15.

ON A LINEAR METHOD OF ELIMINATING BETWEEN DOUBLE, *TREBLE*, AND OTHER SYSTEMS OF ALGEBRAIC EQUATIONS.

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PART I. BINARY SYSTEMS.

LET U and V be two integer complete homogeneous functions of x and y, one of the mth, the other of the nth degree; and let it be required to express the condition of the coexistence of the two equations $U=0$, $V=0$ by means of the equation $C=0$, where C is free from all appearances of x or y.

This equation, according to the system of notation developed in a preceding paper, and which has been since adopted and sanctioned by the high authority of M. Cauchy, I call the final derivative: the quantity C is designated the final derivee: and it is our present object to show how this may be obtained in a *prime* form, that is to say, divested of irrelevant factors: in this state it must consist of terms, each containing $m+n$ letters, of which n belong to the coefficients of U, and m to those of V.

Of course in applying this rule it is to be understood that every combination of powers in U or V has a single letter prefixed for its coefficient, and that in the final derivee powers are represented by repetitions of the same character.

Every term in U or V being of the form Cx^py^q, x^py^q is called an argument, C its prefix.

Assume two integer positive numbers r and r', and also two others s and s', such that $r+r'=n-1$, $s+s'=m-1$, and form from $U=0$, $V=0$ two new equations,

$$x^ry^{r'}U=0, \quad x^sy^{s'}V=0.$$

Such equations are termed the augmentatives of the two given ones respectively; also $x^ry^{r'}U$ and its fellow are termed the augmentees of U and V.

r and r' are termed the indices of augmentation belonging to U, s and s' the same belonging to V.

Finally, it will be useful hereafter to call the given polynomials U and V themselves the proposees, and the given equations which assert their nullity, the propositive equations, or, briefly, the propositives.

Now as many augmentees of either proposee can be formed as there are ways of stowing away between two lockers (vacancies admissible) a number of things equal to the index of the other*; hence we shall have n augmentees of U, and m of V: thus there will be $m+n$ augmentatives each of the degree $m+n-1$, and the number of arguments is clearly $m+n$ also, so that they can be eliminated linearly, and the final derivee thus found, containing $m+n$ letters (properly aggregated) in each term, will be in its prime form, that is, incapable of further reduction, and void of irrelevant factors.

It is worthy of remark, that the final derivee obtained by arranging in square battalion the prefixes of the augmentees, permuting the rows or columns, and reading off diagonal products, affected each with the proper sign (according to the well known rule of Duality), will not only be free from factorial irrelevancy, but also of linear redundancy, which latter term I use to signify the reappearance of the same combination of prefixes, sometimes with positive and sometimes with negative signs: furthermore, it follows obviously from the nature of the process that no numerical quantity in the final derivee will be greater than the higher of the indices of the two given polynomials.

Part II. Ternary Systems.

Case A. *Indices all equal.*

Method 1.

Let there be now three proposees, U, V, W, integer complete homogeneous functions of x, y, z, each of the degree n: let

$$r+r'+r''=n-1, \quad s+s'+s''=n-1, \quad t+t'+t''=n-1,$$

$$x^r y^{r'} z^{r''} U, \; x^s y^{s'} z^{s''} V, \; x^t y^{t'} z^{t''} W,$$

will, as above, be called the augmentees of U, V, W, and every other part of the notation previously described is to be preserved.

* "Tot Augmenta utriusvis ex æquationibus propositis formari possunt quot modi sint inter duo receptacula (utrivis vel ambobus omnino vacare licet) rerum, quarum numerus indicem alterius æquat, distributionem faciendi."

Suppose now

$$U=0, \quad V=0, \quad W=0,$$

we shall have as many augmentative equations formed from each proposee as there are ways of stowing away n things between *three* lockers (vacancies admissible)*, that is, $n\frac{n+1}{2}$ of each kind; in all, therefore, $3\frac{n(n+1)}{2}$, and every one of these will be of the degree $2n-1$, so that the number of arguments to be eliminated is equal to the number of ways of stowing away $2n-1$ things between three lockers (empty ones counting), that is

$$\frac{2n(2n+1)}{2}.$$

As yet, then, we have not *enough* equations for eliminating these linearly.

Make, however,

$$\alpha+\beta+\gamma=n+1,$$

and write

$$U=x^\alpha F+y^\beta F'+z^\gamma F''=0,$$
$$V=x^\alpha G+y^\beta G'+z^\gamma G''=0,$$
$$W=x^\alpha H+y^\beta H'+z^\gamma H''=0,$$

it will always be possible to make the multipliers of x^α, y^β, z^γ integer functions: for if we look to any argument in U, V, or W, it is of the form $x^ay^bz^c$, and one of the letters a, b, c must be *not less* than its correspondent α, β, γ, for otherwise $a+b+c$ would be not greater than $\alpha+\beta+\gamma-3$, that is, n would be not greater than $(n+1)-3$, or $n-2$, which is absurd: if now any one, as a, be equal to or greater than α, it may be made to supply an integer part to the multiplier of x^α.

Here it may be asked what is to be done with such terms as $Kx^ay^bz^c$, when two letters a, b are each not less than their correspondents α, β: the answer is, such terms may be made to enter under the multiplier of x^α, or of x^β, or to supply a part to both in any proportion at pleasure†.

From the equations above we get, by linear elimination,

$$FG'H''+GH'F''+HF'G''-GF'H''-HG'F''-FH'G''=0.$$

This may be denoted thus : $\Pi(\alpha, \beta, \gamma)=0$, which equation I call a *secondary* derivative, and the left side of it a *secondary* derivee; α, β, γ may likewise be termed the indices of derivation (as r, s, t, &c. are of augmentation).

Now since $\alpha+\beta+\gamma=n+1$, it is clear that the index of $\Pi(\alpha, \beta, \gamma)$ is always $n+n+n-(n+1)$; that is, $2n-1$.

* See for Latin translation the preceding note.

† The prefixes of any such terms (say K) may be conceived as made up of two parts, an arbitrary constant, as e and $(K-e)$; e will disappear spontaneously from the final derivee.

1st. Let any two of the indices of derivation be taken zero, then it is easily seen that all the terms in $\Pi(\alpha, \beta, \gamma)$ vanish, and consequently the secondary derivative equations obtained upon this hypothesis become mere identities, and are of no use.

2nd. Let any *one* of them become zero.

It is manifest, from the doctrine of simple equations, that $\Pi(\alpha, \beta, \gamma)$ may be made equal to

$$\left\{\lambda U + \mu V + \nu W\right\} \frac{1}{x^\alpha},$$

or

$$\left\{\lambda' U + \mu' V + \nu' W\right\} \frac{1}{x^\beta},$$

or

$$\left\{\lambda'' U + \mu'' V + \nu'' W\right\} \frac{1}{x^\gamma},$$

upon the understanding that

$$\lambda = G'H'' - G''H', \quad \mu = H'F'' - H''F', \quad \nu = F'G'' - F''G',$$
$$\lambda' = G''H - GH'', \quad \mu' = H''F - HF'', \quad \nu' = F''G - FG'',$$
$$\lambda'' = GH' - G'H, \quad \mu'' = HF' - H'F, \quad \nu'' = FG' - F'G.$$

The three rows of coefficients will be respectively of the degrees

$$(n-\beta) + (n-\gamma), \quad (n-\gamma) + (n-\alpha), \quad (n-\alpha) + (n-\beta).$$

Thus if any one of the indices α, β, γ be zero, $\Pi(\alpha, \beta, \gamma)$ becomes identical with $\lambda^{\prime} U + \mu^{\prime} V + \nu^{\prime} W$, where the multipliers of U, V, W are of $2n - (\alpha + \beta + \gamma)$ dimensions, that is of $(n-1)$ dimensions, and may accordingly be put under the form

$$\Sigma A\, x^r y^{r'} z^{r''} U + \Sigma B x^s y^{s'} z^{s''} V + \Sigma C x^t y^{t'} z^{t''} W,$$

that is to say, becomes a *linear function* of the augmentatives, and therefore if combined with them in the process of linear elimination would *give rise* to the identity $0 = 0$.

Hence we must reject all such secondary derivatives as have zero for one of the indices of derivation. But all others, it may be shown, will be *linearly* independent of one another, and of the augmentees previously found. Hence, besides $3\dfrac{n(n+1)}{2}$ equations of augment of the degree $2n-1$, we shall have of the same degree so many equations of derivation as there are ways of stowing away between three lockers $(n+1)$ things, under the condition that no locker shall ever be left *empty*, that is $\dfrac{n(n-1)}{2}$ *.

Thus, then, in all we have $n\dfrac{n-1}{2} + 3\dfrac{n(n+1)}{2} = \dfrac{2n(2n+1)}{2}$ equations, which is exactly equal to the number of arguments to be eliminated. Hence

* Vide page 76 for the Latin version.

the final derivee can be obtained by the usual explicit rule of permutation, and moreover will be *its lowest form*, for it will contain in each term $\frac{n(n+1)}{2}$ prefixes belonging to the augmentatives of U, and a like number pertaining to those of V and of W, as well as $n\frac{n-1}{2}$ belonging to the secondary derivatives, each prefix in any one of which is triliteral, containing a prefix drawn out of those belonging to each of the proposees.

Thus every member containing $n\frac{n+1}{2}+n\frac{n-1}{2}$, that is n^2 of the original prefixes belonging to U, V, W, singly and respectively, the final derivee evolved by this process will be in its lowest terms; as was to be proved.

CASE A. *Indices all equal.*

Method 2.

It is remarkable that we may vary the method just given by making

$$r+r'+r''=n-2,\quad s+s'+s''=n-2,\quad t+t'+t''=n-2.$$

The augmentatives will thus be of the degree $2n-2$.

Furthermore, we must make $\alpha+\beta+\gamma=n+2$. It will still be possible to satisfy by integer multipliers the equations

$$U=x^\alpha F+y^\beta F'+z^\gamma F'',$$
$$V=x^\alpha G+y^\beta G'+z^\gamma G'',$$
$$W=x^\alpha H+y^\beta H'+z^\gamma H'',$$

[these it will be useful in future to term the *equations*, x^α, y^β, z^γ being the *arguments*, and F, G, H, &c. the *factors* of decomposition] for otherwise calling the indices of x, y, z in any original argument a, b, c, their sum or n would be not greater than $(n+2)-3$, that is $(n-1)$, which is absurd.

For the same reasons as in the last case no index of augmentation must be made zero: the degree of each will be $(n-\alpha)+(n-\beta)+(n-\gamma)$, that is $(2n-2)$, and their number $\frac{(n+1)n}{2}$; the number of augmentatives will be $\frac{3(n-1)n}{2}$ linearly uninvolved, each of the degree $2n-2$, and therefore containing $\frac{(2n-1)2n}{2}$ arguments.

Now $$\frac{(n+1)n}{2}+\frac{3(n-1)n}{2}=\frac{(2n-1)2n}{2}.$$

Hence the final derivee may be found, and it will be in its *lowest terms*, for every member will contain $\frac{3(n-1)n}{2}$ letters due to the augmentative, and $\frac{3(n+1)n}{2}$ due to the partial derivative equations; in all then there will be $3n^2$ letters in each term.

This second method being applied to three quadratic equations of the most general form, leads to the problem of eliminating between six simple equations which lies within the limits of practical feasibility, and it is my intention to register the final derivee upon the pages of some one of our scientific Transactions as a standing monument for the guidance of hereafter coming explorers*.

Scholium to Case A.

If we attempt to carry forward these processes to quaternary systems, it becomes necessary to make

$$\alpha+\beta+\gamma+\delta=(r-2)n+1$$

or else

$$\alpha+\beta+\gamma+\delta=(r-2)n+2,$$

where r is the number of proposees.

Now if the factors in the equations of decomposition are all integer, one of the indices of derivation must be not greater than the corresponding index in any of the original arguments, which may easily be shown to be always impossible for a system of equations, *complete* in *all* their terms, whenever their number r is greater than three, if $\alpha+\beta+\gamma+\delta=(r-2)n+2$; but if $\alpha+\beta+\gamma+\delta=(r-2)n+1$ only possible for the case of $n=2$.

Particular method applicable to *four Quadratics*.

Let $U=0$, $V=0$, $W=0$, $Z=0$, be four quadratic equations existing between x, y, z, t.

Make

$$\begin{array}{llll} xU=0, & xV=0, & xW=0, & xZ=0, \\ yU=0, & yV=0, & yW=0, & yZ=0, \\ zU=0, & zV=0, & zW=0, & zZ=0, \\ tU=0, & tV=0, & tW=0, & tZ=0. \end{array}$$

* Elimination between *two* quadratics leads to a final derivee made up of *seven* terms only; the final derivee of *three* quadratics is made up of at least several *thousand*; nay, I believe I may safely say, several *myriads* of terms!

Also write
$$U = x^2F + yF' + zF'' + tF''' = 0,$$
$$V = x^2G + yG' + zG'' + tG''' = 0,$$
$$W = x^2H + yH' + zH'' + tH''' = 0,$$
$$Z = x^2K + yK' + zK'' + tK''' = 0.$$

By eliminating linearly we get
$$\Sigma \{F\Sigma G' (H''K''' - H'''K'')\} = 0,$$
which will be of the third degree, since the factors represented by the *unmarked* letters F, G, H, K are of zero, and all the rest of *unit* dimensions.

Similarly we may obtain other equations, so that besides the *sixteen* augmentatives already written down, we have four secondary derivatives, namely,
$$\Pi(2\ 111) = 0, \quad \Pi(12\ 11) = 0, \quad \Pi(11\ 21) = 0, \quad \Pi(111\ 2) = 0.$$
Thus we have *twenty* equations and as many arguments to eliminate, since a perfect cubic function of four letters contains twenty terms.

The final derivee will contain $16 + 4.4$ letters, that is 32, 8 or 2^3 belonging to each system of original prefixes in each member, and will therefore be in its lowest terms: for one of the canons of form teaches us, *à priori*, that every member of the derivee deduced from any number of assumed equations must contain in each member as many prefixes belonging to one equation of the system as there are units in the product of the indices of all the rest taken together.

Corollary to Case A.

Either of the two methods given as applicable to this case enables us to determine integer values of X, Y, Z, which shall satisfy the equation
$$XU + YV + ZW = Fx^py^qz^r,$$
where F is the final derivee and $p + q + r = 3n - 2$. For by the doctrine of simple equations we know how to express F in terms of the linear functions, out of which it is obtained by permutation, that is we are able to assign values of A, B, C, and their antitypes, as also of L and its antitype, which shall satisfy the equation
$$\Sigma(Ax^ry^{r'}z^{r''}U) + \Sigma(Bx^sy^{s'}z^{s''}V) + \Sigma(Cx^ty^{t'}z^{t''}W)$$
$$+ \Sigma\{L\Pi(\alpha, \beta, \gamma)\} = Fx^fy^gz^h, \qquad (1)$$
where A, B, C, as well as L and all the quantities formed after them, are made up of integer combinations of the original prefixes.

Now the functions $\Pi(\alpha, \beta, \gamma)$ may be expressed in three ways in terms of U, V, W, as has been already shown.

We may therefore suppose these functions to be divided into three groups, and make

$$\Sigma L\Pi(\alpha\beta\gamma)=\Sigma\frac{QU+Q'V+Q''W}{x^{\alpha}}+\Sigma\frac{RU+R'V+R''W}{x^{\beta}}$$
$$+\Sigma\frac{SU+S'V+S''W}{x^{\gamma}}. \qquad (2)$$

And it is evident that the equations (1) and (2) lead immediately to the equation

$$XU+YV+ZW=Fx^{a+f}y^{b+g}z^{c+h},$$

if we call a, b, c the *greatest values* attributed respectively to α, β, γ.

Now if we suppose the first method to be followed,

$$f+g+h=2n-1.$$

And it will always be possible to make a, b, c of what values we please *subject* to the condition of $a+b+c=n-1$; for *one* at least of the indices of derivation in $\Pi(\alpha, \beta, \gamma)$ must be not greater than its correspondent among a, b, c; otherwise $\alpha+\beta+\gamma$ would be not less than $(a+b+c)+3$; but

$$\alpha+\beta+\gamma=n+1$$
$$a+b+c=n-1,$$

which is absurd.

Hence we can satisfy $XU+YV+ZW=Fx^py^qz^r$, p, q, r being subject to the condition of $p+q+r=3n-2$, but otherwise arbitrary.

Moreover, we can *not* do so if $p+q+r$ be *less* than $3n-2$, for that would require $a+b+c$ to be less than $n-1$. Now if two of the indices of derivation, as α and β, be made equal to $a+1$, $b+1$ respectively, the third $\gamma=(n+1)-(a+b+2)=(n-1)-(a+b)$, and is therefore greater than c: so that $\alpha+\beta+\gamma$ for this case becomes greater than $a+b+c$, and the method falls to the ground.

In fact, I have discovered a theorem which lets me know this, *à priori*, a law which serves as a staff to guide my feet from falling into error in devising linear methods of solution, and the importance of which all candid judges who have studied the general theory of elimination cannot fail to recognize. To wit, if X_1, X_2, $X_3 \ldots X_n$ be n integer complete polynomial functions of n letters x_1, $x_2 \ldots x_n$, and severally of the degree b_1, b_2, $b_3 \ldots b_n$; then it is always possible to satisfy the identity

$$P_1X_1+P_2X_2+P_3X_3+\ldots+P_nX_n=Fx_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots x_n^{a_n},$$

if $\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n$ be equal to or greater than $b_1+b_2+b_3+\ldots+b_n-n+1$, but otherwise *not**.

This again is founded immediately upon a simple proposition, of which I have obtained a very interesting and instructive demonstration, shortly to appear, and which may be enumerated thus: "*The number of augmentees of the same degree that can be formed, linearly independent of one another, out of any number of polynomial functions of as many variables, may be either equal to or less than the number of distinct arguments contained in such augmentees, but* never *greater. The latter will be the case when the index of the augmentees diminished by unity is less than the sum of the indices of the original unaugmented polynomials each so diminished; the former, when the aforesaid index is equal to or greater than the aforesaid sum.*"

To return to the particular case of finding X, Y, Z to satisfy

$$XU+YV+ZW=Fx^py^qz^r.$$

This has been already done according to the first method; if we employ the *second* method of elimination we shall have

$$f+g+h=2n-2.$$

But, now since $\alpha+\beta+\gamma=n+2$, we shall easily see by the same method as above, that the least value of $a+b+c$ {where a, b, c denote respectively the greatest values of α, β, γ, appearing in the denominator of the fractional *forms* used to express $\Pi(\alpha, \beta, \gamma)$}, will be one greater than before, or n; so that $f+g+h+a+b+c$ will still be equal to $3n-2$, as we might, *à priori*, by virtue of our rule, have been assured.

Ternary Systems.

Case B. *Two of the indices equal; the third* less *by a unit.*

Let $U=0$, $V=0$, $W=0$, be the three given equations severally of the degree n, n, $(n-1)$.

* Hence it is apparent, that in applying the method of multipliers, a curious and important distinction exists between the cases of there being two equations, and there being a greater number to eliminate from: for in the first case the element of arbitrariness needs never to appear; in the *latter* it cannot possibly be excluded from appearing in the multipliers.

This will explain how it comes to pass that the method of the text may be employed to give *various* solutions of the $XU+YV+ZW=Fx^py^qz^r$; thus not only can p, q and r be variously made up of $(f+a)$, $(g+b)$, $(h+c)$, but also $\Pi(\alpha, \beta, \gamma)$ when two of the indices (α, β suppose) are each not greater than the assigned greatest values a, b may be made to figure indifferently either under the form

$$\frac{\lambda U+\mu V+\nu W}{x^\alpha} \text{ or that of } \frac{\lambda' U+\mu' V+\nu' W}{x^\beta}.$$

Make $r + r' + r'' = n - 2, \quad s + s' + s'' = n - 2, \quad t + t' + t'' = n - 1,$

by multiplying U into $x^r y^{r'} z^{r''}$, V into $x^s y^{s'} z^{s''}$, W into $x^t y^{t'} z^{t''}$, we obtain augmentees each of the same, namely, the $(2n-2)$th degree.

The number of these is

$$\frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2} + \frac{n\,(n+1)}{2}.$$

Again, make $\qquad \alpha + \beta + \gamma = n + 1.$

It will still be possible, as before, to form equations of decomposition in which x^α, y^β, z^γ are the arguments, and affected with *integer* factors. For if we look to W *even*, all its arguments are of the form $x^a y^b z^c$, where $a + b + c = (n-1)$, and each of these cannot be less than its correspondent, for that would be to say that $(n-1)$ is not greater $(n+1) - 3$, *à fortiori*, U and V can be decomposed in the manner described. Thus, then, we shall obtain as many secondary derivees as in the last case (Method 1), that is, $\frac{n\,(n-1)}{2}$ {since $\alpha + \beta + \gamma$ is still equal to $(n+1)$}, as before. Moreover, each of these will be of $(n-\alpha) + (n-\beta) + (n-1-\gamma)$, that is of $2n-2$ dimensions.

Altogether, therefore, we have

$$\left\{\frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2} + \frac{n\,(n+1)}{2}\right\} + \frac{(n-1)\,n}{2}$$

linear independent equations of the degree $2n-2$, and the number of arguments to eliminate is $\frac{(2n-1)\,2n}{2}$. Now these two numbers are equal. Thus we obtain a final derivee containing of U's coefficients $\frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2}$, an equal number of V's, but of W's $\frac{n\,(n+1)}{2} + \frac{(n-1)\,n}{2}$; now $n\,(n-1)$, $n\,(n-1)$ and n^2 exactly express the number that ought to appear of each of these respectively: hence the final derivee is clear *of irrelevant factors.*

Ternary Systems.

Case C. *Two of the indices equal; the third one* greater *by a unit.*

Here, calling n the highest index, the augmentees must each be made of the degree $(2n-3)$, their number will evidently be

$$\frac{(n-2)\,(n-1)}{2} + \frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2},$$

making the sum of the indices of derivation now, as before, equal to $(n+1)$; it will be still possible to form integer equations of decomposition, which will give rise to augmentatives of the degree $(n-\alpha)+(n-1)-\beta+(n-1)-\gamma$, that is, of $(2n-3)$ dimensions. The total number of equations, what with augmentatives and secondary derivatives, will be

$$\left\{\frac{(n-2)(n-1)}{2}+\frac{(n-1)n}{2}+\frac{(n-1)n}{2}\right\}+\frac{n(n-1)}{2}=\frac{4n^2-4n+2}{2}=\frac{(2n-2)(2n-1)}{2},$$

that is, is equal to the exact number of distinct arguments contained between them.

Also the final derivative will contain in each member

$$\frac{(n-2)(n-1)}{2}+\frac{n(n-1)}{2},$$

that is, $(n-1)(n-1)$, letters belonging to the first equation, and

$$\frac{(n-1)n}{2}+\frac{n(n-1)}{2},$$

that is, $n(n-1)$ belonging to those of the second and of the third, and will therefore be in its *lowest terms*.

Corollary to Cases B and C.

It is not necessary, after all that has been already said, to do more than just point out that the processes applicable to these cases enable us to determine X, Y, Z, which satisfy the equation

$$XU+YV+ZW=Fx^f y^g z^h,$$

where $\qquad f+g+h=3n-3$ for Case B,

and $\qquad f+g+h=3n-4$ for Case C.