upon one of the fifth may be shown by Mr Jerrard's method to be equivalent to drawing a straight line upon a tritheme of the third degree, just as the reduction of the equation of the fifth degree to a trinomial form may be shown to be dependent upon our being able to draw a straight line upon a ditheme of the second degree. Now at every point of a tritheme straight lines may be drawn, but as they keep together in groups of sixes they cannot be found *in general* at a given point without solving an equation of the sixth degree.

† So in like manner a surface of the third degree with more than one conical point may be generated by the intersection of the tritheme with a pluri-tangent plane; and so too we may get other varieties by taking homaloidal sections of trithemes whose characteristics are minus one or more orders.

Hence then we see, as an instantaneous deduction from our general theorem, that at any conical point (when one exists) of a surface of the third degree *six* right lines may be drawn lying completely upon it. This theorem is thus brought into an immediate and natural connexion with the well-known one, that at *every* point in a surface of the second degree, *two* right lines can be drawn lying wholly upon the surface*.

The last geometrical application of the theorem (3) which I shall make, refers to the equations employed by Mr Salmon in No. XXI. (New Series) of this *Journal*, to obtain the locus of the points on any surface at which tangent lines can be drawn passing through four consecutive points. I may remark in passing that these equations may be obtained by rather simpler considerations than Mr Salmon has employed so to do, and without any reference to Joachimsthal's theorem; for if we take ξ, η, ζ, θ, as the coördinates of any point in one of the tangent lines above described, and if we take the first polar to the surface with this point as origin, three out of the four original points will be found in such polar consecutive but distinct; and consequently in the second polar, referred to the same origin, two will continue consecutive but distinct, and consequently one will remain over in the third polar.

Hence writing the equation to the surface $\omega(x, y, z, t) = 0$, and using D to denote $\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz} + \theta \frac{d}{dt}$, we shall evidently have

$$\omega = 0, \qquad (1)$$

$$D\omega = 0, \qquad (2)$$

$$D^2\omega = 0, \qquad (3)$$

$$D^3\omega = 0, \qquad (4)$$

as obtained by Mr Salmon. And the same kind of reasoning precisely applies to the theory of points of inflexion in curves; three consecutive points in a right line in this case corresponding to four such in the case above considered.

If now we make

$$\xi - \frac{x}{t}\theta = u,$$

$$\eta - \frac{y}{t}\theta = v,$$

$$\zeta - \frac{z}{t}\theta = w,$$

* If we have an indeterminate system of algebraical equations consisting of one quadratic and another n^c function of three variables, this may be completely resolved by considering the first as an equation to a surface of the second degree, finding at any point thereof the two lines which lie upon the surface, and determining their respective intersections with the surface represented by the second equation. This will require therefore the solution only of a quadratic and an n^c equation. In like manner an indeterminate system of two equations of four variables, one of the third and the other of the nth degree, may be completely resolved (with the aid of the theorem in the text) by means of two equations, one of the sixth and the other of the nth degree.

the equations (2), (3), (4), by our theorem, may be expressed in terms of u, v, w, which being eliminated we obtain an equation between x, y, z, t, which will express the surface whose intersection with the given surface $\omega = 0$ serves to determine the locus of the points in question.

Hence if we proceed in the ordinary manner to eliminate two of the four letters, as ξ and η, between the equations (2), (3), (4), the resultant will be of the form $M \times \phi(\zeta, \theta)$, where M does not contain ξ, η, ζ or θ, and where by the general laws of elimination $\phi(\zeta, \theta)$ will be an integral function of the sixth degree in respect to ζ, θ: and it is manifest that $M \times \phi(\zeta, \theta)$ will be identical with the resultant of (2), (3), (4) expressed in terms of u, v, w, when u and v are eliminated *cy-près* of an integralizing factor, showing that $\phi(\zeta, \theta)$ is w^6 integralized, that is, is equal to $(t\zeta - z\theta)^6$. Consequently as $M\phi$ is of the order $(n-1)\,2.3 + (n-2)\,1.3 + (n-3)\,1.2$, that is, $11n - 18$ in respect to x, y, z, t, it follows that $M = 0$, the equation to the second surface spoken of above, will be of the order $11n - 24$, agreeable to Mr Salmon's showing.

I shall conclude this paper by showing the application of our theorem to the subject propounded by Mr Jerrard and Sir William Hamilton, of systems of equations containing a sufficient number of variable letters for effecting the solution without elevation of degree.

If we have n homogeneous equations containing a sufficient number of letters $a_1, a_2 \ldots a_m$ to enable us to express the solution of $(n-1)$ of the equations under the form

$$\begin{aligned} a_1 &= \alpha_1 + \lambda\beta_1, \\ a_2 &= \alpha_2 + \lambda\beta_2, \\ &\ldots\ldots\ldots\ldots \\ a_m &= \alpha_m + \lambda\beta_m, \end{aligned}$$

where $\alpha_1, \alpha_2 \ldots \alpha_m$, $\beta_1, \beta_2 \ldots \beta_m$ are supposed known, and λ is indeterminate, it is evident that by substituting these values in the nth equation, λ may be found by solving an equation of the same degree as that equation contains dimensions of $a_1, a_2 \ldots a_m$.

Let us then propose this question: how many letters $a_1, a_2 \ldots a_r$ are needed to obtain a linear solution of a system of n equations

$$\phi_1 = 0,\ \phi_2 = 0, \ldots \phi_n = 0,$$

of the several degrees $\iota_1, \iota_2 \ldots \iota_n$, without elevation of degree; by a linear solution being understood a solution under the form

$$\begin{aligned} a_1 &= \alpha_1 + \lambda\beta_1, \\ a_2 &= \alpha_2 + \lambda\beta_2, \\ &\ldots\ldots\ldots\ldots \\ a_r &= \alpha_r + \lambda\beta_r, \end{aligned}$$

where λ is left indeterminate.

Let us suppose that $\alpha_1, \alpha_2 \dots \alpha_r$, substituted respectively for $a_1, a_2 \dots a_r$, satisfy the given system of equations. The determination of these values without elevation of degree will, from what has been said before, depend upon the linear solution of a system of equations differing from the given system by the omission of any one of them at pleasure.

Now make

$$D = \alpha_1 \frac{d}{da_1} + \alpha_2 \frac{d}{da_2} + \dots + \alpha_r \frac{d}{da_r},$$

and then write

$$\left.\begin{array}{l} D\phi_1 = 0,\ D^2\phi_1 = 0 \dots D^{\iota_1}\phi_1 = 0 \\ D\phi_2 = 0,\ D^2\phi_2 = 0 \dots D^{\iota_2}\phi_2 = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ D\phi_n = 0,\ D^2\phi_n = 0 \dots D^{\iota_n}\phi_n = 0 \end{array}\right\}. \qquad (\theta)$$

The values of $a_1, a_2 \dots a_r$ derived from this system, say $(a)_1, (a)_2 \dots (a)_r$, give

$$a_1 = \alpha_1 + \lambda(a)_1,\ a_2 = \alpha_2 + \lambda(a)_2, \dots a_r = \alpha_r + \lambda(a)_r,$$

a solution under the required form, where λ is left indeterminate.

The solution of this new system without elevation of degree depends on the *linear solution* of all but one of them; this excepted one may be taken the one whose dimensions ι_r are the highest or as high as any of the quantities $\iota_1, \iota_2 \dots \iota_n$.

Consequently, if we use the symbol $(k_1, k_2 \dots k_r)$ to denote the number of letters required for the linear solution (without elevation of degree) of k_1 equations of the first degree, k_2 of the second, k_3 of the third, ..., k_r of the rth, it would at first sight appear from the preceding reduction that we must have

$$(k_1, k_2 \dots k_r) = \{K_1, K_2 \dots K_{r-1}, K_r'\},$$

where

$$\begin{aligned} K_1 &= k_1 + k_2 + \dots + k_{r-1} + k_r, \\ K_2 &= k_2 + \dots + k_{r-1} + k_r, \\ &\dots\dots\dots\dots\dots\dots \\ K_{r-1} &= k_{r-1} + k_r, \\ K_r' &= k_r - 1. \end{aligned}$$

But now steps in our theorem (3), and shows that the system (θ) may be superseded by another, in which the variables, instead of being $a_1, a_2 \dots a_n$, will be

$$a_1 - \frac{\alpha_1}{\alpha_n} a_n,\ a_2 - \frac{\alpha_2}{\alpha_n} a_n, \dots a_{n-1} - \frac{\alpha_{n-1}}{\alpha_n} a_n;$$

consequently the number of really independent variables is only $(n-1)$; we must therefore have

$$(k_1, k_2 \dots k_r) = 1 + \{K_1, K_2 \dots K_r'\}.$$

Since the introduction of a new simple equation is equivalent to the requirement of one more disposable letter, we may write the above more symmetrically under the form

$$(k_1, k_2 \dots k_r) = ('K_1, K_2 \dots K_{r-1}, K_r'),$$

where

$$'K_1 = 1 + k_1 + k_2 + \dots + k_r,$$

$$K_r' = \qquad k_r - 1.$$

By means of this formula of reduction $(k_1, k_2 \dots k_r)$ may be finally brought down to the form (L), and the value of (L) being the number of letters required for the linear solution of a system of L linear equations is evidently $L+2$.

Thus, to determine the number of letters required for the linear solution of a single quadratic, we write

$$(0, 1) = (2) = 4.$$

For two quadratics, we write

$$(0, 2) = (3, 1) = (5) = 7;$$

for a quadratic and a cubic,

$$(0, 1, 1) = (3, 2) = (6, 1) = (8) = 10;$$

for two cubics,

$$(0, 0, 2) = (3, 2, 1) = (7, 3) = (11, 2) = (14, 1) = (16) = 18.$$

These results coincide with those obtained by Sir William Hamilton in his Report on Mr Jerrard's Transformation of the Equation of the Fifth Degree in the *Transactions* of the British Association. I have much more to say on the subject of the linear solution of a system of indeterminate equations, and am, I believe, able to present the subject in a more general light than has hitherto been done; but my observations on this matter must be deferred until a subsequent communication.