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Stability of three-dimensional natural convection in a porous layer

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THE STABILITY of natural convective flow in a fluid-saturated porous medium heated uniformly from below is studied in order to determine the conditions under which the transition from laminar to fluctuating flow occurs. The Galerkin method is used to investigate three-dimensional convection in a cube, the Fourier series is truncated to eight modes. Two-dimensional steady rolls cannot exist in a cube at a Rayleigh number larger than $18\pi^2$, what is in agreement with the numerical calculations of Schubert and Straus. At Ra = $18\pi^2$, the two-dimensional rolls in the cube begin to oscillate periodically, then the fluctuations become quasi-periodic and for a Ra larger than 242 an exponential growth of amplitude can be observed. The mechanism of the loss of stability of the steady two-dimensional rolls in the cube is different from that occurring at convection in a square cylinder. According to the infinite Prandtl number, the trajectories of the approximate Darcy–Boussinesq equations — after loss of stability — escape to infinity instead of wandering in a bounded region as is the case with the classical Bénard convection.

Badano stabilność konwekcji swobodnej w warstwie porowatej nasączonej płynem, grzanej od spodu, w celu określenia warunków przejścia od przepływu laminarnego do fluktuacyjnego. Analizowano konwekcję w komórce sześciennej, stosując metodę Galerkina i obcinając długość szeregu Fouriera do ośmiu członów. Stwierdzono, że dwuwymiarowe rolki nie mogą istnieć dla liczby Rayleigha większej niż $18\pi^2$, co zgadza się z obliczeniami numerycznymi Schuberta i Strausa. Dla Ra = $18\pi^2$ dwuwymiarowe rolki zaczynają oscylować, następnie oscylacje stają się quasi-periodyczne i dla Ra = 242 obserwuje się eksponencjalny wzrost amplitudy. Mechanizm utraty stabilności dwuwymiarowych rolek jest różny od mechanizmu utraty stabilności przepływu w kwadratowym cylindrze. Zgodnie z założeniem o nieskończonej liczbie Prandtla trajektorie przybliżonych równań Darcy-Boussinesqa uciekają do nieskończoności po utracie stabilności, zamiast błądzić w ograniczonym obszarze jak dla klasycznego problemu Bénarda.

Исследована стабильность свободной конвекции в пористом слое, насыщенном жидкостью и подогреваемом снизу, для определения условий перехода от ламинарного течения к турбулентному течению. Конвекция проанализирована в кубической камере с применением метода Галеркина при сокращенной длине ряда Фурье до восьми членов. Констатировано, что двумерные валики не могут существовать при числе Релея превышающем $18\pi^2$, что соответствует численным расчетам Шуберта и Штрауса. При Ra = $18\pi^2$ двумерные валики начинают колебаться, затем колебания становятся квазипериодическими и при Ra = 242 наблюдается экспоненциальное увеличение амплитуды. Механизм потери стабильности течения двумерных валиков отличается от механизма потери стабильности в цилиндре квадратного сечения. При бесконечном числе Прандтля траектории приближенных уравнений Дарси–Буссинеска отходят в бесконечность после потери стабильности, вместо того, чтобы блуждать в ограниченной области, как в случае классической задачи Бенарда.

1. Introduction

THERMAL convection in porous media has received considerable attention mainly because of its geophysical interest. Since the Navier–Stokes equation is replaced by Darcy's law, the analysis of this phenomenon seems to be easier than the classical Bénard problem. Although numerous investigations have recently been carried out on thermal convection in porous media, the understanding of some aspects of this phenomenon is still far from satisfactory [1].

The first critical Rayleigh number $\mathrm{Ra}_1 = 4\pi^2$ for the onset of convective flow in a porous medium was determined theoretically by LAPWOOD [2]. This criterion is very well documented by experiments. Small amplitude solutions emanating at $\mathrm{Ra} = 4\pi^2$ were investigated by means of the perturbation method for two-dimensional physical space by PALM *et al.* [3] and JOSEPH [4], for three-dimensional space by ZEBIB and KASSOY [5] and by KORDYLEWSKI and BORKOWSKA-PAWLAK [6].

A classification of the steady-state solutions of the Darcy–Boussinesq equations bifurcating from the trivial solution was given by BECK [7]. However, not all of these solutions are interesting from the physical point of view because only stable ones are observable.

The stability of two-dimensional rolls in the presence of three-dimensional perturbations was analysed first by STRAUS [8] using the LORTZ [9] method. An extensive perturbation analysis of this problem for Ra close to $4\pi^2$ was given by JOSEPH [4]. The stability of finite-amplitude solutions was analysed by means of numerical methods. STRAUS [8] and KVERNVOLD [10] have shown that on the plane Rayleigh number — wave number there is an envelope inside which the two-dimensional rolls are stable. However, the assumption made by these authors that the largest eigenvalue which crosses the imaginary axis is real, is probably not always true for the finite-amplitude flows.

At present the most interesting problem is to determine the second critical Rayleigh number Ra_2 for the transition from laminar to turbulent flow. COMBARNOUS and LE FUR [11] established experimentally that Ra_2 is in the range of 240–280. Some numerical calculations were also made of the critical conditions for the onset of fluctuactions. However, between the particular numerical results obtained by different authors there is no complete agreement. The interested reader can find a discussion of these results in [12].

The probable reason for these diverging numerical results is the multiplicity of steady solutions of the Darcy–Boussinesq equations, the transition to fluctuating convection depending on the realization of a particular flow. The situation is similar to the classical Bénard problem where many routes to turbulent convection were observed [13].

This paper is a continuation of a previous work [14] in which the stability of two-dimensional convection in the square box was analysed in accordance with Lorenz [15]. At present we are investigating the stability of three-dimensional convection in a cube. Although only eight modes are assumed, we hope to be able to explain some qualitative features of the transition from time-independent to fluctuating flow in porous media.

2. Role of the Prandtl number

Consider an infinite extended porous layer saturated with fluid between two non-permeable horizontal plates. The lower plate is warmer than the upper one. Assume the Darcy–Boussinesq equations in the dimensional form

$$\frac{1}{\mathrm{Pr}}\frac{\partial \bar{u}}{\partial t} = -\bar{u} - \nabla p + \mathrm{Ra}\,\theta\bar{k}, \qquad \nabla\cdot\bar{u} = 0, \tag{2.1}$$

$$\frac{\partial\theta}{\partial t} = \nabla^2\theta + u_z - \bar{u}\cdot\nabla\theta, \tag{2.2}$$

where $\bar{u}:(u_x, u_y, u_z), \theta, p$ denote the velocity vector, temperature and pressure, respectively, and $\bar{k}$ is the vertical unit vector. The Rayleigh number and the Prandtl number Pr are defined in [16] where they are denoted by R and B^{-1}, respectively.

The problem will be considered only with regard to the cube; hence the following boundary conditions are added:

on the upper and lower planes

$$z = 1,0: u_z = 0 \quad \text{and} \quad \theta = 0, \tag{2.3}$$

on the sidewalls

$$\begin{aligned} x &= 1,0: u_x = 0 \quad \text{and} \quad \partial\theta/\partial x = 0,\\ y &= 1,0: u_y = 0 \quad \text{and} \quad \partial\theta/\partial y = 0. \end{aligned} \tag{2.4}$$

Introducing the new variables

$$\theta' = \theta - (1+\mathrm{Pr\,Ra})z, \qquad u' = u, \qquad p' = p$$

we obtain Eqs. (2.1) and (2.2) in the form

$$\frac{1}{\mathrm{Pr}}\frac{\partial\bar{u}'}{\partial t} = -\bar{u}' - \nabla p' + \mathrm{Ra}\,\theta'\bar{k} + z\,\mathrm{Ra}(1+\mathrm{Pr\,Ra})\bar{k}; \tag{2.5}$$

$$\frac{\partial\theta'}{\partial t} = \nabla^2\theta' - \bar{u}'\cdot\nabla\theta' - u'_z\mathrm{Pr\,Ra}; \tag{2.6}$$

$$\begin{aligned} z &= 0: u'_z = 0 \quad \text{and} \quad \theta' = 0,\\ z &= 1: u'_z = 0 \quad \text{and} \quad \theta' = -(1+\mathrm{Pr\,Ra}); \end{aligned} \tag{2.7}$$

$$\begin{aligned} x &= 1,0: u'_x = 0 \quad \text{and} \quad \partial\theta'/\partial x = 0,\\ y &= 1,0: u'_y = 0 \quad \text{and} \quad \partial\theta'/\partial y = 0. \end{aligned} \tag{2.8}$$

Multiplying Eqs. (2.5) and (2.6) by $\bar{u}'$ and θ' and integrating over the cube domain V, we get

$$\begin{aligned} 1/2\partial/\partial t(\|\bar{u}'\|^2 + \|\theta'\|^2) &= -\mathrm{Pr}\|\bar{u}'\|^2 - \|\nabla\theta'\|^2 + \mathrm{Pr\,Ra}(1+\mathrm{Pr\,Ra})\int_V z u'_z dv\\ &- (1+\mathrm{Pr\,Ra})\int_0^1\int_0^1 \partial\theta'(x,y,z=1)/\partial z dx dy \leqslant -\mathrm{Pr}\|\bar{u}'\|^2 - \|\nabla\theta'\|^2\\ &+ \mathrm{Pr\,Ra}(1+\mathrm{Pr\,Ra})\int_V |u'_z| dv + (1+\mathrm{Pr\,Ra})\int_0^1\int_0^1 |\partial\theta'(x,y,z=1)/\partial z| dx dy, \end{aligned} \tag{2.9}$$

where $\|\cdot\|$ denotes the norm in $L_2(V)$.

For the finite Prandtl number the right-hand side of the above inequality becomes negative when the trajectory $(\bar{u}', \theta')$ escapes too far from the origin. We conclude that there is a bounded region B in $L_2(V)$ so that every solution of Eqs. (2.5)–(2.8) eventually becomes trapped by B. This property, noticed by LORENZ [15] and emphasized by RUELLE [17], caused the chaotic behaviour of the trajectories of the Lorentz system. At present we know that the appearance of a strange attractor in the extended Lorenz system may be preceded by a few bifurcations of periodic flows [18]; however, the role of the trap is still important. When the Prandtl number increases, the "boundaries" of the trap shift away from this origin, and for the infinite Prandtl number the trap disappears. Such a situation is observed for the two-dimensional convection in a porous layer. When Ra exceeded $30\pi^2$, the trajectories escaped to infinity [14]. For the porous media the Prandtl number assumes large values; hence in Eq. (2.1) we assume the infinite Prandtl number which is common in literature. The consequence of this assumption is the lack of trap for the system of ordinary differential equations obtained from Eqs. (2.1) and (2.2) by Galerkin's method.

3. Approximate time-independent solutions

To determine the approximate finite-amplitude solutions of the Darcy-Boussinesq equations, we will use the Galerkin method. However, first we transform Eqs. (2.1) and (2.2) into the form given by STRAUS and SCHUBERT [12]:

$$\nabla^2\phi = -\mathrm{Ra}\,\theta, \tag{3.1}$$

$$\frac{\partial\theta}{\partial t}+\frac{\partial^2\phi}{\partial x\partial z}\frac{\partial\theta}{\partial x}+\frac{\partial^2\phi}{\partial y\partial z}\frac{\partial\theta}{\partial y}-\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\frac{\partial\theta}{\partial z}=\nabla^2\theta-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}, \tag{3.2}$$

where ϕ satisfies the boundary conditions

$$\begin{aligned} x &= 0,1: \quad \partial^2\phi/\partial x\partial z = \partial(\nabla^2\phi)/\partial x = 0,\\ y &= 0,1: \quad \partial^2\phi/\partial y\partial z = \partial(\nabla^2\phi)/\partial y = 0,\\ z &= 0,1: \quad \partial^2\phi/\partial x_2+\partial^2\phi/\partial y^2 = \partial^2\phi/\partial z^2 = 0. \end{aligned}$$

Expand θ and ϕ in the Fourier series satisfying the boundary conditions

$$\phi = \sum_{n=1}^{\infty}\sum_{j,m=0}^{\infty}\phi_{njm}F_{njm}, \qquad \theta = \sum_{n=1}^{\infty}\sum_{j,m=0}^{\infty}\theta_{njm}F_{njm}, \tag{3.3}$$

where

$$F_{njm} = \begin{cases} \sqrt{2}\sin(n\pi z) & \text{for} \quad m=j=0,\\ 2\sin(n\pi z)\cos(j\pi x) & \text{for} \quad m=0,\\ 2\sin(n\pi z)\cos(m\pi y) & \text{for} \quad j=0,\\ 2\sqrt{2}\sin(n\pi z)\cos(j\pi x)\cos(m\pi y) & \end{cases}$$

and introduce the relations (3.3) into Eqs. (3.1) and (3.2). By multiplying these equations by F_{njm} and integrating over the cube, we obtain an infinite set of first-order, ordinary differential equations for the unknown θ_{njm} (or for ϕ_{njm}).

To reduce the number of equations, we truncate the series (3.3) to the first eight terms. The resulting differential equations have the following form:

$$
\begin{aligned}
\frac{d\theta_{101}}{dt} &= \frac{\mathrm{Ra}-4\pi^2}{2}\theta_{101}+\frac{\pi\mathrm{Ra}}{\sqrt{2}}(\theta_{101}\theta_{200}-\theta_{201}\theta_{100}/5)\\
&\qquad +\frac{6\pi\mathrm{Ra}}{5\sqrt{2}}\theta_{111}\theta_{210}+\frac{\pi\mathrm{Ra}}{2\sqrt{2}}\theta_{110}\theta_{211},\\
\frac{d\theta_{211}}{dt} &= \frac{\mathrm{Ra}-18\pi^2}{3}\theta_{211}-\frac{\pi\mathrm{Ra}}{\sqrt{2}}\theta_{101}\theta_{110}-\frac{\sqrt{2}\pi\mathrm{Ra}}{3}\theta_{111}\theta_{100},\\
\frac{d\theta_{110}}{dt} &= \frac{\mathrm{Ra}-4\pi^2}{2}\theta_{110}+\frac{\pi\mathrm{Ra}}{\sqrt{2}}(\theta_{110}\theta_{200}-\theta_{210}\theta_{100}/5)\\
&\qquad +\frac{6\pi\mathrm{Ra}}{5\sqrt{2}}\theta_{111}\theta_{201}+\frac{\pi\mathrm{Ra}}{2\sqrt{2}}\theta_{101}\theta_{211},\\
\frac{d\theta_{200}}{dt} &= -4\pi^2\theta_{200}-\frac{\pi\mathrm{Ra}}{\sqrt{2}}(\theta_{110}^2+\theta_{101}^2)-\frac{2\sqrt{2}\pi\mathrm{Ra}}{3}\theta_{111}^2,\\
\frac{d\theta_{100}}{dt} &= -\pi^2\theta_{100}+\frac{7\pi\mathrm{Ra}}{10\sqrt{2}}(\theta_{110}\theta_{210}+\theta_{101}\theta_{201})+\frac{\pi\mathrm{Ra}}{\sqrt{2}}\theta_{211}\theta_{111},\\
\frac{d\theta_{210}}{dt} &= \frac{\mathrm{Ra}-25\pi^2}{5}\theta_{210}-\frac{\pi\mathrm{Ra}}{\sqrt{2}}(\theta_{110}\theta_{100}/2+2\theta_{101}\theta_{111}),\\
\frac{d\theta_{201}}{dt} &= \frac{\mathrm{Ra}-25\pi^2}{5}\theta_{201}-\frac{\pi\mathrm{Ra}}{\sqrt{2}}(\theta_{101}\theta_{100}/2+2\theta_{110}\theta_{111}),\\
\frac{d\theta_{111}}{dt} &= \frac{2\mathrm{Ra}-9\pi^2}{3}\theta_{111}+\frac{\pi\mathrm{Ra}}{\sqrt{2}}\Big(\frac{4}{3}\theta_{111}\theta_{200}-\frac{1}{3}\theta_{100}\theta_{211}\\
&\qquad +\frac{4}{5}\theta_{110}\theta_{201}+\frac{4}{5}\theta_{101}\theta_{210}\Big).
\end{aligned}
\tag{3.4}
$$

We denote the right-hand side of these equations as a vector field F: $(F_{101}, F_{211}, F_{110}, F_{200}, F_{100}, F_{210}, F_{201}, F_{111})$.

Eight branches of steady-state solutions calculated for the above equations are presented in Fig. 1. Only three of them: A, B and C represent stable flows in a suitable range of the Rayleigh number. At $\mathrm{Ra} = 4\pi^2$ three branches emanate: A and B corresponding to stable two-dimensional rolls, and E which corresponds to unstable three-dimensional flow. Branch C representing three-dimensional flow emanates at $4.5\pi^2$ as a curve of unstable steady solutions and then crosses the point of secondary bifurcation $\mathrm{Ra} \cong 4.78\pi^2$, becoming a stable branch.

The corrdinates of the steady-state solutions belonging to curves A, B and C are

$$A: \quad \big(0, 0 \pm 2\sqrt{\mathrm{Ra}-4\pi^2}/\mathrm{Ra},\ -(\mathrm{Ra}-4\pi^2)/(\sqrt{2}\pi\mathrm{Ra}), 0, 0, 0, 0\big),$$

$$B: \quad \big(\pm 2\sqrt{\mathrm{Ra}-4\pi^2}/\mathrm{Ra}, 0, 0,\ -(\mathrm{Ra}-4\pi^2)/(\sqrt{2}\pi\mathrm{Ra}), 0, 0, 0, 0\big),$$

$$C: \quad \big(0, 0, 0, (\mathrm{Ra}-4.5\pi^2)/(\sqrt{2}\pi\mathrm{Ra}), 0, 0, 0,\ \pm\sqrt{3(\mathrm{Ra}-4.5\pi^2)}/\mathrm{Ra}\big).$$

The solutions cooresponding to the two-dimensional rolls (branches A and B) lose stability at $\mathrm{Ra} = 18\pi^2$ when the largest eigenvalue of the Jacoby matrix DF $(18\pi^2)$ crosses the imaginary axis

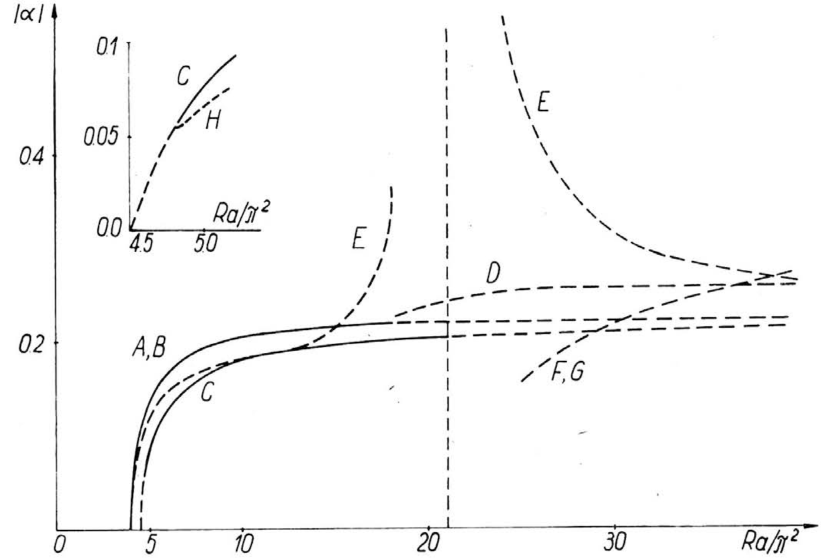

FIG. 1. Variation of the norm of steady-state solutions with the Rayleigh number.

$$\lambda_{AB} = \pm i\pi^2 \sqrt{14}.$$

This is in agreement with SCHUBERT's and STRAUS's [12] statement saying that steady, unicellular rolls cannot exist in a cube at a Rayleigh number of a value higher than 200.

The stable three-dimensional flow (branch C) loses stability at $\mathrm{Ra} = 21\pi^2$ when the largest eigenvalue of $DF\ (21\pi^2)$ becomes purely imaginary

$$\lambda_C = \pm i\pi^2 \sqrt{4.5}.$$

Since we have

$$\frac{d\mathrm{Re}(\lambda_{A,B})}{d\mathrm{Ra}}(18\pi^2) \neq 0 \quad \text{and} \quad \frac{d\mathrm{Re}(\lambda_C)}{d\mathrm{Ra}}(21\pi^2) \neq 0$$

then $\mathrm{Ra} = 18\pi^2$ and $21\pi^2$ are the Hopf bifurcation points. In the next section we analyse the properties of periodic solutions branching at these points.

4. Time-dependeing flows

4.1. Two-dimensional rolls

The stability of the small amplitude periodic solutions emanating at $\mathrm{Ra} = 18\pi^2$ was analysed by means of the MARSDEN and McCRACKEN [18] algorithm. While not losing generality, we assume that the periodic solutions emanate at the A branch.

Usually when the number of differential equations is more than two, a stability analysis leads to long and complicated calculations. In this case the Jacobian matrix $DF\ (18\pi^2)$

assumes the form of a block diagonal matrix, what greatly simplifies calculations. Introducing a new variable in the system of Eq. (3.4)

$$\theta'_{101} = \sqrt{2}\,\theta_{101} \tag{4.1}$$

we get $DF'(18\pi^2)$ in the form

$$DF'(18\pi^2) = \begin{bmatrix} 0 & \pi^2\sqrt{14} & 0 \\ \pi^2\sqrt{14} & 0 & 0 \\ 0 & 0 & S \end{bmatrix}, \tag{4.2}$$

where F' denotes the vector field F after the transformation (4.1), and S is also a block diagonal matrix. Such a form of the Jacoby matrix (4.2) allows us to use immediately the vector field F' for calculating the expression $V'''(18\pi^2)$ (see the formula (4.2) in MARSDEN and McCRACKEN [19]).

If $V'''(18\pi^2) < 0$, then $\mathrm{Ra} = 18\pi^2$ is the supercritical Hopf bifurcation point (periodic orbits are attracting), $V'''(18\pi^2) > 0$ determines the subcritical Hopf bifurcation point when there are no stable solutions for $\mathrm{Ra} = 18\pi^2$. We shall omit lengthy calculations and show the final results

$$V'''(18\pi^2) = -\frac{243\pi^3}{4\sqrt{14}}.$$

Hence the periodic solutions emanating at $\mathrm{Ra} = 18\pi^2$ are stable.

The stability of the finite-amplitude periodic flow was investigated numerically, the set of equations (3.4) being integrated by the Runge–Kutte method of second order. The numerical calculations are presented on the plane $(\theta_{101}, \theta_{211})$ which is tangent to the central manifold on which the above mentioned orbits lie [19].

Figure 2 shows the periodic orbits for $\mathrm{Ra} = 200$. The period of oscillations $T \cong 0.18$ is close to the period T which follows from the Hopf bifurcation theory:

$$T_0 = \frac{2\pi}{|\lambda_{AB}|} = \frac{1}{\pi\sqrt{3.5}}.$$

(It is interesting that CURRY [18] also obtained the period of the first closed orbit $T \cong 0.176$).

When Ra crosses approximately the value of 205, the amplitude of oscillations increases rapidly (Fig. 3). The period of oscillations also increases suddenly but a doubling of orbits does not take place — there rather follows a bifurcation towards torus. For $\mathrm{Ra} > 222.5$ we have a stable quasi-periodic flow (Fig. 4). A plot of the amplitude and the period of oscillations against the Rayleigh number is shown in Fig. 5.

When the Rayleigh number crosses approximately the value of 242, the quasi-periodic flow loses stability. The numerical calculations have shown that the coordinates θ_{110} and θ_{101} become almost equal and grow together with θ_{211} until they suddenly escape to infinity in the neighbourhood of the fixed point lying on branch E and corresponding to the unstable three-dimensional flow.

It would be interesting to compare our time-dependent solutions with periodic solutions of other authors. SCHUBERT and STRAUS [12] made numerical studies of unsteady convection by means of the Galerkin method using a large number of modes. However,

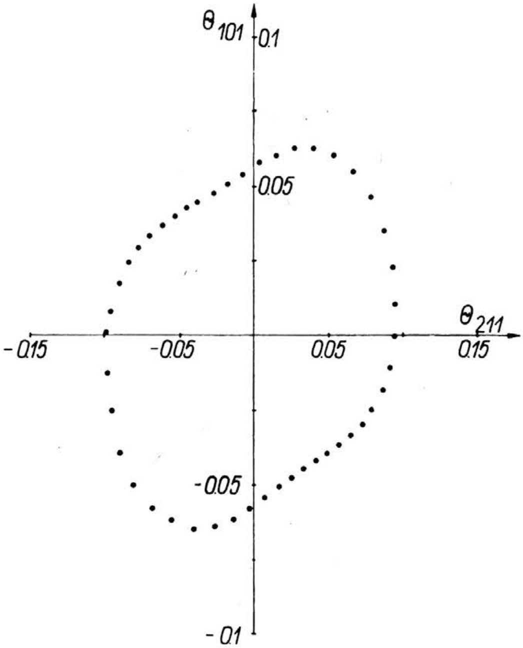

FIG. 2. Projection of periodic oscillations on the $\theta_{101}-\theta_{211}$ plane for Ra = 200.

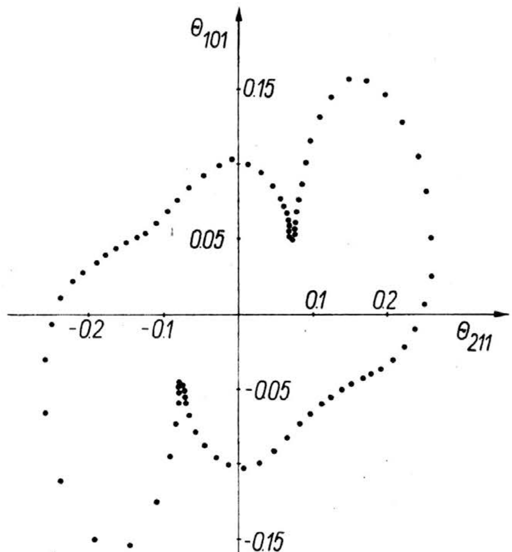

FIG. 3. Projection of periodic oscillations on the $\theta_{101}-\theta_{211}$ plane for Ra = 220.

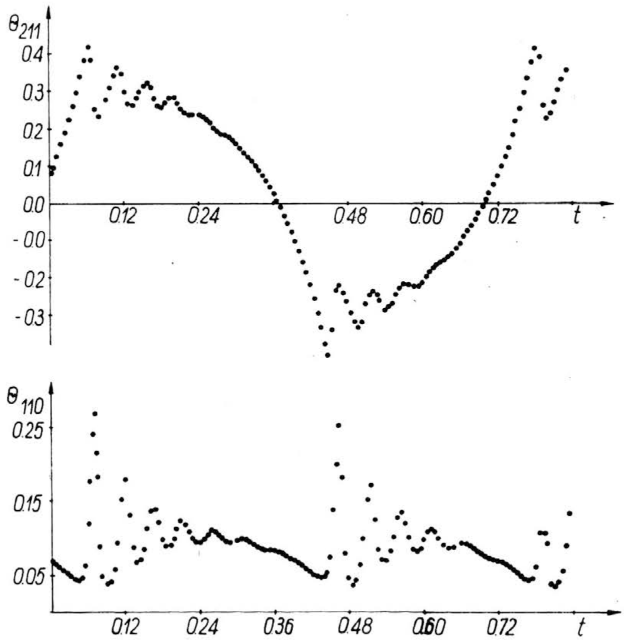

FIG. 4. Variation of the θ_{211} and θ_{110} with time for Ra = 240.

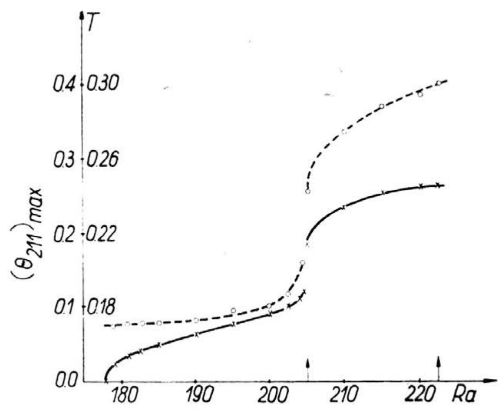

FIG. 5. Variation of the maximum of θ_{211} and the period of oscillations with the Rayleigh number.

they present only three-dimensional oscillations in a cube for which the period is almost fifty times shorter than our T_0.

4.2. Stability of three-dimensional flow

We do not have any analytical proof as to the kind of Hopf bifurcation involved in this case. The calculations have turned out to be so long and complicated that it seems impossible to avoid a mistake (for example the Russian editor of MARSDEN's and Mc-

CRACKEN'S [19] book does not give calculations of V''' for the Lorenz system because an error was found).

The numerical calculations have shown that the trajectories starting from a neighbourhood of fixed points belonging to branch C are attracted to these points at $\mathrm{Ra} < 21\pi^2$ and escape to infinity at $\mathrm{Ra} > 21\pi^2$. Their behaviour was similar to that taking place at two-dimensional convection for the infinite Prandtl number [14]. No periodic orbits or nonperiodic attractor were observed. Hence we conclude that $\mathrm{Ra} = 21\pi^2$ is the supercritical Hopf bifurcation point for branch C.

The similarity to the two-dimensional case results also from another fact. Numerical observations have shown that the subspace $\theta_{100}, \theta_{211}, \theta_{200}, \theta_{111}$ is attracted in the neighbourhood of the fixed points belonging to branch C. Assuming the remaining coordinates to be equal to zero, we obtain a system of four differential equations of the same structure as that in the two-dimensional case for four modes [14]. Probably, in both cases the transition to an unstable solution by the subcritical Hopf bifurcation point was caused by a very small number of modes as was also the case with the Lorenz system.

5. Discussion

A considerable difference is shown in the transition from time-independent flow to a fluctuating state for the two-dimensional rolls in the two- and three-dimensional spaces. In the first case the second critical Rayleigh number $30\pi^2$ was distinctly marked and the loss of stability followed at the subcritical Hopf bifurcation point. In the three-dimensional case it is difficult to establish a single value of Ra_2 because for $\mathrm{Ra} > 18\pi^2$ we have the periodic flow which, with Ra increasing, becomes quasi-periodic and at $\mathrm{Ra} > 242$ loses stability. However, the numerical difference is not the most important. Of great interest is the route to fluctuating convection. In the three-dimensional case the oscillations are caused by the modes creating the rolls with their axes being perpendicular to the axes of the original rolls. Such a mechanism of destabilization of two-dimensional rolls was also suggested for the classical Bénard problem by CLEVER and BUSSE [20]. The sharp transition to instability in the two-dimensional space [14] was probably caused by too small a number of modes. An increase in the number of modes should change the picture as it was observed in the extended Lorenz system [18].

There exists the problem of choosing the minimum number of modes which would ensure a qualitatively correct picture of the flow under study. A strict answer to this problem with regard to general flow is now unavailable, yet there are some attempts to estimate the finite-dimensional space for particular flows. For example, MANLEY and TREVE [21] made an estimate of the minimal number of modes for the Bénard problem based on the mathematical results of FOIAS and PRODI [22] for twodimensional physical space.

Finally, we would like to interpret the obtained results on the basis of recent theories of transition from laminar to turbulent flow. We will consider only branch A or B. The basic prediction of RUELLE and TAKENS [23] and NEWHOUSE, RUELLE and TAKENS [24] that nonperiodic motion occurs after a small number of bifurcations to T^m ($m \geq 3$) torus is consistent with our calculations. The nonoccurrence of chaotic behaviour on the part of the trajectories after stability loss results from the lack of trap. However, this fact can be helpful in determining the minimal number of modes necessary to describe the transition to turbulence. Maybe a minimal number of modes would ensure the existence of a strange local attractor.

We did not observe a cascade of subharmonic bifurcations of periodic orbits which, according to the Feigenbaum [25] theory, leads to chaotic motion at the cumulative point. In a recent study by Coste and Peyraud [26] such a process has been shown in the model of the two-dimensional Bénard convection in which five modes interact. It seems that this effect can be obtained also for the Darcy–Boussinesq equations if suitable modes in the Fourier series (3.3) are chosen.

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