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ON A NEW CLASS OF THEOREMS IN ELIMINATION BETWEEN QUADRATIC FUNCTIONS.

[*Philosophical Magazine*, XXXVII. (1850), pp. 213—218.]

IN a forthcoming memoir on determinants and quadratic functions, I have demonstrated the following remarkable theorem as a particular case of one much more general, also there given and demonstrated.

Let U and V be respectively quadratic functions of the same $2n$ letters, and let it be supposed possible to institute n such linear equations between these letters as shall make U and V both simultaneously become identically zero*. Then the determinant of $\lambda U + \mu V$, which is of course a function of λ and μ of the $2n$th degree, will become the *square* of a function of λ and μ of the nth degree; and conversely, if this determinant be a perfect square, U and V may be made to vanish simultaneously by the institution of n linear equations between the $2n$ letters†.

Let now P and Q be respectively quadratic functions of three letters only, say x, y, z; and let

$$U = P + (lx + my + nz)\,t,$$

$$V = Q + k\,(lx + my + nz)\,t.$$

The determinant of $\lambda U + \mu V$ in respect to x, y, z, t is easily seen to be $(\lambda + k\mu)^2 \times$ the determinant of

$$\lambda P + \mu Q + (lx + my + nz)\,t$$

in respect to x, y, z, t. Hence if we call

$$\lambda P + \mu Q + (lx + my + nz)\,t = W,$$

and make $\underset{xyzt}{\square} W$ a squared function of λ, μ or which is the same thing, if

$$\underset{\lambda\mu}{\square}\;\underset{xyzt}{\square}\{W\} = 0,$$

* In the more general theorem above alluded to, the number of letters is any number m, the number of linear equations being any number not exceeding $\frac{m}{2}$.

† When $n=1$, we obtain a theorem of elimination between two quadratics, which has been already given by Professor Boole.

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U and V will vanish simultaneously when two linear relations are instituted between the quantities (all or some of them) x, y, z, t.

In order that this may be the case, it will be seen to be sufficient that

$$P = 0, \quad Q = 0, \quad (lx + my + nz) = 0,$$

shall coexist; for then two equations between x, y, z of which $lx+my+nz=0$ will be one, will suffice to make U and V each identically zero. Hence we have the following theorem:

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square}\{\lambda U + \mu V + (lx + my + nz)\,t\}$$

is a factor of the resultant of

$$P = 0, \quad Q = 0, \quad lx + my + nz = 0.$$

A comparison of the orders of the resultant and the determinant shows that they must be identical, *à-ci-près*, of a numerical factor, which, if the resultant be taken in its *general* lowest terms, may no doubt be easily shown to be unity.

As an illustration of our theorem, let

$$P = xy + yz + zx,$$
$$Q = cxy + ayz + bzx.$$

Then

$$\underset{xyzt}{\square}\{\lambda P + \mu Q + (lx + my + nz)\,t\} = \begin{vmatrix} 0, & \lambda + c\mu, & \lambda + b\mu, & l \\ \lambda + c\mu, & 0, & \lambda + a\mu, & m \\ \lambda + b\mu, & \lambda + a\mu, & 0, & n \\ l, & m, & n, & 0 \end{vmatrix}$$

$$= n^2(\lambda + c\mu)^2 + m^2(\lambda + b\mu)^2 + l^2(\lambda + a\mu)^2$$
$$- 2lm(\lambda + b\mu)(\lambda + a\mu) - 2mn(\lambda + c\mu)(\lambda + b\mu) - 2nl(\lambda + a\mu)(\lambda + c\mu)$$
$$= \lambda^2\{n^2 + m^2 + l^2 - 2lm - 2mn - 2nl\}$$
$$+ 2\lambda\mu\{cn^2 + bm^2 + al^2 - lm(a+b) - mn(b+c) - nl(c+a)\}$$
$$+ \mu^2\{c^2n^2 + b^2m^2 + a^2l^2 - 2ablm - 2bcmn - 2canl\}.$$

And we thus obtain, finally,

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square}\{\lambda P + \mu Q + (lx + my + nz)\,t\}$$
$$= (n^2 + m^2 + l^2 - 2lm - 2mn - 2nl)$$
$$\times (c^2n^2 + b^2m^2 + a^2l^2 - 2ablm - 2bcmn - 2canl)$$
$$- \{(cn^2 + bm^2 + al^2 - lm(a+b) - mn(b+c) - nl(c+a)\}^2$$
$$= -4lmn\{(a-b)(a-c)\,l + (b-a)(b-c)\,m + (c-a)(c-b)\,n\}.$$

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Now to obtain the resultant of

$$xy + yz + zx = 0,$$
$$cxy + azy + bxz = 0,$$
$$lx + my + nz = 0,$$

we need only find the four systems *in their lowest terms* of $x : y : z$, which satisfy the first two equations, and multiply the four linear functions obtained by substituting these values of x, y, z in the fourth: the product will contain the resultant of the system affected with some numerical factor. In the present case, the four systems of x, y, z are

$$x = 0, \quad y = 0, \quad z = 1,$$
$$y = 0, \quad z = 0, \quad x = 1,$$
$$z = 0, \quad x = 0, \quad y = 1,$$
$$x = (a-b)(a-c), \quad y = (b-a)(b-c), \quad z = (c-a)(c-b),$$

and accordingly the product of

$$lx_1 + my_1 + nz_1,$$
$$lx_2 + my_2 + nz_2,$$
$$lx_3 + my_3 + nz_3,$$
$$lx_4 + my_4 + nz_4,$$

becomes

$$lmn\{(a-b)(a-c)l + (b-a)(b-c)m + (c-a)(c-b)n\},$$

agreeing with the result obtained by my theorem,—a *special* numerical factor 4, arising from the peculiar form of the equations, having disappeared from the resultant.

A geometrical demonstration may be given of the theorem which is instructive in itself, and will suggest a remarkable extension of it to functions containing more than three letters; the equation

$$\underset{xyzt}{\square}\{\lambda U + \mu V + (lx + my + nz)t\} = 0,$$

which is a quadratic equation in $\lambda : \mu$, may easily be shown to imply that the conic $\lambda U + \mu V$ is touched by the straight line

$$lx + my + nz = 0.$$

And we thus see that in general two conics,

$$\lambda U + \mu V = 0,$$

passing through the intersections of two given conics,

$$U = 0, \quad V = 0,$$

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may be drawn to touch a given line. If, however, the given line passes through any of the four points of intersection, in such case only one conic can be drawn to touch it; accordingly

$$\square\,\square\,\{\lambda U + \mu V + (lx + my + nz)\,t\}$$

must be zero when l, m, n are so taken as to satisfy this condition, that is, if

$$lx_1 + my_1 + nz_1 = 0,$$

or $$lx_2 + my_2 + nz_2 = 0,$$

or $$lx_3 + my_3 + nz_3 = 0,$$

or $$lx_4 + my_4 + nz_4 = 0,$$

whence the theorem.

Now suppose U and V to be each functions of four letters, x, y, z, t; when

$$\underset{xyztu}{\square}\{\lambda U + \mu V + (lx + my + nz + pt)\,u\} = 0,$$

the conoid $\lambda U + \mu V$ touches the plane

$$lx + my + nz + pt = 0\,;$$

and $\square = 0$ being a cubic equation, in general three such conoids can be drawn.

Considerations of analogy make it obvious to the intuition, that in the particular case of two of these becoming coincident, the given plane

$$lx + my + nz + pt$$

must be a tangent plane to those two coincident conoids at one of the points where it meets the intersections of $U = 0$, $V = 0$; that is

$$lx + my + nz + pt = 0$$

will pass through a tangent line to, or in other words, may be termed a tangent plane to the intersections. Hence the following analytical theorem, derived from supposing q, r, s, t to be proportional to the areas of the triangular faces of the pyramid cut out of space by the four coordinate planes to which x, y, z, t refer. As these planes are left indefinite, q, r, s, t are perfectly arbitrary.

Theorem. The resultant of

1. $U = 0$
2. $V = 0$ $\Big\}$, where U and V are functions of x, y, z, t;
3. $$lx + my + nz + pt = 0\,;$$
4. $$\begin{vmatrix} \dfrac{dU}{dx}, & \dfrac{dU}{dy}, & \dfrac{dU}{dz}, & \dfrac{dU}{dt} \\ \dfrac{dV}{dx}, & \dfrac{dV}{dy}, & \dfrac{dV}{dz}, & \dfrac{dV}{dt} \\ l, & m, & n, & p \\ q, & r, & s, & t \end{vmatrix} = 0\,;$$

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which system, it will be observed, consists of three quadratic functions, and one linear function of x, y, z, t, *contains* the factor

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square}\ \{\lambda U + \mu V + (lx + my + nz + pt)\, u\}.$$

This last quantity is of the 4×3th, that is, the 12th order in respect of the coefficients in U and V combined; of the 4×2th, that is, the 8th order in respect of l, m, n, p; and of the zero order in respect of q, r, s, t.

The resultant which contains it is of the $(4 + 4 + 2 . 4)$th, that is, 16th order in respect to the coefficients in U and V; of the $(4 + 8)$th, that is, the 12th, in respect of l, m, n, p; and of the 4th in respect of q, r, s, t. Hence the special (and, as far as the geometry of the question is concerned, the unnecessary, I may not say extraneous or irrelevant) factor which enters into the resultant is of the 4th order in respect to the combined coefficients of U and V*; and of the same order in respect to l, m, n, p, and in respect to q, r, s, t.

I have not yet succeeded in divining its general value.

In the very particular example, of the system,

$$\alpha x^2 + \beta y^2 = 0,$$
$$cz^2 + dt^2 = 0,$$
$$lx + my + nz + pt = 0,$$

$$\begin{vmatrix} \alpha x, & \beta y, & 0, & 0 \\ 0, & 0, & cz, & dt \\ l, & m, & n, & p \\ q, & 0, & 0, & 0 \end{vmatrix} = 0,$$

I find that the double determinant is

$$c^2d^2\alpha^2\beta^2 (cp^2 + dn^2)^2 (m^2\alpha + l^2\beta)^2,$$

and the resultant is

$$q^4c^2d^2\alpha^2\beta^4 (cp^2 + dn^2)^4 (m^2\alpha + l^2\beta)^2,$$

giving as the special factor

$$q^4\beta^2 (cp^2 + dn^2)^2.$$

I believe that the theorem which I have here given for determining the condition that $lx + my + nz + pt$ shall be a tangent plane to the intersection of two conoids U and V, namely, that the determinant of

$$\lambda U + \mu V + (lx + my + nz + pt)\, u$$

shall have two equal roots, is altogether novel.

* And consequently of the second in respect to the separate coefficients of each.

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What is the meaning of all three roots of this determinant becoming equal, that is, of only one conoid being capable of being drawn through the intersection of U and V to touch the plane

$$lx + my + nz + pt\,?$$

Evidently (*ex vi analogiæ*) that this plane shall pass through three consecutive points of the curve of intersection, that is, that it shall be the osculating plane to the curve.

If we return to the intersection of two co-planar conics, and if we suppose a line to be drawn through two of the points of intersection, the conics capable of being drawn through the four points of intersection to touch the line, besides becoming coincident, evidently degenerate each into a pair of right lines. It would seem, therefore, by analogy, that if a plane be drawn including any two tangent lines to the curve of intersection of two surfaces of the second degree, this should be touched by two coincident cones drawn through the curve of intersection, and consequently every such double tangent plane to the intersection of two conoids (and it is evident that one or more of these can be taken at every point of the curve) must pass through one of the vertices of the four cones in which the intersection may also be considered to lie; and it would appear from this, that in general four double tangent planes admit of being drawn to the curve, which is the intersection of two conoids, at each point thereof. At particular points a tangent plane may be drawn passing through more than one of the vertices, and then of course the number of double tangent planes that can be drawn will be lessened. These results, indicated by analogy, become immediately apparent on considering the curve in question as traced upon any one of the four containing cones. For the plane drawn through a tangent at any point, and the vertex of the cone being a tangent plane to the cone, must evidently touch the curve again where it meets it. We thus have an additional confirmation of the analogy between a point of intersection of two curves and the tangent at any point of the intersection of two surfaces.

I might extend the analytical theorems which have been established for functions of three and four to functions of a greater number of variables; but enough has been done to point out the path to a new and interesting class of theorems at once in elimination and in geometry, which is all that I have at present leisure or the disposition to undertake.

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