26.

ON THE SOLUTION OF A SYSTEM OF EQUATIONS IN WHICH THREE HOMOGENEOUS QUADRATIC FUNCTIONS OF THREE UNKNOWN QUANTITIES ARE RESPECTIVELY EQUATED TO NUMERICAL MULTIPLES OF A FOURTH NON-HOMOGENEOUS FUNCTION OF THE SAME.

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LET U, V, W be three homogeneous quadratic functions of x, y, z, and let ω be any function of x, y, z of the nth degree, and suppose that there is given for solution the system of equations

$$U = A\omega,$$
$$V = B\omega,$$
$$W = C\omega.$$

Theorem. The above system can be solved by the solution of a cubic equation, and an equation of the nth degree.

For let D be the determinant in respect to x, y, z of

$$fU + gV + hW,$$

then D is a cubic function of f, g, h. Now make

$$D = 0, \quad Af + Bg + Ch = 0;$$

the ratios of $f : g : h$ which satisfy the last two equations can be determined by the solution of a cubic equation, and there will accordingly be three systems of f, g, h which satisfy the same, as

$$f_1, \quad g_1, \quad h_1,$$
$$f_2, \quad g_2, \quad h_2,$$
$$f_3, \quad g_3, \quad h_3.$$

Now $D = 0$ implies that $fU + gV + hW$ breaks up into two linear factors; accordingly we shall find

$$(l_1x + m_1y + n_1z)(\lambda_1x + \mu_1y + \nu_1z) = 0,$$
$$(l_2x + m_2y + n_2z)(\lambda_2x + \mu_2y + \nu_2z) = 0,$$
$$(l_3x + m_3y + n_3z)(\lambda_3x + \mu_3y + \nu_3z) = 0,$$

in which the several sets of l, m, n; λ, μ, ν can be expressed without difficulty in terms of the several values of $\sqrt{f}, \sqrt{g}, \sqrt{h}$.

Let the above equations be written under the form

$$PP' = 0,$$
$$QQ' = 0,$$
$$RR' = 0.$$

Since the given equations are perfectly general, it is readily seen that the equations

$$(P = 0,\ P' = 0), \quad (Q = 0,\ Q' = 0), \quad (R = 0,\ R' = 0),$$

will severally represent pairs of opposite sides of a quadrangle expressed by general coordinates x, y, z; so that one of the two functions R, R' will be a linear function of P and Q and also of P' and Q', and the other will be a linear function of P and Q' and also of P' and Q*.

In order to solve the equations, we need only consider two such pairs as $PP' = 0$, $QQ' = 0$; we then make

$$P = 0, \quad Q = 0,$$

or

$$P = 0, \quad Q' = 0,$$

or

$$P' = 0, \quad Q = 0,$$

or

$$P' = 0, \quad Q' = 0.$$

Any one of these four systems will give the ratios of $x:y:z$; and then, by substitution in any one of the given equations, we obtain the values of x, y, z by the solution of an ordinary equation of the nth degree. The number of systems x, y, z is therefore always $4n$.

The equations connected with the solution of Malfatti's celebrated problem, "In a given triangle to inscribe three circles such that each circle touches the remaining two circles and also two sides of the triangle," given by Mr Cayley in the November Number for 1849 of the *Cambridge and Dublin Mathematical Journal*, to wit,

$$by^2 + cz^2 + 2fyz = \theta^2 a\,(bc - f^2) = A,$$
$$cz^2 + ax^2 + 2gzx = \theta^2 b\,(ca - g^2) = B,$$
$$ax^2 + by^2 + 2hxy = \theta^2 c\,(ab - h^2) = C,$$

come under the general form which has just been solved. It so *happens*, however, that in this particular case

$$\left.\begin{matrix} f_1, & g_1, & h_1 \\ f_2, & g_2, & h_2 \\ f_3, & g_3, & h_3 \end{matrix}\right\},$$

* Were it not for this being the case, the number of solutions would be n times the number of ways of obtaining duads out of three sets of two things, excluding the duads forming the sets, that is, the number of solutions would be $12n$ in place of $4n$, the true number.

become respectively

$$\left.\begin{matrix} 0, & \frac{1}{B}, & -\frac{1}{C} \\ -\frac{1}{B}, & 0, & \frac{1}{C} \\ -\frac{1}{C}, & \frac{1}{B}, & 0 \end{matrix}\right\},$$

and the cubic equation is resolved without extraction of roots.

It follows from my theorem that the eight intersections of three concentric surfaces of the second order can be found by the solution of one cubic and one quadratic equation; and in general, if we have ϕ, ψ, θ any three quadratic functions of x, y, z, and $\phi = 0$, $\psi = 0$, $\theta = 0$ be the system of equations to be solved, provided that we can by linear transformations express ϕ, ψ, θ under the form of

$$U - aw,$$
$$V - bw,$$
$$W - cw,$$

U, V, W being homogeneous functions, and w a non-homogeneous function of three new variables, x', y', z', we can find the eight points of intersection of the three surfaces, of which U, V, W are the characteristics, by the solution of one cubic and one quadratic. But (as I am indebted to Mr Cayley for remarking to me) that this may be possible, implies the coincidence of the vertices of one cone of each of the systems of four cones in which the intersections of the three surfaces taken two and two are contained.

I may perhaps enter further hereafter into the discussion of this elegant little theory. At present I shall only remark, that a somewhat analogous mode of solution is applicable to two equations,

$$U = aP^2,$$
$$V = bP^2,$$

in which U, V are homogeneous quadratic functions, and P some non-homogeneous function of x, y.

We have only to make the determinant of $fU + gV$ equal to zero, and we shall obtain two systems of values of f, g, wherefrom we derive

$$l_1 x + m_1 y = \pm \sqrt{(af_1 + bg_1)}\, P,$$
$$l_2 x + m_2 y = \pm \sqrt{(af_2 + bg_2)}\, P,$$

from which x and y may be determined.