# 17.

## ELEMENTARY RESEARCHES IN THE ANALYSIS OF COMBINATORIAL AGGREGATION.

[*Philosophical Magazine*, XXIV. (1844), pp. 285—296.]

THE ensuing inquiries will be found to relate to combination-systems, that is, to combinations viewed in an aggregative capacity, whose species being given, we shall have to discover rules for ranging or evolving them in classes amenable to certain prescribed conditions. The question of numerical amount will only appear incidentally, and never be made the primary object of investigation*.

The number of things combined will be termed the modulus of the system to which they belong. The elements taken singly, or combined in twos, threes, &c., will be denominated accordingly the monadic, duadic, triadic elements, or simply the monads, duads, or triads of the system.

Let us agree to denote by the word syntheme† any aggregate of combinations in which all the monads of a given system appear once, and once only.

It is manifest that many such synthemes totally diverse in every term may be obtained for a given system to any modulus, and for any order of combination.

Let us begin with considering the case of duad synthemes. Take the modulus 4 and call the elements $a$, $b$, $c$, $d$.

$(ab, cd)$, $(ac, bd)$, $(ad, cb)$ constitute three perfectly independent synthemes, and these three synthemes include between them all the duad elements, so that no more independent synthemes can be obtained from them.

* The present theory may be considered as belonging to a part of mathematics which bears to the combinatorial analysis much the same relation as the geometry of position to that of measure, or the theory of numbers to computative arithmetic; number, place, and combination (as it seems to the author of this paper) being the three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred.

† From σὺν and τίθημι.

Again, let $a, b, c, d, e, f$ be the monads; we can write down five independent synthemes, to wit,

$$\left.\begin{array}{l} ab,\ cd,\ ef \\ ad,\ cf,\ eb \\ ac,\ de,\ fb \\ af,\ bd,\ ce \\ ae,\ df,\ bc \end{array}\right\}.$$

We can write no more than these without repeating duads which have already appeared*.

We propose to ourselves this problem:—*A system to any even*† *modulus being given, to arrange the whole of its duads*‡ *in the form of synthemes*; or in other words, *to evolve a Total of duad synthemes to any given even modulus* §.

When the modulus is odd, as before remarked, the formation of a duad syntheme is of course impossible, for any number of duads must necessarily contain an even number of monadic elements; but there is nothing to prevent us from forming in *all* cases what may be termed a bisyntheme or diplotheme, that is, an aggregate of combinations, where each element occurs twice and no more.

For instance, if the elements be called after the letters of the alphabet, we have $\begin{pmatrix} ab, & bc, & cd, & de, & ea \\ ac, & ce, & eb, & bd, & da \end{pmatrix}$, the bisynthematic total to modulus 5; and in

* Such an *aggregate* of synthemes may be therefore termed a Total.

† The modulus must be even, as otherwise it is manifest no single syntheme can be formed. We shall before long extend the scope of our inquiry so as to take in the case of odd moduli.

‡ Triadic systems will be treated of hereafter.

§ It is scarcely necessary to advert here to the fact of the problem being in general indeterminate and admitting of a great variety of solutions.

When the modulus is four there is only one synthematic arrangement possible, and there is no indeterminateness of any kind; from this we can infer, *à priori*, the reducibility of a biquadratic equation; for using $\phi, f, F$ to denote rational symmetrical forms of function, it follows that

$$F\left\{\begin{array}{l} f\{\phi(a, b),\ \phi(c, d)\} \\ f\{\phi(a, c),\ \phi(b, d)\} \\ f\{\phi(a, d),\ \phi(b, c)\} \end{array}\right\} \text{ is itself a rational symmetric function of } a, b, c, d.$$

Whence it follows that if $a, b, c, d$ be the roots of a biquadratic equation, $f\{\phi(a, b), \phi(c, d)\}$ can be found by the solution of a cubic: for instance, $(a+b)\times(c+d)$ can be thus determined, whence immediately the sum of any two of the roots comes out from a quadratic equation.

To the modulus 6 there are fifteen different synthemes capable of being constructed; at first sight it might be supposed that these could be classed in natural families of three or of five each, on which supposition the equation of the sixth degree could be depressed; but on inquiry this hope will prove to be futile, not but what natural affinities do exist between the totals; but in order to separate them into families each will have to be taken twice over, or in other words, the fifteen synthemes to modulus 6 being reduplicated subdivide into six natural families of five each. Again, it is true that the triads to modulus 6 (just like the duads to modulus 4) admit of being thrown into but one synthematic total, but then this will contain ten synthemes, a number greater than the modulus itself.

like manner

$$\left.\begin{array}{l} ab,\ bc,\ cd,\ de,\ ef,\ fg,\ ga \\ ac,\ ce,\ eg,\ gb,\ bd,\ df,\ fa \\ ad,\ dg,\ gc,\ cf,\ fb,\ be,\ ea \end{array}\right\} \text{ the total to modulus 7.}$$

In general, if $n$ be the modulus, the number of duads is $n\dfrac{n-1}{2}$; $n$ being even, $\dfrac{n}{2}$ duads go to each syntheme, and therefore the total contains $(n-1)$ of these. If $n$ be odd, then, since always $n$ duads go to a bisyntheme, the number of such in the total is $\dfrac{n-1}{2}$.

Before proceeding to the solution of the problem first proposed, let us investigate the theory of diplothematic arrangement. Here we shall find another term convenient to employ. By a cyclotheme, I designate a fixed arrangement of the elements in one or more circles, in which, although for typographical purposes they are written out in a straight line, the last term is to be viewed as contiguous and antecedent to the first; the recurrence may be denoted by laying a dot upon the two opened ends of the circle; $\dot{a}.b.c.d.\dot{e}$ will thus denote a cyclotheme to modulus 5; $\dot{a}.b.c.d.e.f.g.h.\dot{k}$ the same to modulus 9; so also is $\dot{a}.b.\dot{c},\ \dot{d}.e.\dot{f},\ \dot{g}.h.\dot{k}$ a cyclotheme of another species to the same modulus. In general the number of terms will be alike in each division of a cyclotheme.

Now it is evident that every cyclotheme, on taking together the elements that lie in conjunction, may be developed into a diplotheme. Thus

$$\dot{1}.2.\dot{3} = 12,\ 23,\ 31,$$

$$\dot{1}.2.3.\dot{4} = 12,\ 23,\ 34,\ 41,$$

$$(\dot{1}.2.\dot{3};\ \dot{4}.5.\dot{6};\ \dot{7}.8.\dot{9}) = \begin{pmatrix} 12, & 23, & 31 \\ 45, & 56, & 64 \\ 78, & 89, & 97 \end{pmatrix}.$$

Hence we shall derive a rule for throwing the duads of any system into bisynthemes.

Let $m = 3$, we have simply $\dot{a}b\dot{c}$,

$m = 5$, we write $\dot{a}.b.c.d.\dot{e}$,

$\dot{a}.c.e.b.\dot{d}$,

the second being derived from the first by omitting every alternate term; similarly below, the lines are derived each from its antecedent.

$m = 7$, we have $\dot{a}.b.c.d.e.f.\dot{g}$,

$\dot{a}.c.e.g.b.d.\dot{f}$,

$\dot{a}.e.b.f.c.g.\dot{d}$.

A very little consideration will serve to prove that in this way, $m$ being a *prime number*, $\frac{m-1}{2}$ cyclothemes may be formed, such that no element will ever be found more than once in contact on either side with any other; whence the rule for obtaining the diplothematic total to any prime-number modulus is apparent.

For example, to modulus 7 the total reads thus:—

$$\left.\begin{array}{ll}\text{1st.} & ab,\ bc,\ cd,\ de,\ ef,\ fg,\ ga \\ \text{2nd.} & ac,\ ce,\ eg,\ gb,\ bd,\ df,\ fa \\ \text{3rd.} & ae,\ eb,\ bf,\ fc,\ cg,\ gd,\ da\end{array}\right\},$$

and no more remains to be said on this special case.

Let us now return to the theory of even moduli, and show how to apply what has been just done to constructing a synthematic total to a modulus which is the double of a prime number.

Suppose the modulus to be six, the number of synthemes is five. Let the six elements, $a, b, c, d, e, f$, be taken in three parts, so that each part contains two of them; let these parts be called $A, B, C$, where $A$ denotes $ab$, $B$, $cd$, and $C$, $ef$.

Now the duads will evidently admit of a distinction into two classes, those that lie in one part, and those that lie between two; thus $ab$, $cd$, $ef$ will be each *unipartite* duads, the rest will be *bipartite*.

The unipartite duads may be conveniently formed into a syntheme by themselves; it only remains to form the four remaining bipartite duad synthemes.

Write the parts in cyclothematic order, as below:

$$\dot{A}B\dot{C}.$$

It will be observed that each part may be written in two positions; thus

$$\begin{array}{llll} A \text{ may be expressed by} & \begin{matrix}a\\b\end{matrix} & \text{or by} & \begin{matrix}b\\a\end{matrix}, \\ B \quad ,, \quad ,, & \begin{matrix}c\\d\end{matrix} & ,, & \begin{matrix}d\\c\end{matrix}, \\ C \quad ,, \quad ,, & \begin{matrix}e\\f\end{matrix} & ,, & \begin{matrix}f\\e\end{matrix}. \end{array}$$

Now we may form a cyclic table of positions as below:

| $\dot{A}$ | $B$ | $\dot{C}$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 2 |
| 2 | 1 | 2 |
| 2 | 2 | 1 |

Here the numbers in each horizontal line denote the synchronic positions of the parts.

On inspection it will be discovered that $A$ will be found in each of its two positions, with $B$ in each of its two; similarly $B$ with $C$, and $C$ with $A$. In fact the four permutations, 11, 12, 21, 22, occur, though in different orders, in any two assigned vertical columns.

Now develope the preceding table, and we have

$$\dot{a}c\dot{e} \quad \dot{a}d\dot{f} \quad \dot{b}c\dot{f} \quad \dot{b}d\dot{e},$$
$$bdf \quad bce \quad ade \quad acf;$$

and these being read off (the *superior* of each antecedent with the *inferior* of each consequent*) must manifestly give the four independent bipartite synthemes which we were in quest of, *videlicet*

$$(ad,\ cf,\ eb),\quad (ac,\ de,\ fb),\quad (bd,\ ce,\ fa),\quad (bc,\ df,\ ea);$$

these four, together with the syntheme first described ($ab$, $cd$, $ef$), constitute a duad synthematic total to modulus 6.

Before proceeding further let us take occasion to remark that the foregoing table of positions may evidently be extended to any odd number of terms by repetition of the second and third places, as seen in the annexed tables of position.

$$\dot{1}.1.1.1.\dot{1} \quad \dot{1}.1.1.1.1.1.\dot{1},$$
$$\dot{1}.2.2.2.\dot{2} \quad \dot{1}.2.2.2.2.2.\dot{2}\dagger,$$
$$\dot{2}.1.2.1.\dot{2} \quad \dot{2}.1.2.1.2.1.\dot{2},$$
$$\dot{2}.2.1.2.\dot{1} \quad \dot{2}.2.1.2.1.2.\dot{1}.$$

Now let 10 be the modulus.

As before divide the elements into five parts, which call $A$, $B$, $C$, $D$, $E$.

The unipartite duads fall into a single syntheme; the eight remaining bipartite synthemes may be found as follows:—

Arrange in cyclothemes $\left(\frac{n-1}{2} \text{ in number}\right)$ the odd modulus system $A, B, C, D, E$. We have thus

$$\dot{A}BCD\dot{E},$$
$$\dot{A}CEB\dot{D}.$$

* Any other *fixed* order of successive conjunction would answer equally well.

† It will not fail to be borne in mind that in operating with these tables only *contiguous* elements are taken in conjunction: the first with the second, the second with the third, the third with the fourth, &c., and the last with the first; no two terms but such as lie together are in any manner conjugated with one another.

Let each cyclotheme be taken in the four positions given in the table above, we have thus 2 × 4, that is, eight arguments.

$$\dot{a}\,b\,c\,d\,\dot{e}\,.\,\dot{a}\,\beta\,\gamma\,\delta\,\dot{\epsilon}\,.\,\dot{a}\,b\,\gamma\,d\,\dot{\epsilon}\,.\,\dot{a}\,\beta\,c\,\delta\,\dot{e},$$

$$\alpha\,\beta\,\gamma\,\delta\,\epsilon,\ \ a\,b\,c\,d\,e,\ \ a\,\beta\,c\,\delta\,e,\ \ a\,b\,\gamma\,d\,\epsilon,$$

$$\dot{a}\,c\,e\,b\,\dot{d}\,.\,\dot{a}\,\gamma\,\epsilon\,\beta\,\dot{\delta}\,.\,\dot{a}\,c\,\epsilon\,b\,\dot{\delta}\,.\,\dot{a}\,\gamma\,e\,\beta\,\dot{d},$$

$$\alpha\,\gamma\,\epsilon\,\beta\,\delta,\ \ a\,c\,e\,b\,d,\ \ a\,\gamma\,e\,\beta\,d,\ \ a\,c\,\epsilon\,b\,\delta.$$

And each of these arguments will furnish one bipartite syntheme, by reading off, as before, the *superior* of each antecedent with the *inferior* of each consequent; and the least reflection will serve to show that the same duad can never appear in two distinct arguments.

In like manner, if the modulus be 14 and seven parts be taken, the bipartite synthemes, twelve in number, may be expressed symbolically thus:

$$\left\{\begin{array}{l} \dot{1}.1.1.1.1.1.\dot{1} \\ +\,\dot{1}.2.2.2.2.2.\dot{2} \\ +\,\dot{2}.1.2.1.2.1.\dot{2} \\ +\,\dot{2}.2.1.2.1.2.\dot{1} \end{array}\right\} \times \left\{\begin{array}{l} \dot{A}.B.C.D.E.F.\dot{G} \\ +\,\dot{A}.C.E.G.B.D.\dot{F} \\ +\,\dot{A}.E.B.F.C.G.\dot{D} \end{array}\right\}.$$

Nay more, from the above table, if we agree to name the elements $\begin{matrix} A_1 & B_1 \\ A_2 & B_2 \end{matrix}$, &c., we can at once proceed to calculate each of the twelve synthemes in question by an easy algorithm. For instance,

$$(\dot{1}.2.2.2.2.2.\dot{2}) \times (\dot{A}.C.E.G.B.D.\dot{F})$$

$$= (A_1C_1,\ C_2E_1,\ E_2G_1,\ G_2B_1,\ B_2D_1,\ D_2F_1,\ F_2A_2).$$

And again

$$(\dot{2}.1.2.1.2.1.\dot{2}) \times (\dot{A}.E.B.F.C.G.\dot{D})$$

$$= A_2E_2,\ E_1B_1,\ B_2F_2,\ F_1C_1,\ C_2G_2,\ G_1D_1,\ D_2A_1;$$

each figure occurring once unchanged as an antecedent and once changed as a consequent.

If it were thought worth while it would not be difficult, by using numbers instead of letters, to obtain a general analytical formula, from which all similarly constituted synthemes to any modulus might be evolved.

But the rule of proceeding must be now sufficiently obvious; the modulus being $2p$, we divide the elements into $p$ classes; these may be arranged into $\frac{p-1}{2}$ distinct forms of cyclothematic arrangement, and each of the cyclothemes taken in four positions, thus giving $4 \times \frac{p-1}{2}$, that is, $2p-2$ bipartite synthemes, the whole number that can be formed to the given modulus $2p$.

I shall now proceed to the theory of bipartite synthemes to the modulus $2m \times p$, by which it is to be understood that we have $p$ parts each containing $2m$ terms, and $p$ is at present supposed to be a prime number; the total number of synthemes to the modulus $2mp$ being $2mp-1$, and $2m-1$ of these evidently being capable of being made unipartite; the remainder, $2mp-2m$, that is, $(p-1)\,2m$, will be the number of bipartites to be obtained*:

$$2m\,(p-1) = \frac{p-1}{2} \times 4m;$$

$\frac{p-1}{2}$ denotes the total number of cyclothemes to modulus $p$; $4m$, as will be presently shown, the number of lines or syzygies in the *Table of position.*

To fix our ideas let the modulus be $4 \times 3$, and let $A$, $B$, $C$ be three parts:

$$\left.\begin{matrix} a_1\,a_2\,a_3\,a_4 \\ b_1\,b_2\,b_3\,b_4 \\ c_1\,c_2\,c_3\,c_4 \end{matrix}\right\} \text{ their constituents respectively.}$$

Give a *fixed* order to the constituents of each part, then each of them may be taken in four positions; thus $A$ may be written

$$a_1\,a_2\,a_3\,a_4,$$
$$a_2\,a_3\,a_4\,a_1,$$
$$a_3\,a_4\,a_1\,a_2,$$
$$a_4\,a_1\,a_2\,a_3.$$

Assume some particular position for each, as, for instance,

$$a_1\,b_1\,c_1,$$
$$a_2\,b_2\,c_2,$$
$$a_3\,b_3\,c_3,$$
$$a_4\,b_4\,c_4,$$

and read off by coupling the first and third vertical places of each antecedent with the second and fourth respectively of each consequent; we have accordingly,

$$a_1b_2,\ b_1c_2,\ c_1a_2,$$
$$a_3b_4,\ b_3c_4,\ c_3a_4.$$

It is apparent that the same combinations will recur if any two contiguous parts revolve simultaneously through two steps; or in other words, that $A_rB_s = A_{r+2}B_{s+2}$, where $\mu$ is any number, odd or even.

* In general, if there be $\pi$ parts of $\mu$ terms each, and $\mu\pi$ be even, the number of bipartite synthemes is $(\pi-1)\,\mu$, as is easily shown from dividing the whole number of bipartite duads by the semi-modulus.

Symbolically speaking, therefore, as regards our table of position,

$$r : s = r + 2 : s + 2,$$

or more generally,

$$= r + 2 \pm 4i : s + 2 \pm 4i.$$

So that

$$1:1 = 3:3, \qquad 2:1 = 4:3,$$
$$1:2 = 3:4, \qquad 2.2 = 4:4,$$
$$1:3 = 3.1, \qquad 2.3 = 4:1,$$
$$1:4 = 3.2, \qquad 2.4 = 4:2.$$

There are therefore no more than eight independent unequivalent permutations to every pair of parts. Now inspect the following table of position:—

$$\dot{1}.1.\dot{1}, \qquad \dot{2}.1.\dot{2},$$
$$1.2.3, \qquad 2.2.4,$$
$$1.3.2, \qquad 2.3.1,$$
$$1.4.4, \qquad 2.4.3.$$

It will be seen that in the first and second, second and third, third and first places, all the eight independent permutations occur under different *names*; the law of formation of such and similar tables will be explained in due time; enough for our present object to see how, by means of this table, we are able to obtain the bipartite synthemes to the given modulus $4 \times 3$; the number according to our formula is $2 \times 4 \times \frac{3-1}{2} = 8$, and they may be denoted symbolically as follows:—

$$(\dot{A}.B.\dot{C})\begin{pmatrix}1.1.1 + 1.2.3 + 1.3.2 + 1.4.4 \\ + 2.1.2 + 2.2.4 + 2.3.1 + 2.4.3\end{pmatrix}.$$

Each of the eight terms connected by the sign of + gives a distinct syntheme; for example, let us operate on

$$\dot{A}.B.\dot{C} \times (2.3.1).$$

2.3.1 denotes 2.3, 3.1, 1.2.

2.3 gives rise to $2(3+1) + (2+2).(3+3) = 2.4 + 4.2$.

3.1 gives rise to $3(1+1) + (3+2).(1+3) = 3.2 + 1.4$.

1.2 gives rise to $1(2+1) + (1+2).(2+3) = 1.3 + 3.1$.

The syntheme in question is therefore

$$A_2B_4,\ A_4B_2,\ B_3C_2,\ B_1C_4,\ C_1A_3,\ C_3A_1,$$

and so on for all the rest, the rule being that

$$r : s = r(s+1) + (r+2)(s+3).$$

Now, as before, it is evident that if we look only to contiguous terms, the above table of position may be extended to any number of odd terms, simply by repetition of the second and third figures in each syzygy; and hence the rule for obtaining the bipartite synthemes to the modulus $4 \times p$ is apparent. For instance, let $p = 7$, there will be $8 \times \frac{7-1}{2}$, that is, $8 \times 3$ of them denoted as follows:—

$$\left\{\begin{array}{l} \dot{A}.B.C.D.E.F.\dot{G} \\ +\dot{A}.C.E.G.B.D.\dot{F} \\ +\dot{A}.E.B.F.C.G.\dot{D} \end{array}\right\} \times \left\{\begin{array}{l} 1.1.1.1.1.1.1+2.1.2.1.2.1.2 \\ +1.2.3.2.3.2.3+2.2.4.2.4.2.4 \\ +1.3.2.3.2.3.2+2.3.1.3.1.3.1 \\ +1.4.4.4.4.4.4+2.4.3.4.3.4.3 \end{array}\right\}.$$

As an example of the mode of development, let us take the term

$$\dot{A}.E.B.F.C.G.\dot{D} \times \dot{2}.4.3.4.3.4.\dot{3},$$

$$\dot{2}.4.3.4.3.4.\dot{3} = (2:4,\ 4:3,\ 3:4,\ 4:3,\ 3:4,\ 4:3,\ 3:2)$$

$$= \left(\begin{matrix} 2.1 \\ +4.3 \end{matrix}\right\} \left.\begin{matrix} 4.4 \\ +2.2 \end{matrix}\right\} \left.\begin{matrix} 3.1 \\ +1.3 \end{matrix}\right\} \left.\begin{matrix} 4.4 \\ +2.2 \end{matrix}\right\} \left.\begin{matrix} 3.1 \\ +1.3 \end{matrix}\right\} \left.\begin{matrix} 4.4 \\ +2.2 \end{matrix}\right\} \left.\begin{matrix} 3.3 \\ +1.1 \end{matrix}\right\}\Bigg),$$

$$\dot{A}.E.B.F.C.G.\dot{D} = A.E,\ E.B,\ B.F,\ F.C,\ C.G,\ G.D,\ D.A,$$

and the product

$$= \left(\begin{matrix} A_2E_1, & E_4B_4, & B_3F_1, & F_4C_4, & C_3G_1, & G_4D_4, & D_3A_3 \\ A_4E_3, & E_2B_2, & B_1F_3, & F_2C_2, & C_1G_3, & G_2D_2, & D_1A_1 \end{matrix}\right).$$

Let the modulus be $6 \times 3$; as before, give a *fixed* cyclic order to the constituents of each part, and each will admit of being exhibited in six positions.

Write similarly as before,

$$\begin{array}{c} \dot{a}_1 b_1 \dot{c}_1, \\ a_2 b_2 c_2, \\ a_3 b_3 c_3, \\ a_4 b_4 c_4, \\ a_5 b_5 c_5, \\ a_6 b_6 c_6, \end{array}$$

and take the odd places of each antecedent with the even places of each consequent; it will now be seen that

$$r:s = r+2:s+2 = r+4:s+4,$$

and the number of independent permutations is $\frac{6.6}{3} = 2.6$; and so in general, if there be $2m$ constituents in a part, the number of independent permutations is $\frac{2m.2m}{\frac{m}{2}} = 4m$.

The rule for the formation of the table will be apparent on inspection. I suppose only three parts, as the rule may always be extended to any number by reiteration of the second and third terms. The table will be found to resolve itself naturally into four parts, each containing $m$ lines.

Let $m = 1$, we have

1 . 1 . 1  2 . 1 . 2
1 . 2 . 2  2 . 2 . 1

$m = 2$, we have

1 . 1 . 1  2 . 1 . 2
1 . 2 . 3  2 . 2 . 4
1 . 3 . 2  2 . 3 . 1
1 . 4 . 4  2 . 4 . 3

$m = 3$, we have

1 . 1 . 1  2 . 1 . 2
1 . 2 . 3  2 . 2 . 4
1 . 3 . 5  2 . 3 . 6
1 . 4 . 2  2 . 4 . 1
1 . 5 . 4  2 . 5 . 3
1 . 6 . 6  2 . 6 . 5

$m = 4$, we have

1 . 1 . 1  2 . 1 . 2
1 . 2 . 3  2 . 2 . 4
1 . 3 . 5  2 . 3 . 6
1 . 4 . 7  2 . 4 . 8
1 . 5 . 2  2 . 5 . 1
1 . 6 . 4  2 . 6 . 3
1 . 7 . 6  2 . 7 . 5
1 . 8 . 8  2 . 8 . 7

So that $x$, going through all its values from 1 to $m$, the general expression for the four parts is

$$\Sigma \begin{Bmatrix} 1 \,.\, x\,(2x-1) + 1\,(m+x)\,2x \\ +\,2 \,.\, x \,.\, 2x \quad + 2\,(m+x)\,(2x-1) \end{Bmatrix}.$$

To show the use of this formula, let us suppose that we have seven parts, each containing ten terms, the general expression for the bipartite duad synthemes is

$$\begin{Bmatrix} \dot{A}.B.C.D.E.F.\dot{G} \\ +\,A.C.E.G.B.D.F \\ +\,A.E.B.F.C.G.D \end{Bmatrix} \times \Sigma \begin{Bmatrix} 1 \,.\, x\,(2x-1)\,x\,(2x-1)\,x\,(2x-1) \\ +\,2 \,.\, x \,.\, 2x \,.\, x \,.\, 2x \,.\, x \,.\, 2x \\ +\,1\,(5+x)\,2x\,(5+x)\,2x\,(5+x)\,2x \\ +\,2\,(5+x)(2x-1)\,(5+x)(2x-1)\,(5+x)\,(2x-1) \end{Bmatrix}.$$

Make, for example, $x = 3$, one of the synthemes in question out of the twelve corresponding to this value will be

$$A\,.\,C\,.\,E\,.\,G\,.\,B\,.\,D\,.\,F \times 2\,.\,3\,.\,6\,.\,3\,.\,6\,.\,3\,.\,6.$$

Here

$$A\,.\,C\,.\,E\,.\,G\,.\,B\,.\,D\,.\,F = AC,\ CE,\ EG,\ GB,\ BD,\ DF,\ FA,$$

$\dot{2}\,.\,3\,.\,6\,.\,3\,.\,6\,.\,3\,.\,\dot{6} =$

$$
\left.\begin{array}{l} = 2\,.\,4 \\ +\ 4\,.\,6 \\ +\ 6\,.\,8 \\ +\ 8\,.\,10 \\ +10\,.\,2 \end{array}\right\}
\left.\begin{array}{l} \ \ 3\,.\,7 \\ +5\,.\,9 \\ +7\,.\,1 \\ +9\,.\,3 \\ +1\,.\,5 \end{array}\right\}
\left.\begin{array}{l} \ \ \ 6\,.\,4 \\ +\ 8\,.\,6 \\ +10\,.\,8 \\ +\ 2\,.\,10 \\ +\ 4\,.\,2 \end{array}\right\}
\left.\begin{array}{l} \ \ 3\,.\,7 \\ +5\,.\,9 \\ +7\,.\,1 \\ +9\,.\,3 \\ +1\,.\,5 \end{array}\right\}
\left.\begin{array}{l} \ \ \ 6\,.\,4 \\ +\ 8\,.\,6 \\ +10\,.\,8 \\ +\ 2\,.\,10 \\ +\ 4\,.\,2 \end{array}\right\}
\left.\begin{array}{l} \ \ 3\,.\,7 \\ +9\,.\,9 \\ +7\,.\,1 \\ +9\,.\,3 \\ +1\,.\,5 \end{array}\right\}
\left.\begin{array}{l} \ \ \ 6\,.\,3 \\ +\ 8\,.\,5 \\ +10\,.\,7 \\ +\ 2\,.\,9 \\ +\ 4\,.\,1 \end{array}\right\}
$$

and the product

$$= A_2C_4,\ C_3E_7,\ E_6G_4,\ G_3B_7,\ B_6D_4,\ D_3F_7,\ F_6A_3$$
$$A_4C_6,\ C_5E_9,\ E_8G_6,\ G_5B_9,\ B_8D_6,\ D_5F_9,\ F_8A_5,$$

&c. &c. &c.

To *prove* the rule for the table of formation, it will be sufficient to show that no two contiguous duads ever contain the same or *equivalent* permutations; the equation of equivalence it will be remembered is

$$r : s = r + 2i \pm 2m : s + 2i \pm 2m.$$

Now, as regards the first and second terms, it is manifest that $1 : x$ cannot be equivalent, either to $1 : x'$ nor to $2 : x$, nor to $2 : x'$, where $x'$ is any number differing from $x$.

Similarly, as regards the last and first terms, $x : 1$ cannot be equivalent to $x' : 1$, nor to $x : 2$, nor to $x' : 2$; therefore there is no danger as far as the first term is concerned, either as antecedent or consequent.

Again, it is clear that $x : (2x - 1)$ cannot interfere with $x' : 2x'$, nor $(m + x) : 2x$ with $(m + x') : (2x' - 1)$; neither can $(2x - 1) : x$ with $2x' : x'$, nor $2x : (m + x)$ with $(2x' - 1) : (m + x')$.

Again, if possible, let

$$x : (2x - 1) = (m + x') : (2x' - 1);$$

then

$$m + x' - x = 2i,$$

and

$$2x' - 2x = 2i,$$

therefore

$$2m = 2i,$$

or

$$m = i,$$

which is impossible, since $+ i$ is the difference between two indices, each less than $m$.

Similarly,

$$m + x : 2x \text{ cannot} = x' : 2x',$$

and *vice versâ* with the terms changed

$$2x : (m + x) \text{ cannot} = 2x' : x',$$

and

$$(2x - 1) : x \text{ cannot} = (2x' - 1) : (m + x'),$$

which proves the rule for the table of formation.

So much for the bipartite duad synthemes. As regards the unipartite synthemes little need be said, for every part may be treated as a separate system, and as each will produce an equal number of synthemes, these being taken one with another, will furnish just as many unipartite synthemes of the whole system as there are synthemes due to each part. Thus then the synthematic resolution of the modulus $2m \times p$ may be made to depend on the synthematization of $2m$ and the cyclothematization of $p$. This has been already shown (whatever $m$ may be) for the case of $p$ being a prime number; but I proceed now to extend the rule to the more general case of $p$ being any number whatever.