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Mathematics in the alternative set theory as a tool of Newtonian mechanics

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WE ARE TO SHOW that the concepts of the alternative set theory, [1], can be taken as the analytical basis for the modelling of motions and interactions within mechanics. In this way the foundations of Newtonian mechanics can be formulated without using any infinite sets and limit passages. At the same time the notion of a "material continuum" and that of "local interaction" can be derived directly from the mass-point system mechanics on the basis of purely phenomenological assumptions.

Wykazano, że koncepcje alternatywnej teorii mnogości [1] można wykorzystać jako analityczną bazę dla modelowania ruchów i oddziaływań w mechanice. W ten sposób formułuje się podstawy mechaniki newtonowskiej bez użycia jakichkolwiek nieskończonych zbiorów i przejść granicznych. Również pojęcia "materialnego kontinuum" oraz "lokalnego oddziaływania" można wýprowadzić bezpośrednio z mechaniki systemów dyskretnych na podstawie czysto fenomenologicznych założeń.

Показано, что концепции альтернативной теории множеств [1] можно использовать как аналитический базис для моделирования движений и взаимодействий в механике. Таким образом формулируются основы ньютоновой механики без использования каких-нибудь бесконечных множеств и предельных переходов. Тоже понятия „материальный континуум" и „локальное взаимодействие" можно вывести непосредственно из механики систем дискретних на основе чисто феноменологических предположений.

1. Introduction

THE KNOWN formulations of classical mechanics involve different mathematical structures and concepts (such as infinite sets and sequences, passages to the limit etc.) the relation of which to the physical objects is questionable. On the basis of experiments and observations we are not able to give any example of an infinite set of individual real objects and we are not able to verify that continuous physical structures and processes really exist. Moreover, the investigation of physical objects by means of certain analytical methods (mainly in continuum mechanics) may provide certain information about the mathematical tools of mechanics rather than about the physical problems under consideration. Such situations take place, for example, if we analyse existence problems in continuum mechanics. A source of these facts is that "contemporary mathematics... studies a construction whose relation to the real world is at least problematic.... Mathematics can be degraded to a mere game played in some specific artificial world", [1] p. 10. An attempt to reformulate the foundations of mathematics on a phenomenological basis was made in [1] and called the alternative set theory. Using this theory we eliminate all actually infinite sets from mathematics and, at the same time, we treat infinity as "a phenomenon involved in the observation of large, incomprehensible sets", [1] p. 11. In the alternative set theory we also postulate the existence of a class which is included in a set but itself is not a set; such class is said to be a proper semiset (each set is trivially a semiset). Hence all "large" sets (sets which cannot be "grasped") include a proper semiset. The foundations of the alternative set theory are due to P. VOPĚNKA and can be found in [1].

The aim of the note is to show that the alternative set theory constitutes a suitable background for the mathematical modelling of the basic concepts of Newtonian mechanics. From a purely formal point of view, the mathematical approach to the physical concepts of mass-point mechanics presented in the note can also be realized using the methods of the non-standard analysis, [2–5], or via the Ω-calculus, [6]. Howewer, the phenomenological sense of the alternative set theory seems more adequate to the physical ideas of mechanics we are to investigate. All concepts and denotations of the alternative set theory which are not explained throughout the note coincide with those used in [1], cf. also Appendix to the paper.

2. Galilean space-time

By the Galilean space-time we shall mean here fourtouple $\langle E, \sim, \tau, h\rangle$ of classes (from the extended universe or codable) ([1]), the objects of which will be explained below:

1. E is the class of events e, $e \in E$ (from the extended universe), e being elements of the universal class $V = \{x;\ x = x\}$.

2. The symbol $\sim$ stands for the equivalence class (defined here for elements of E^2, but in general dom $(\sim) = V$), such that the factorization of E modulo $\sim$, i.e.

$$U \equiv E/\sim\ = \{u;\ (\exists \langle e_1, e_2\rangle \in E^2)\,[u = E^2 \cap (\sim'' \{\langle e_1, e_2\rangle\}]\},$$

is a four-dimensional vector space over the set RN which is also a translational space of E([2]).

Elements of U (translations in the class of events) are classes and U is a class codable by $\langle U, \sim\rangle$.

Hence for every $(e, u, u_1, u_2) \in E \times U^3$

$$(e+u_1)+u_2 = e+(u_1+u_2), \qquad e+O = e, \qquad e+u = e \Rightarrow u = 0$$

and for every $\langle e_1, e_2\rangle$ there is exactly one u and $e_1+u = e_2$. At the same time, for any basis $\langle e_0, \langle a_1, a_2, a_3, a_4\rangle\rangle$, $a_i \in U$, there is

$$e = e_0+\zeta^i(e)a_i, \qquad \zeta^i(e) \in RN, \qquad i = 1, 2, 3, 4,$$

i.e. each basis determines a mapping $\zeta\colon E \to (RN)^4$.

3. τ is the class from the extended universe, which is a linear function $\tau\colon U \to RN$, interpreted as a time-form. Hence $S \equiv \operatorname{Ker}\tau$ is a class of space translation and S is a three-dimensional vector space over RN.

([1]) Absolute elements of Newton's mechanics can be represented by classes (from the extended universe or codable), while dynamical elements can be given by sets (from the universe of sets). The physical scope of this section is based on Chapter III of [7].

([2]) $X''Y \overset{\mathrm{df}}{=} \{u;\ u = \operatorname{Set}(u),\ (\exists v \in Y)[\langle u, v\rangle \in X]\}$.

4. h is the class from the extended universe, which represents a scalar product $h:S\times \times S \to RN$. We shall interpret $h(e_1-e_2, e_1-e_2)$ as the square of the distance between the events e_1, e_2 provided that $\tau(e_1-e_2) = 0$, by a distance we mean any rational number $|e_1-e_2| \in RN$ such that $|e_1-e_2|^2 \doteq h(e_1-e_2, e_1-e_2)$. Mind that the distance is not uniquely defined ([3]).

From now on we shall assume that there is the pair $r = \langle e_0, \langle a_1, a_2, a_3, a_4\rangle\rangle$ determining what can be called the inertial reference frame in E, i.e. the pair satisfying the known conditions: $\tau(a_4) = 1$, $\tau(a_\alpha) = 0$ for $\alpha = 1, 2, 3$, $h(a_\alpha, a_\beta) = \delta_{\alpha\beta}$, $e_0 \in E$. Hence for every $e \in E$ we obtain $e = e_0 + x^\alpha a_\alpha + ta_4$, where x^α, $t \in RN$.

REMARK. The choice of classes τ and h depends on the choice of the unit measure of time and space distances, respectively.

3. Indiscernibility and tolerance of events

One of the main features of mathematics in the alternative set theory as the tool of mechanics is the possibility of describing the concept of indiscernibility of events in the Gallilean space-time.

It can be easily observed that every time form $\tau(\cdot)$ and every scalar product $h(.,.)$ leads to the indiscernibility relations $\underset{T}{\doteq}$ = and $\underset{S}{\doteq}$ in time and space, respectively, by means of the following definitions:

DEFINITION 1. *Define* $u \underset{T}{\doteq} 0$ *for every* $u \in U$, *if* $\tau(u) \doteq 0$. *Then putting for every* $e_1, e_2 \in E$ *and* $u_1, u_2 \in U$:

$$(3.1)\qquad \begin{gathered} e_1 \underset{T}{\doteq} e_2 \Leftrightarrow e_1 - e_2 \underset{T}{\doteq} 0, \\ u_1 \underset{T}{\doteq} u_2 \Leftrightarrow \tau(u_1) \doteq \tau(u_2), \end{gathered}$$

we shall refer $\underset{T}{\doteq}$ *to as the time indiscernibility relation.*

DEFINITION 2. *Define* $u \underset{S}{\doteq} 0$ *for every* $u \in U$ *and* $\tau(u) = 0$, *iff* $h(u, u) \doteq 0$. *Then putting for every* $e_1, e_2 \in E$:

$$(3.2)\qquad e_1 \underset{S}{\doteq} e_2 \Leftrightarrow \tau(e_1-e_2) = 0 \quad \text{and} \quad e_1 - e_2 \underset{S}{\doteq} 0,$$

we shall refer $\underset{S}{\doteq}$ *to as the space indiscernibility relation.*

Let us observe that the concepts of the time- and space-indiscernibility, via the concepts of time and space monads defined by

$$(3.3)\qquad \begin{gathered} \mathrm{Mon}_T(e_0) \equiv \{e; e \underset{T}{\doteq} e_0\}, \\ \mathrm{Mon}_S(e_0) \equiv \{e; e \underset{S}{\doteq} e_0\}, \end{gathered}$$

respectively, constitute a certain alternative to the wellknown treatment of the space-time as a four-dimensional differentiable manifold.

Now assume that $\{R_n^T, n \in FN\}$ is the generating sequence for $\underset{T}{\doteq}$; here R_n^T is assumed to be a tolerance relation on E (or on U) for every $n \in FN$, where $R_0^T = E^2$ (or $R_0^T = U^2$)

([3]) Here $\alpha \doteq \beta$ if $\alpha \in BRN$, $\beta \in BRN$ and $|\alpha-\beta| < \dfrac{1}{n}$ for every $n \in FN$.

Analogously, let $\{R_n^S; n \in FN\}$ be the generating sequence for $\doteq_S$; here R_n^S is assumed to be a tolerance relation on S, where $R_0^S = S^2$. By virtue of the known definitions, [1], we have

$$\doteq_T \equiv \bigcap_{n \in FN} R_n^T, \quad \doteq_S \equiv \bigcap_{n \in FN} R_n^S.$$

Every R_n^T can be treated as a certain "n-th tolerance approximation" of $\doteq_T$ and every R_n^S as a certain "n-th tolerance approximation" of $\doteq_S$. At the same time the sets $\{R_n^T; n \in FN\}$, $\{R_n^S; n \in FN\}$ constitute tolerance systems in the sense of [8]. Hence we see that the tolerance approach to problems of mechanics, which has been proposed in [8], can be interpreted as a certain approximation of the "exact" approach to the mechanics, based on the concepts of the alternative set theory. For example, the concept of monads (3.3) can be approximated by the sequences of sets defined by

$$(3.4) \qquad \begin{aligned} Ap_n^T(e_0) &\equiv \{e; \langle e, e_0 \rangle \in R_n^T\}, \\ Ap_n^S(e_0) &\equiv \{e; \langle e, e_0 \rangle \in R_n^S, \tau(e-e_0) = 0\}, \end{aligned}$$

which will be called the tolerance approximations of $\mathrm{Mon}_T(e_0)$, $\mathrm{Mon}_S(e_0)$, respectively. The concept of indiscernibility can also be used here in the sense given in [9].

4. Mass-point mechanics

Let $E \to (RN)^3 \times RN$ be an arbitrary but fixed inertial reference frame (cf. Sect. 2). As the primitive concepts of the single mass-point mechanics we postulate: 1) mass $m \in (RN)_+$ 2) motion $RN \ni t \to p(t) \in (RN)^3$, 3) impulse of a force $F([t_1, t_2])$, defined for every non-empty closed time interval $[t_1, t_2] \subset RN$. Let ν be the known infinite natural number, $\nu \in N \setminus FN$ and define $\varepsilon \equiv \nu^{-1}$. Moreover, let $I_\varepsilon \equiv \{t \in RN,\ t = \pm n\varepsilon,\ n \in N\}$ be the subset of the "time-axis" RN.

We shall assume that the Newtonian mass-point mechanics can be developed if the domain RN of motion is restricted to the subset I_ε of RN for some infinitely small ε. This means that all information about the motion (which is required in order to formulate the dynamics of any mass-point) can be obtained, roughly speaking, from a "rapid series of photographs", (cf. [1], p. 97), of physical objects, i.e. "series of photographs" taken at time instants $t \in I_\varepsilon$. Hence also all impulses of forces can be treated as concentrated exclusively at time instants $t \in I_\varepsilon$; i.e. for every $t_1, t_2 \in RN$ we assume that $[t_1, t_2] \cap I_\varepsilon = \emptyset$ implies $F([t_1, t_2]) = 0$ and that $[t_1, t_2] \cap I_\varepsilon = \{t\}$ implies $F([t_1, t_2]) = F(\{t\})$. The forementioned assumptions correspond to the general idea given in [10] that "a world in which all motions consisted of a series of small finite jerks would be empirically indistinguishable from one in which motion was continuous" (cf. [10], p. 140). It follows that Newton's law of motion has to be postulated in the form

$$(4.1) \qquad m[v^+(t) - v^-(t)] = F(\{t\}), \qquad t \in I_\varepsilon,$$

where

$$(4.2) \qquad v^+(t) \equiv \frac{p(t+\varepsilon) - p(t)}{\varepsilon}, \quad v^-(t) \equiv \frac{p(t) - p(t-\varepsilon)}{\varepsilon}, \qquad t \in I_\varepsilon,$$

stand for the *RHS*-velocity and the *LHS*-velocity at $t \in I_\varepsilon$, respectively. Obviously, $v^+(t) = v^-(t+\varepsilon)$ is the (constant) velocity in every time interval $(t, t+\varepsilon)$, $t \in I_\varepsilon$. Setting

$$(4.3) \qquad a(t) \equiv \frac{v^+(t)-v^-(t)}{\varepsilon}, \quad f(t) \equiv \frac{F(\{t\})}{\varepsilon}, \quad t \in I_\varepsilon,$$

we obtain Eq. (4.1) in the equivalent form

$$(4.4) \qquad ma(t) = f(t), \quad t \in I_\varepsilon.$$

From the formulas (4.2)–(4.4) it follows that the mass-point mechanics under consideration can be analysed in terms of the functions

$$p(\cdot): I_\varepsilon \ni t \to p(t) \in (RN)^3, \quad f(\cdot): I_\varepsilon \ni t \to f(t) \in (RN)^3,$$

which will be referred to as a mass-point motion and an evolution of the resultant force, respectively. In order to eliminate "unphysical" situations (which cannot be observed and measured), we have to introduce, however, certain extra regularity conditions.

DEFINITION 3. *Mass-point motion* $p(\cdot)$ *will be called micro-regular if* $p(t)$, $v^+(t)$, $a(t) \in (BRN)^3$ *for every* $t \in I_\varepsilon \cap BRN$.

PROPOSITION 1. For every micro-regular motion $p(\cdot)$ the conditions $p(t) \doteq p(t+\varepsilon)$, $v^+(t) \doteq v^-(t)$ hold for every $t \in I_\varepsilon \cap BRN$(4).

PROPOSITION 2. If $m \in (FRN)_+$, $f(t) \in (BRN)^3$ for every $t \in I_\varepsilon \cap BRN$ and $v^+(t_0)$, $p(t_0) \in (BRN)^3$ for some $t_0 \in I_\varepsilon \cap BRN$, then the mass-point motion is micro-regular.

The mirco-regular mass-point motions constitute a subclass of what will be called regular mass-point motions in which, for every $t_1, t_2 \in I_\varepsilon \cap BRN$, the condition $t_1 \doteq t_2$ implies that

$$p(t_1) \doteq p(t_2), \quad v^+(t_1) \doteq v^+(t_2), \quad a(t_1) \doteq a(t_2).$$

Let us observe that every micro-regular motion $p(\ ): I_\varepsilon \to (RN)^3$ is a motion in the sense given in [1], i.e. "is a phenomenon which we perceive when we are presented with a sequence of states in which each state differs indistinguishably from the preceding state in time and substance" (cf. [1], p. 97).

Moreover, the known continuity and smoothness assumptions concerning motions are replaced here by a more physical concept of regular motions. Hence we see that the notion of rational numbers *RN* in the sense of the alternative set theory makes it possible to describe the physical concept of motion and to formulate Newton's law of motion without using any limit passages.

5. From a mass-point system to a material continuum

Now assume that a system of mutually interacting mass-points is given. Let ϑ, $\vartheta \in N \setminus FN$ be a number of points in the system under consideration; thus we deal here with a certain "large" set of mass points, cf. Sect. 1. Moreover, let

(4) Here $a \doteq b$ if $a = (a_1, a_2, a_3) \in (BRN)^3$, $b = (b_1, b_2, b_3) \in (BRN)^3$, and $\max |a_i - b_i| < \frac{1}{n}$, $i = 1, 2, 3$ for every $n \in FN_+$.

$$I_\varepsilon \ni t \to p_\gamma(t) \in (RN)^3,$$

$$I_\varepsilon \ni t \to f_\gamma(t) \in (RN)^3, \qquad \gamma = 1, 2, \ldots, \vartheta,$$

stand for a motion and an evolution of forces, respectively, in the mass point system under consideration, cf. Sect. 4. Let the governing relations of a mass-point system be given by

$$m_\gamma a_\gamma(t) = f_\gamma(t),$$

$$\text{(5.1)} \qquad f_\gamma(t) = b_\gamma\big(t, p_\gamma(t)\big) + \sum_{\delta=1}^{\vartheta} \varphi_{\gamma\delta}(|p_\gamma(t) - p_\delta(t)|) \frac{p_\gamma(t) - p_\delta(t)}{|p_\gamma(t) - p_\delta(t)|},$$

$$\gamma = 1, 2, \ldots, \vartheta, \qquad t \in I_\varepsilon,$$

where $b_\gamma(\cdot)$, $\varphi_{\gamma\delta}(\cdot)$ are the known functions with values in $(BRN)^3$ and BRN, respectively, and where $\varphi_{\delta\gamma}(\cdot) \equiv \varphi_{\gamma\delta}(\cdot)$, $\varphi_{\gamma\gamma}(\cdot) \equiv 0$ for every $\gamma, \delta \in \{1, 2, \ldots, \vartheta\}$.

Using an approach analogous to that given in [4, 5], we can pass from Eqs. (5.1) to the governing relations of a certain "material continuum". Here a "material continuum" will be interpreted as a certain phenomenon due to the effect of the space indiscernibility. This means that the mass-point system under consideration, for every $t \in I_\varepsilon$, is "observed" as a part B_t of the "physical space" $(RN)^3$ given by

$$\text{(5.2)} \qquad B_t := \{p \in (RN)^3; p \doteq p_\gamma(t) \quad \text{for some } \gamma \in \{1, \ldots, \vartheta\}\},$$

provided that $\{p_1(t), \ldots, p_\vartheta(t)\}$ is a connected set ([1], p. 93). Mind that

$$B_t = \text{Fig}\big(\{p_1(t), \ldots, p_\vartheta(t)\}\big),$$

where Fig (X) is a figure of a set X, cf. [1], p. 94. Moreover, if $B_t \cap (FRN)^3$ represents a certain "three-dimensional" part of $(FRN)^3$, then $\{p_1(t), \ldots, p_\vartheta(t)\}$ will be called a "material continuum".

Now assume that $\varphi_{\gamma\delta}(|p_\gamma(t) - p_\delta(t)|) \neq 0$ implies $p_\gamma(t) \doteq_s p_\delta(t)$ for every $\gamma, \delta \in \{1, 2, \ldots, \vartheta\}$ and every $t \in I_\varepsilon$. Then the (nonlocal) interactions in the mass-point system under consideration will be called macro-local. Hence the local interactions, which in the classical approach to continuum mechanics are postulated a priori, in the approach presented in the note have a clear physical sense being defined as the interactions between the mass points situated in one space monad. The exact analysis leading from Eqs. (5.1) to the equations of continuum mechanics will be given separately in [11].

At the end of the note let us also observe that the time- and space-indiscernibility depends on the choice of the time form $\tau(\cdot)$ and the scalar product $h(.,.)$. Hence the ideas of "material continuum" or "macro-local" interactions depend on the accuracy of a measure or observations of the real objects and phenomena. Thus the approach to mechanics based on the mathematics in the alternative set theory has certain common features with that given in [8]. A more detailed analysis of the problems under consideration will be developed in [12].

Appendix. On some basic concepts of the alternative set theory

Among the primitive concepts of an alternative set theory (*AST*), [1], we mention here the following three: a class, a set and a semiset. The axioms for classes are analogous to those of the Kelley–Morse's theory of classes [13–14]. All sets are classes but not all classes are sets, for example the class of all sets — the universal class $V = \{x;\ x = x\}$ — is not a set. The fact that X is a set is denoted by Set (X). A semiset is a subclass of a set. We write Sms (X) for "X is a semiset". Thus $\mathrm{Sms}(X) \equiv (\exists \mathrm{Set}(Y))[X \subset \mathrm{Set}(Y)]$. Each set is trivially a semiset. A proper semiset is a semiset which is not a set. A very important axiom is an axiom of existence of proper semisets, namely

$$(\exists X)\,[\mathrm{Sms}(X) \wedge \sim \mathrm{Set}(X)].$$

A class X is finite (notation: Fin (X)) if and only if each subclass of X is a set. Classes that are not finite are called infinite. Obviously, each finite class is a set. Thus all proper classes, in particular, all proper semisets, are infinite.

A x is *a natural number* if it satisfies the following conditions: 1) each element of x is a subset of x, i.e. $(\forall y \in x)[y \subset x]$, 2) is connected on x i.e. $(\forall y, z \in x)\,[y \in z \vee y = z \vee z \in y]$.

The class of all natural numbers is denoted by N. FN will denote the class of all finite natural numbers, i.e. $FN = \{x;\ \mathrm{Fin}\,(x)\}$. Evidently, $FN \subset N$. In *AST* we postulate that there exists an infinite natural number. Thus $N \setminus FN \neq \emptyset$. Hence we also have two kinds of rational numbers. Let $FN^{(-)} = FN \cup \{\langle 0, n\rangle, n \neq 0, n \in FN\}$. The class of finite rational numbers FRN is defined by

$$FRN = \left\{\frac{x}{y};\ x, y \in FN^{(-)} \wedge y \neq 0\right\}.$$

The class of all rational numbers (of all rationals) is denoted by RN. The class of bounded rationals is given by

$$BRN = \{x;\ x \in RN \wedge (\exists n \in FN)[|x| < n]\}.$$

Obviously $FRN \nsubseteq BRN \nsubseteq RN$.

A sequence $\{R_n, n \in FN\}$ where $R_n \subset V^2$, $R_0 = V^2$ is called a generating sequence of a certain equivalence class which will be denoted by $\doteq$, if and only if the following conditions hold: 1) for each n, R_n is a set-theoretically definable tolerance, i.e. reflexive and symmetric relation, 2) for each $n \in FN$ and each $x, y, z \in V$, $\langle x, y\rangle \in R_{n+1}$ and $\langle y, z\rangle \in R_{n+1}$ implies $\langle x, z\rangle \in R_n$, 3) $\doteq$ is the intersection of all the classes R_n.

An equivalence $\doteq$ is said to be compact if for each infinite set $u \in V$, there are $x, y \in u$ such that $x \neq y$ and $x \doteq y$. A relation is called an indiscernibility equivalence if and only if $\doteq$ is a compact and has a generating sequence.

For a detailed discussion of the mentioned concepts cf [1].

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