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On the non-uniqueness of solutions for boundary value problems in transonic flows

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It is shown that the critical points of the functional (3.6) may be interpreted as a "weak" solution of the continuity equation (2.1) which fulfils the boundary conditions (3.2) and (3.3) in the sense of traces. Generally speaking, the critical points are saddle points of the deduced functional. This fact brings many interesting consequences in transonic flows as shown in examples.

Pokazano, że krytyczne punkty funkcjonału (3.6) można interpretować jako "słabe" rozwiązania ciągłości (2.1) spełniające warunki brzegowe (3.2) i (3.3) w sensie śladów. Mówiąc ogólnie, punkty krytyczne są punktami siodłowymi funkcjonału. Jak pokazano na przykładach, fakt ten pociąga za sobą liczne interesujące konsekwencje w przepływach przydźwiękowych.

Показано, что критические точки функционала (3.6) можно интерпретировать как ,,слабое'' решение уравнения непрерывности (2.1), удовлетворяющее граничным условиям (3.2) и (3.3) в смысле следов. В общем говоря, критические точки являются седлообразными точками функционала. Как показано на примерах этот факт влечет за собой многие интересные последствия в околозвуковых течениях.

1. List of symbols

$\mathbf{R}^2$	two-dimensional Euclidean space,
Ω	(open) domain in $\mathbf{R}^2$,
$\partial\Omega$	boundary of the domain Ω,
$\overline{\Omega}$	closure of the domain Ω,
$\mathscr{C}^{0,1}$	set of domains Ω having so-called "Lipschitz's" boundary,
$W^{1,p}(\Omega)$	Sobolev's space of functions defined in Ω and having the first derivative integrable with the p-th power,
$\{v \in V, \eta(v)\}$	set of all elements having the property $\eta(v)$ and lying in V,
$dF(u, v)$	the first derivative of the functional $F(u)$ understood in the Gâteaux's sense,
$d^2F(u, v, w)$	the second derivative of the functional $F(u)$ understood in the Gâteaux's sense,
ϱ	density of fluid,
φ	velocity potential.

2. Introduction

Recently, the application of modern functional-theoretical methods has been used more and more in the mathematical theory of transonic flows. The application of these methods also in the field which may not be quite prepared for this is wholly understandable. The continuity equation

(2.1) $$\operatorname{div} \varrho \operatorname{grad} \varphi = 0$$

can be considered as Euler's equation of a functional (i.e. in the so-called "weak" sense) and in this way one can study such complicated effects as shock waves. In the classical numerical calculations of transonic flows one had to assess the positions of shock waves and then to treat the resulting Neumann's problem as a contact problem whose solution fulfils the so-called Prandtl's shock wave conditions. On applying the functional-theoretical methods, one need not assess the position of the shock wave; across the shock wave the Prandtl's conditions are satisfied automatically. But in the theory of transonic flows the functional-theoretical methods have many disadvantages and in solving the implied boundary value problems many difficulties have to be overcome; some of these troubles are discussed in this paper.

3. Functional for transonic flow fields

Considerations in this chapter are based on the so-called generalized Green's theorem (cf. [2]): Let $\Omega \in \mathscr{C}^{0,1}$.

If $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$ where $p > 1$, $q > 1$ and $1/p+1/q = 1$, then follows

(3.1) $$\int_{\Omega} \frac{\partial u}{\partial x_i}(x) v(x)\, dx = -\int_{\Omega} u(x) \frac{\partial v}{\partial x_i}\, dx + \int_{\partial\Omega} uv\nu_i\, ds,$$

where the derivatives $\partial u/\partial x_i$, $\partial v/\partial x_i$ are the so-called generalized derivatives, ν_i is the i-th component of the outside normal vector to $\partial\Omega$ and the functions u, v in the curve integral (i.e. in the last integral at the right-hand side of Eq. (3.1)) are considered as traces of functions u, v on $\partial\Omega$.

The proof of the generalized Green's theorem can be found for example in [3].

In what follows we assume the density ϱ of flowing medium to be a non-negative decreasing function of the variable $u = |\operatorname{grad}\varphi|^2$ only. We are able to deduce this fact from Euler's equations of motion for an inviscid barotropic fluid assuming that the fluid density increases with increasing pressure.

Let Ω be an element of $\mathscr{C}^{0,1}$. In a standard way the corresponding boundary value problem for ideal gas flow may be formulated as follows: We look for a solution of Eq. (2.1) which fulfils the boundary conditions

(3.2) $$\varrho \operatorname{grad}_n \varphi/\partial\Omega = \Phi,$$

(3.3) $$\int_{\partial^*\Omega} \operatorname{grad}_t \varphi\, ds^* = \Gamma$$

and Prandtl's shock wave conditions.

In Eq. (3.2) Φ is a given function defined on the boundary $\partial\Omega$ of the domain Ω in such a way that on solid walls, not moving with the fluid, Φ equals zero and the symbol $\varrho \operatorname{grad}_n \varphi/\partial\Omega$ means the component of mass flow normal to the curve $\partial\Omega$ in the curve points $\partial\Omega$. In Eq. (3.3) Γ is the circulation past the swept body and $\partial^*\Omega$ is the boundary of the solid body swept by the fluid; in points $\partial^*\Omega \subset \partial\Omega$ Φ evidently equals zero. The symbol $\text{grad}_t \varphi / \partial^* \Omega$ means the velocity component tangential to the curve $\partial^* \Omega$ in the curve points $\partial^* \Omega \subset \partial \Omega$.

Assuming a nonzero circulation value, the sought velocity potential φ is a multivalued function of variables x_i ($i = 1, 2$). But choosing the domain Ω in a suitable way, one can determine the single-valued branch of this multivalued potential. In the case of a profile cascade, the boundary $\partial \Omega$ of the domain Ω is chosen in the way shown in Fig. 1. In the limit $h \to 0$ the abscissa $\overline{AB}$ identifies itself with the abscissa $\overline{CD}$. The domain Ω be-

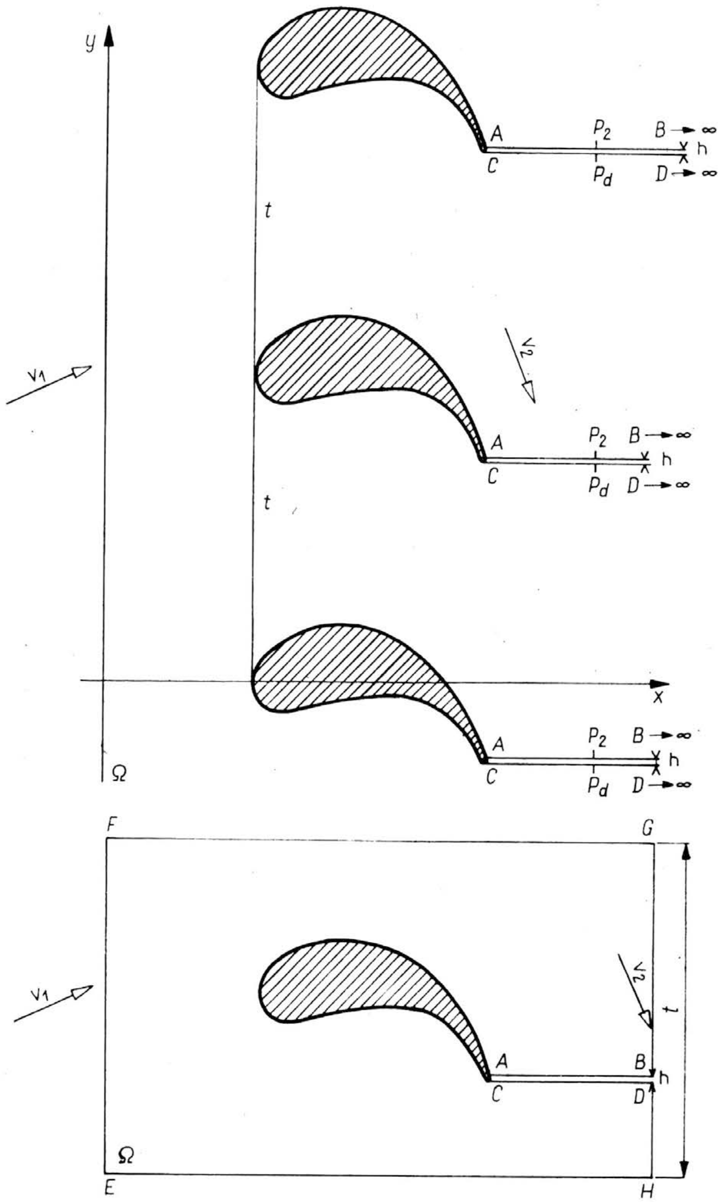

FIG. 1.

comes simply connected; in the limit case $h \to 0$ the difference of the potential values in any point P_h of the abscissa $\overline{AB}$ and the point P_d (corresponding to the point P_h on the abscissa $\overline{CD}$) equals

$$\varphi(P_h)-\varphi(P_d) = \Gamma. \tag{3.4}$$

On crossing the abscissa $\overline{AB} \equiv \overline{CD}$ the velocity potential is changed by jump; the jump value in the potential value is then given by Eq. (3.4). Equation (3.4) can be used for an arbitrary pair of corresponding points P_h, P_d and for that reason the equality of derivatives $\partial u/\partial s$ in the tangential direction holds for each such pair P_h, P_d. The abscissae identifying themselves in the limit case $h \to 0$ are a part of the boundary for the simply connected domain Ω; therefore one has to prescribe the boundary conditions on this part of the boundary, too. It is usually enough to take the equality

$$\frac{\partial\varphi}{\partial n}(P_h) = \frac{\partial\varphi}{\partial n}(P_d), \tag{3.5}$$

where the symbol $\partial/\partial n$ means the derivative in the direction normal to the abscissa $\overline{AB} \equiv \overline{CD}$.

In what follows we consider the domain Ω modified in the above mentioned way.

Applying the generalized Green's theorem, we will show that the boundary value problem for potential flow formulated in the standard way is identical with the following problem: One has to look for the critical point of the functional

$$F(\varphi) = \int_{\Omega} M(|\operatorname{grad}\varphi|^2)\,dx - \int_{\partial\Omega} \Phi\varphi\,ds + \lambda \int_{\partial^*\Omega} \operatorname{grad}_t\varphi\,ds^* \tag{3.6}$$

$\left(\text{where } u = |\operatorname{grad}\varphi|^2,\ M(u) = \frac{1}{2}\int_0^u \varrho(t)\,dt\right)$ on the set $V = \{\varphi \in W^{1,2}(\Omega),\ \varrho(|\operatorname{grad}\varphi|^2) > 0$ fulfilling the condition (3.2)$\}$. The set V is evidently bounded and convex. The value of the parameter λ (Lagrange's multiplicator) may not be chosen arbitrarily; one has to choose such a value of λ to satisfy the condition (3.3) with such a value Γ for which the Kutta–Zhukovsky condition of the swept outlet flow holds.

It is clear that the functional defined by Eq. (3.6) has a bounded value for the bounded domain Ω only. For this reason the domain Ω has to be modified as shown in Fig. 1a. The inlet conditions are defined on the part EF of the boundary upstream, the outlet conditions are defined on the part GH of the boundary downstream. The boundary conditions on FG or EH follow from the periodicity of velocity as long as the multiple of the pitch remains an integer (the whole number).

To study the functional (3.6) with such values of λ to which the Kutta–Zhukovsky condition of swept outlet flow does not apply, one has to modify the function $\varrho(u)$ in the following manner: A satisfactorily large value of u_m is prescribed so that, for $0 \leqslant u \leqslant u_m$, the function $\varrho^*(u)$ coincides with the above mentioned function $\varrho(u)$ and, for $u > u_m$, one puts $\varrho^*(u) = \varrho(u_m)$. This new function $\varrho^*(u)$ replaces the function $\varrho(u)$ in all our considerations.

DEFINITION 3.1. *Let $F(\varphi)$ be a functional defined with the aid of* Eq. (3.6) *on the open nonempty set V of the normed linear space $W^{1,2}(\Omega)$; let u be an element from V and $v \in W^{1,2}(\Omega)$ be such that*

$$\int_{\partial^*\Omega} \operatorname{grad}_t v\, ds^* = 0. \tag{3.7}$$

Then, for an arbitrary v satisfying the above mentioned condition, the function

$$f_v(t) = F(u+tv) \tag{3.8}$$

is a real function of the real variable t defined on an open interval $(-\delta, \delta)$ where the number $\delta > 0$ is sufficiently small. One assumes the function $f_v(t)$ to have a derivative in the point $t = 0$, i.e. there exists a bounded limit

$$f_v'(0) = \lim_{t\to 0} \frac{f_v(t)-f_v(0)}{t} = \lim_{t\to 0} \frac{F(u+tv)-F(u)}{t}. \tag{3.9}$$

Let the equation

$$dF(u, v) = f_v'(0) \tag{3.10}$$

hold. Then the number $dF(u, v)$ is named the first Gâteaux's derivative of the functional $F(u)$ in the point $u \in V$.

NOTE. Similarly one can define the second Gâteaux's derivative of the functional $F(u)$ and Gâteaux's derivatives of a higher orders.

DEFINITION 3.2. *The point $\varphi_0 \in V$ is called a critical point of the functional $F(\varphi)$, defined on the V, if there is $dF(\varphi_0, v)$ for an arbitrary $v \in W^{1,2}(\Omega)$ satisfying* Eq. (3.7) *and if the equation*

$$dF(\varphi_0, v) = 0 \tag{3.11}$$

holds.

The condition for the critical point of the functional (3.6) is evidently

$$0 = dF(\varphi, v) = \int_{\Omega} \varrho(|\operatorname{grad}\varphi|^2) \left\{ \frac{\partial\varphi}{\partial x_1} \cdot \frac{\partial v}{\partial x_1} + \frac{\partial\varphi}{\partial x_2} \cdot \frac{\partial v}{\partial x_2} \right\} dx_1\, dx_2 - \int_{\partial\Omega} \Phi v\, ds + \lambda \int_{\partial^*\Omega} \operatorname{grad}_t v\, ds^*. \tag{3.12}$$

Since the function $v \in W^{1,2}(\Omega)$ satisfies the condition $\int_{\partial^*\Omega} \operatorname{grad}_t v ds = 0$, the last term on the righ-hand side of Eq. (3.12) is omitted. Applying the generalized Green's theorem, the term

$$\int_{\Omega} \varrho(|\operatorname{grad}\varphi|^2) \left\{ \frac{\partial\varphi}{\partial x_1} \cdot \frac{\partial v}{\partial x_1} + \frac{\partial\varphi}{\partial x_2} \cdot \frac{\partial v}{\partial x_2} \right\} dx_1\, dx_2$$

is transformed to the form

$$-\int_{\Omega} v \left\{ \frac{\partial}{\partial x_1} \left(\varrho(|\operatorname{grad}\varphi|^2) \frac{\partial\varphi}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\varrho(|\operatorname{grad}\varphi|^2) \frac{\partial\varphi}{\partial x_2} \right) \right\} dx_1\, dx_2 \tag{3.13}$$

$$\text{(3.13) [cont.]} \qquad + \int_{\partial\Omega} \varrho(|\operatorname{grad}\varphi|^2)\cdot v\cdot \operatorname{grad}_n \varphi\, ds;$$

for $\varrho(|\operatorname{grad}\varphi|^2)\operatorname{grad}_n\varphi$ equals Φ in the points of $\partial\Omega$ and thus the expression (3.13) equals

$$(3.14) \qquad -\int_{\Omega} v\left\{\frac{\partial}{\partial x_1}\left(\varrho(|\operatorname{grad}\varphi|^2)\frac{\partial\varphi}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\varrho(|\operatorname{grad}\varphi|^2)\frac{\partial\varphi}{\partial x_2}\right)\right\}dx_1\,dx_2+\int_{\partial\Omega}\Phi v\,ds.$$

Substituting the expression (3.14) into Eq. (3.12), the condition (3.11) becomes identical with the equation

$$(3.15) \qquad 0=\int_{\Omega} v\left\{\frac{\partial}{\partial x_1}\left(\varrho(|\operatorname{grad}\varphi|^2)\frac{\partial\varphi}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\varrho(|\operatorname{grad}\varphi|^2)\frac{\partial\varphi}{\partial x_2}\right)\right\}dx,\,dx_2.$$

The function $\varphi\in V$ that satisfies Eq. (3.15) for all functions $v\in W^{1,2}(\Omega)$ within the condition $\int_{\partial^*\Omega}\operatorname{grad}_t v\,ds^*=0$ may be interpreted as a "weak" solution of Eq. (1.1) which fulfils the boundary conditions (3.2) and (3.3) in the sense of traces. It is very easy to show that the solution of Eq. (3.15) satisfies the so-called Prandtl's shock wave conditions (cf. [1]).

But this advantage is, however, compensated by the following complication: In many cases the critical point of the functional (3.6) is not its local extreme point. Let us determine $d^2F(\varphi,v,v)$; it will be seen that

$$(3.16) \qquad d^2F(\varphi,v,v)=\int_{\Omega}\left[\left\{2\frac{d\varrho(u)}{du}\left(\frac{\partial\varphi}{\partial x_1}\right)^2+\varrho(u)\right\}\left(\frac{\partial v}{\partial x_1}\right)^2\right.$$
$$\left.+4\frac{d\varrho(u)}{du}\frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial x_2}\frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2}+\left\{2\frac{d\varrho(u)}{du}\left(\frac{\partial\varphi}{\partial x_2}\right)^2+\varrho(u)\right\}\left(\frac{\partial v}{\partial x_2}\right)^2\right]dx_1\,dx_2.$$

The discriminant D of the quadratic form

$$(3.17) \qquad \left\{2\frac{d\varrho(u)}{du}\left(\frac{\partial\varphi}{\partial x_1}\right)^2+\varrho(u)\right\}\xi^2+4\frac{d\varrho(u)}{du}\frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial x_2}\xi\eta+\left\{2\frac{d\varrho(u)}{du}\left(\frac{\partial\varphi}{\partial x_2}\right)^2+\varrho(u)\right\}\eta^2$$

evidently equals

$$(3.18) \qquad D=-\frac{d}{du}[u\varrho^2(u)].$$

Consequently, as long as the derivative $\frac{d}{du}[u\varrho^2(u)]$ is positive in the whole Ω, the discriminant D is negative and the quadratic form (3.17) is positive definite so that the critical point is the minimum point of the functional (3.6).

If the derivative $\frac{d}{du}[u\varrho^2(u)]$ is negative in a nonempty part of Ω, the quadratic form (3.17) has a saddle point in the point $\xi=\eta=0$ and thus the critical point of the functional (3.6) is its saddle point, too. This fact brings many interesting consequences in the theory of transonic flows as shown in the following examples.

4. Examples

EXAMPLE 1. **The Ko–Tamada gas**

If the velocity of the flowing medium is related to the square sound velocity in the stagnation point (where the local Mach number equals zero), the fluid density ϱ is determined by the following equation:

$$\varrho(u) = \frac{1}{(1+u)^{1/2}}, \tag{4.1}$$

where u is the square of the (nondimensional) velocity. (The Ko–Tamada gas is defined by the expression (4.1)). In this case the inequality

$$\frac{d}{du}\left(u\varrho^2(u)\right) = \frac{d}{du}\,\frac{u}{1+u} = \frac{1}{(1+u)^2} > 0 \tag{4.2}$$

remains.

Since the set $V = \{\varphi \in W^{1,2}(\Omega),\ u = |\mathrm{grad}\,\varphi|^2,\ \varrho(u) > 0$ with the fulfiling boundary conditions (3.2)$\}$ is convex, one can prove with the aid of the inequality (4.2) and our considerations presented in Chapter 3 that the functional (3.6) is convex as well. Consequently the solution of the boundary value problem for the Ko–Tamada gas not only exists, but is uniquely determined from the boundary conditions.

EXAMPLE 2. **Isentropic subsonic flow of ideal gas**

In this case the following equation ($\varkappa > 1$)

$$\varrho(u) = \left\{1 - \frac{\varkappa-1}{2}\,u\right\}^{\frac{1}{\varkappa-1}}, \quad 0 \leqslant \sqrt{u} < \sqrt{\frac{2}{\varkappa+1}} = \text{nondimensional sound speed} \tag{4.3}$$

is to be used where u is the square ratio of the local fluid velocity related to the sound speed in the stagnation point. Then the following inequality

$$\frac{d}{du}\,u\varrho^2(u) = \left\{1 - \frac{\varkappa-1}{2}u\right\}^{\frac{2}{\varkappa-1}} \left\{\frac{1-\dfrac{\varkappa+1}{2}u}{1-\dfrac{\varkappa-1}{2}u}\right\} > 0 \tag{4.4}$$

is evidently true as far as $0 \leqslant u < \dfrac{2}{\varkappa+1}$. In this case again the set

$$V = \{\varphi \in W^{1,2}(\Omega)\}, \quad u = |\mathrm{grad}\,\varphi|^2, \quad 0 \leqslant \sqrt{u} < \sqrt{\frac{2}{\varkappa+1}}$$

with the boundary conditions (3.2) is convex. Using the inequality (4.4) and our considerations presented in Sect. 3, one can deduce that the inequality

$$d^2F(\varphi, v, v) > 0$$

holds. Thence it follows: the functional $F(\varphi)$ defined by Eq. (3.6) in the points of the set V is convex. Consequently the sought solution of the boundary value problem minimizes the functional (3.4) on the set V. The convex property of the functional (3.4) leads to the conclusion that in subsonic flows the solution of the boundary value problem not only exists but is uniquely determined from the boundary conditions.

Example 3. **Isentropic flow of ideal gas in a channel**

In this case the following equation ($\varkappa > 1$)

$$\varrho(u) = \left\{1 - \frac{\varkappa-1}{2}u\right\}^{\frac{1}{\varkappa-1}}, \tag{4.5}$$

$$0 \leqslant \sqrt{u} < \sqrt{\frac{2}{\varkappa-1}} = \text{maximum fluid velocity}$$

is to be used. The function $u\varrho^2(u)$ is not monotonous; as far as $0 \leqslant \sqrt{u} < \sqrt{\frac{2}{\varkappa+1}}$ the function $u\varrho^2(u)$ increases, for $\sqrt{u} = \sqrt{\frac{2}{\varkappa+1}}$ it reaches its maximum value and while $\sqrt{\frac{2}{\varkappa+1}} < \sqrt{u}$ the function $u\varrho^2(u)$ decreases till $\sqrt{u} = \sqrt{\frac{2}{\varkappa-1}}$, when $u\varrho^2(u)$ equals zero. For the transonic channel flow of ideal gas there are the following boundary conditions:

$$\operatorname{grad}_x\varphi|_{x=-l} = q, \quad \sqrt{\frac{2}{\varkappa+1}} < q < \sqrt{\frac{2}{\varkappa-1}}, \tag{4.6}$$

$$\operatorname{grad}_y\varphi|_{y=0} = \operatorname{grad}_y\varphi|_{y=h} = 0,$$

where $y = 0$, $-l \leqslant x \leqslant l$ and $y = h > 0$, $-l \leqslant x \leqslant l$ are equations of the channel walls. Requirements due to physical reasons: $|\operatorname{grad}\varphi|$ is changed by jump only from its higher to its lower value. (If $|\operatorname{grad}\varphi|$ jumps from its lower to its higher value, the entropy decreases — though very little; this contradicts the second law of thermodynamics).

The velocity potential for a two-dimensional channel is only a linear function of the variables x, y; using the boundary conditions (4.6) one can deduce that the velocity potential is a linear function of the variable x only; so the solution of the given boundary value problem is defined by the equations

$$\frac{\partial\varphi}{\partial y} = 0,$$

$$\varrho(|\operatorname{grad}\varphi|^2)\frac{\partial\varphi}{\partial x} = \text{const}. \tag{4.7}$$

As long as q satisfies the inequality $\sqrt{\frac{2}{\varkappa+1}} < q < \sqrt{\frac{2}{\varkappa-1}}$, one can adopt all solutions defined in the following way:

$$\text{(4.8)} \qquad \begin{aligned} &\text{if} \quad -l \leqslant x \leqslant x_0, \quad &\varphi = qx, \\ &\text{if} \quad x_0 < x \leqslant l, \quad &\varphi = q_1 x + k_0, \end{aligned}$$

where q_1 is determined by the postulates

$$\text{(4.9)} \qquad \begin{gathered} q_1 \varphi(q_1^2) = \varrho(q^2) q, \\ q_1 < q \end{gathered}$$

and the constant k_0 fulfils the conditions that the velocity potential is continuous in the point $x = x_0$; thence

$$\text{(4.10)} \qquad k_0 = (q - q_1) x_0 .$$

Considering that the function $\sqrt{u}\varrho(u)$ has no monotone course with respect to the variable u, one can solve Eq. (4.9) at any time. Our consideration thus results in the position (i.e. the value x_0) being undetermined. It may be of interest to note that in the studied problem when the velocity potential and the velocity are not uniquely determined, the stream function is defined in only one way.

A question may be raised if all solutions of the considered boundary value problem are physically relevant. The answer is as follows: The solution involving the jump in the velocity course for $x_0 = -l$, $0 \leqslant y \leqslant h$ (i.e. with subsonic speed inside the channel) minimizes the value of the functional (3.6) on the set $V = \{\varphi \in W^{1,2}(\Omega),\ u = |\operatorname{grad} \varphi|^2,$ $0 \leqslant \sqrt{u} \leqslant \sqrt{\dfrac{2}{\varkappa - 1}}$ and on the conditions (3.2)$\}$; consequently this solution is stable for small perturbations of the Ω-boundary form.

All other solutions in question (i.e. solutions defined by Eqs. (4.8), (4.9) and (4.10) including the solution with the jump in the velocity course for $x_0 = l$, $0 \leqslant y \leqslant h$) define saddle points of the functional (3.6). It follows that all such solutions are not stable for small perturbations of the Ω-boundary form.

If the channel walls were perfectly parallel and perfectly smooth, the flow of the inviscid ideal gas inside the channel would be supersonic with no shock waves. But as soon as a small obstacle (cf. Fig. 2) narrows the cross-section of the channel, the velocity within

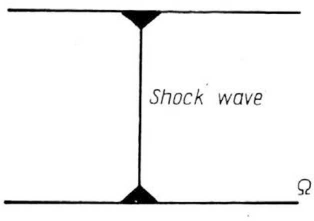

FIG. 2.

a close distance sinks to the corresponding subsonic value (to maintain the flow volume given by the inlet conditions). In the limit case this is idealized as though the velocity of the flowing ideal gas changed by jump in the place of the narrowing cross-section. For this reason the shock wave position is defined by the perturbation of the channel wall form. If real viscid gas is used, the channel cross-section will be narrowed successively with the rising displacement thickness of the boundary layer. With the increasing displacement thickness the gas velocity inside the potential flow core decreases; this velocity drop results in a secondary displacement thickness rising faster than that acceptable for the constant pressure gradient in the channel. When the displacement thickness increases so that the channel cross-section is near to its critical value (the velocity of the flowing medium equaling the gas sound speed), then within a close distance the gas velocity changes from a supersonic to a subsonic value. In the limit case this is idealized in such a way that the gas velocity changes by jump from a supersonic to a subsonic value never changing the flow volume given by the inlet conditions. In a subsonic flow (i.e. behind the shock wave) the gas velocity in the potential flow core increases with the increasing displacement thickness of the boundary layer. This increasing velocity makes the displacement thickness rise more slowly than that acceptable for the constant pressure gradient in a channel. This results in two facts:

a) The transonic channel flow of real viscid gas with one normal shock wave is able to keep up for a "longer" time than a supersonic flow with no shock waves.

b) Since the increase of the displacement boundary-layer thickness is essentially given by the pressure drop in the channel (in our symbolics by the expression $p_2-p_1/2l$) and therefore (assuming l to be constant) by the pressure difference p_2-p_1, the shock wave position in the channel is given by the pressure difference, too. If the calculated transonic flow of the ideal gas had to correspond to the real gas flow, the position of the shock wave must be determined by an additional condition, e.g. by prescribing the velocity potential value in the place of the shock wave.

EXAMPLE 4. **A two-dimensional cascade flow with abscissae-shaped profiles and with circulation $\Gamma = 0$**

This case is represented in Fig. 3; let us denote the "stagger", i.e. the angle between the cascade front and the vector of the inlet velocity by α. If M_1 is the inlet Mach number,

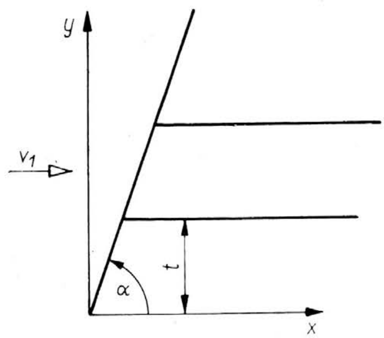

FIG. 3.

then there exists one solution with constant velocity over the whole flow field. The condition of Kutta–Zhukovsky at the trailing edge of profiles has been fulfilled.

EXAMPLE 5. **A two-dimensional cascade transonic flow with abscissae-shaped profiles with such a non-zero circulation that the inlet angle equals the outlet angle**

This case is shown in Fig. 4; let us denote "the stagger" by α. Let M_1 be the value of the local inlet Mach number and M_2 that of the outlet Mach number. Then evidently (the velocity being parallel to the x-axis) the equation

$$M_1\left(1+\frac{\varkappa-1}{2}M_1^2\right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} = M_2\left(1+\frac{\varkappa-1}{2}M_2^2\right)^{-\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}} \tag{4.11}$$

is correct. The condition (4.11) expresses the law of mass conservation (i.e. the ratio of the inlet flow volume divided by the pitch equals that of the outlet). If $M_1 > 1$, Eq.

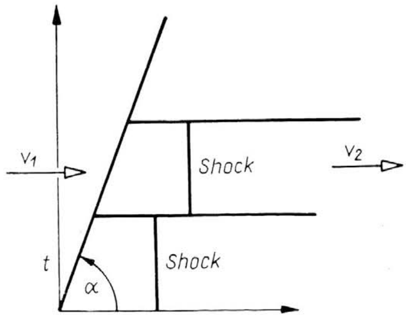

FIG. 4.

(4.11) has only one root $M_2 < 1$. Let us denote the cascade pitch by the symbol t; then the circulation Γ is given by the following equation:

$$\frac{\Gamma}{a_0 t} = (\operatorname{tg}\alpha)^{-1}\left[\frac{M_1}{\left(1+\frac{\varkappa-1}{2}M_1^2\right)^{1/2}} - \frac{M_2}{\left(1+\frac{\varkappa-1}{2}M_2^2\right)^{1/2}}\right], \tag{4.12}$$

where a_0 means the sound speed in the stagnation point of the flow field. Since $\Gamma > 0$, a lifting force works on the profiles, which is proportional to Γ.

Since the velocity of the flowing medium is periodic in the direction parallel to the cascade front with its period equal to $t(\sin\alpha)^{-1}$, it follows that the suitable solution is that with only one normal shock wave whose position is undetermined. The shock wave is inside the cascade space — neither outside nor behind the cascade. In this case the Kutta–Zhukovsky condition of a smooth outlet on the trailing edges has been fulfilled as well. Adopting the explanation that the constant velocity supersonic flow on a cascade of the considered type is unstable due to small perturbations of swept wall and to the boundary layer development, one can see why forces caused by instability in transonic flow can often cause cascade damage.

5. Conclusion

To conclude our discussion, one can make the following observations:

a) In the transonic flow of ideal gas the ambiguity of the boundary value problem solutions is generally caused not only by the shape of the domain Ω but also by the boundary conditions; for a given domain Ω and prescribed boundary conditions the considered boundary value problem may have one solution. But if Ω remains unchanged and only the boundary conditions (e.g. the value of Γ) change, the considered boundary value problem may have many solutions.

b) Nøt all solutions are equally relevant from the physical point of view; the increasing thickness of the boundary layer may result in only one of all possible solutions effective for a longer period of time.

c) The numerical methods developed to solve the boundary value problems in transonic flow will have to account for the effects of nonzero viscosity.

NOTE. It will be understood that the solutions — except those given in example 3, 4, 5 — might be unambiguously determined by the boundary conditions. This hypothesis is supported by the fact that in curvilinear channels solutions with no shock waves as well as with normal shock waves can only be successful if gradϱ is parallel to gradφ; this means that such a case in isentropic flows is possible only if ϱ is constant on equipotential curves.

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