14.

A NEW AND MORE GENERAL THEORY OF MULTIPLE ROOTS.

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I SHALL begin with developing the theory of polynomials containing perfect *square factors*, one or more.

First, let us proceed to determine the relations which must exist between the coefficients of such polynomials, and afterwards show how they may be broken up into others of an inferior degree.

A parallelogram filled with letters standing in *one* row is intended to express the product of the squared difference of the quantities contained. Thus $(\overline{ab})$ indicates $(a-b)^2$, $(\overline{abc})$ is used to indicate $(a-c)^2(a-c)^2(b-c)^2$, and so forth.

Suppose now that two of the roots $e_1, e_2 \ldots e_n$ belonging to the equation $fx = 0$ are equal to one another, it is clear that $(\overline{e_1, e_2 \ldots e_n}) = 0$; and moreover is a symmetric function, and can be calculated in terms of the coefficients of fx.

Next let us suppose that we have two couples of equals (as for instance a and b, two of the roots equal, as also c and d two others), it is clear, that on leaving any one of the roots out, the $(n-1)$ that are left will still contain one equality, and therefore we have

$$(\overline{e_2, e_3 \ldots e_n}) = 0, \quad (\overline{e_1, e_3 \ldots e_n}) = 0 \ldots (\overline{e_1, e_2 \ldots e_{n-1}}) = 0.$$

None of the parallelogrammatic functions above taken *singly*, are symmetric functions of the coefficients, but their sum is; so also is the sum of the product of each into the quantity left out.

Now in general, suppose that the polynomial fx contains r perfect square factors, so that we have r couples of equal roots belonging to the equation $fx = 0$, it is clear that $(\overline{e_r, e_{r+1} \ldots e_n})$ and all the other $\dfrac{n(n-1)\ldots(n-r+2)}{1 \,.\, 2 \ldots (r-1)}$ functions of which it is the type are severally zero. Moreover, the sum of these or the sum of the products of each by *any* symmetrical function of the $(r-1)$ letters left out will be a symmetrical function of the coefficients of the powers of x in fx. To express now the *affirmative** conditions corresponding to the case of there being r pairs of equal roots, we *might* employ the r equations,

$$\left(\overline{\underline{e_1, e_2 \ldots e_n}}\right) = 0,$$

$$\Sigma \left(\overline{\underline{e_2, e_1 \ldots e_n}}\right) = 0,$$

$$\Sigma \left(\overline{\underline{e_3 \ldots e_n}}\right) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Sigma \left(\overline{\underline{e_r, e_{r+1} \ldots e_n}}\right) = 0.$$

But these, except the last, are not the *simplest* that can be employed; that is to say, we can write down r others, the terms of which shall be of lower dimensions in respect to the roots.

Let $\int_\mu$ denote that any rational symmetrical function of the μth degree is to be taken of the quantities which it precedes.

Then the r equations in question are all contained in the general equation

$$\Sigma \left\{\int_\mu (e_1, e_2 \ldots e_{r-1}) \times \left(\overline{\underline{e_r, e_{r+1} \ldots e_n}}\right)\right\} = 0;$$

μ being taken from 0 up to $(r-1)$ we obtain r equations, which in respect to the roots are respectively of all degrees between

$$\frac{n(n-1)\ldots(n-r+2)}{1.2\ldots(r-1)} \text{ and } \frac{n(n-1)\ldots(n-r+2)}{1.2\ldots(r-1)} + (r-1)$$

reckoned inclusively.

Now at this stage it is important to remark that the above r equations, although *necessary*, are not *sufficient*; and indeed, no mere affirmations of equality can be sufficient to ensure there being r pairs of equal roots.

To make this manifest, suppose $r = 2$. Then in order that an equation *may* have two pairs of equal roots, we must have by the above formula

$$\Sigma \left(\overline{\underline{e_2, e_3 \ldots e_n}}\right) = 0, \quad \Sigma \left\{e_1 \left(\overline{\underline{e_2, e_3 \ldots e_n}}\right)\right\} = 0.$$

But if instead of there being two perfect square factors there be one perfect *cube* factor in fx, it may be shown by the same reasoning as above, that the very same two equations apply. In fact, it may be shown in general that no such equations as those given above can be *affirmed* in consequence of there being an amount r of multiplicity consisting of unit parts which may not be affirmed with equal truth as necessary consequences of the same

* The importance of the restriction hinted at by the use of the word affirmative will appear hereafter.

amount distributed *in any other manner* whatever. How to obtain affirmative equations sufficient as well as necessary (under certain limitations) will appear at the close of this present paper.

It is worthy of being remarked, that if we make $\int_\mu$ denote the sum of the products of the quantities to which it is prefixed, taken μ and μ together, the equations of affirmation become identical with those obtained by eliminating between fx and $\dfrac{dfx}{dx}$ *.

It can scarcely be doubted that the illustrious Lagrange, had he chosen to perfect the incomplete theory of equal roots given in the *Résolution Numérique*, by applying to it his own favourite engine of symmetric functions, could scarcely have failed of stumbling by a back passage upon Sturm's memorable theorem.

Let us now proceed to show how a polynomial known to contain one or more perfect square factors may be decomposed.

Let us begin with supposing that it contains but one such factor; so that $fx = \phi x\,(x-a)^2$.

I shall show how to obtain the equations

$$C\,(x-a)=0, \quad D\phi x\,(x-a)=0, \quad E\,(x-a)^2=0, \quad F(\phi x)=0,$$

each in its lowest terms.

1. To form the equation $Lx + M = 0$, where $x = a$, it is easy to see that if we write down in general the expression $(x-e_1)\left(\overline{e_2, e_3 \ldots e_n}\right)$ this will become zero whenever the root e_1 left out is not one of the equal roots (a): so that in fact (calling the two equal roots e_1, e_2 respectively)

$$\Sigma\left\{(x-e_1)\times\left(\overline{e_2, e_3 \ldots e_n}\right)\right\} = (x-e_1)\times\left(\overline{e_2, e_3 \ldots e_n}\right) + (x-e_2)\times\left(\overline{e_1, e_3 \ldots e_n}\right),$$

or simply $$= 2\,(x-a)\left(\overline{e_2, e_3 \ldots e_n}\right).$$

Hence by making

$$x\Sigma\left(\overline{e_2, e_3 \ldots e_n}\right) - \Sigma\left\{e_1\times\left(\overline{e_2, e_3 \cdots e_n}\right)\right\} = 0,$$

we have an equation for finding the equal roots e_1, e_2.

Again, it is easily seen upon the same hypothesis, that

$$\Sigma\left\{(x-e_2)(x-e_3)(x-e_4)\ldots(x-e_n)\times\left(\overline{e_2, e_3 \ldots e_n}\right)\right\}$$
$$= 2\,(x-e_2)(x-e_3)\ldots(x-e_n)\times\left(\overline{e_2, e_3 \ldots e_n}\right).$$

* See my note on Sturm's Theorem, *Phil. Mag.*, December, 1839 [p. 45 above. Ed.].

Hence, to form the equation having the same roots as $(x-a)\,\phi x$, we have only to make

$$x^{n-1}\Sigma\left(\overline{e_2, e_3 \ldots e_n}\right) - x^{n-2}\Sigma\left\{(e_2+e_3+\ldots e_n)\times\left(\overline{e_2, e_3 \ldots e_n}\right)\right\}\ldots\ldots$$
$$\pm\,\Sigma\left\{(e_2 e_3 \ldots e_n)\times\left(\overline{e_2, e_3 \ldots e_n}\right)\right\} = 0.$$

Suppose now in general that we have r perfect square factors, so that

$$fx = \phi x\,(x-a_1)^2\,(x-a_2)^2 \ldots (x-a_r)^2.$$

To form the equation $C(x-a_1)(x-a_2)\ldots(x-a_r)=0$, we have only to make

$$\Sigma\left\{(x-e_1)(x-e_2)\ldots(x-e_r)\times\left(\overline{e_{r+1}, e_{r+2} \ldots e_n}\right)\right\} = 0.$$

And to obtain

$$D\phi x \times (x-a_1)(x-a_2)\ldots(x-a_r) = 0,$$

we must make

$$\Sigma\left\{(x-e_{r+1})(x-e_{r+2})\ldots(x-e_n)\times\left(\overline{e_{r+1}, e_{r+2} \ldots e_n}\right)\right\} = 0.$$

The theory of perfect square factors is not yet complete until it has been shown how to obtain constructively ϕx, and, as analogy suggests, the complementary part $D'(x-a_1)^2(x-a_2)^2\ldots(x-a_r)^2$, each in its lowest terms. To effect the latter it might be said that it is only necessary to take the square of $C(x-a_1)(x-a_2)\ldots(x-a_r)$. It is true the polynomial so formed would contain every pair of equal factors, but not in the lowest terms as regards the coefficients (as we shall presently show).

To solve this last part of the problem, let it be agreed that two rows of letters inclosed in a parenthesis shall indicate the product of the squares of the differences got by subtracting each in the row from each in the other, so that

$$\begin{pmatrix}a\\ b\end{pmatrix} = (a-b)^2,\quad \begin{pmatrix}a\\ b\ c\end{pmatrix} = (a-b)^2(a-c)^2,\quad \begin{pmatrix}a\ b\\ c\ d\end{pmatrix} = (a-c)^2(a-d)^2(b-c)^2(b-d)^2.$$

Let us begin with supposing that fx has one pair only of equal roots; to form the simplest quadratic equation containing this pair, write down

$$(x-e_1)(x-e_2)\times\left(\overline{e_3, e_4 \ldots e_n}\right)\times\begin{pmatrix}e_1, e_2\\ e_3, e_4 \ldots e_n\end{pmatrix}.$$

Now if e_1 and e_2 are the two equal roots in question neither of the multipliers of $(x-e_1)(x-e_2)$ vanishes.

If e_1 and e_2 are neither of them equal roots $\left(\overline{e_3, e_4 \ldots e_n}\right) = 0$.

If one of the two only belong to the pair of equal roots

$$\begin{pmatrix}e_1, e_2\\ e_3, e_4 \ldots e_n\end{pmatrix} = 0.$$

Hence it is clear that

$$\Sigma \left\{(x-e_1)(x-e_2) \times \left(\overline{e_3, e_4 \ldots e_n}\right) \times \begin{pmatrix} e_1, e_2 \\ e_3, e_4 \ldots e_n \end{pmatrix}\right\} = 0$$

is the equation desired.

In like manner if there be r pairs of equal roots the equation of the $(2r)$th degree which contains them all may be written

$$\Sigma \left\{(x-e_1)(x-e_2) \ldots (x-e_{2r}) \times \left(\overline{e_{2r+1} \ldots e_n}\right) \times \begin{pmatrix} e_1, e_2 \ldots e_{2r} \\ e_{2r+1} \ldots e_n \end{pmatrix}\right\} = 0.$$

The coefficient of x^{2r} in this equation is clearly of

$$(n-2r)(n-2r-1) + 4r(n-2r),$$

that is, of $(n+2r-1)(n-2r)$ dimensions. The coefficient of x^r in the equation which contains the r equal roots unyoked together is of $(n-r)(n-r-1)$ dimensions, and consequently the coefficient of x^{2r} in the square of this equation would be of $2(n-r)(n-r-1)$ dimensions, that is, would be $n^2 + 6r^2 - (4r+1)n$ dimensions higher than needful.

Finally, to obtain an equation clear of *simple* as well as double appearances of the equal roots, we have only to write the complementary form

$$\Sigma \left\{(x-e_{2r+1})(x-e_{2r+2}) \ldots (x-e_n) \times \left(\overline{e_{2r+1} + e_n}\right) \times \begin{pmatrix} e_1, e_2 \ldots e_{2r} \\ e_{2r+1} \ldots e_n \end{pmatrix}\right\} = 0.$$

Let us, now that we are more familiarized with the notation essential to this method, revert to the question with which we set out, and endeavour to obtain r such equations as shall imply *unambiguously* the existence of r pairs of equal roots.

The existence of r such pairs enables us to assert the following disjunctive proposition, which cannot be asserted when the *same amount* of multiplicity is distributed in any other way.

To wit, on selecting any r roots out of the entire number, either these r will all be found again in those that are left, *or* those that are left will contain *inter se*, one repetition at least; so that except on the latter supposition any $(r-1)$ may be absolutely sunk out of those that are left, and there will still be *one* root common to the $(n-2r+1)$ remaining, and to the r originally selected to be left out.

Wherefore calling the roots $e_1, e_2 \ldots e_n$, and giving μ any value whatever, we have

$$\Sigma \left\{ \int_\mu (e_1, e_2 \ldots e_r) \times \left(\overline{e_{r+1}, e_{r+2} \ldots e_n}\right) \times \Sigma \begin{pmatrix} e_1, e_2 \ldots e_r \\ e_{2r}, e_{2r+1} \ldots e_n \end{pmatrix}\right\} = 0.$$

Hence the simplest distinctive equations indicative of the existence of r pairs of equal roots are to be found by putting μ equal in succession to all values from 0 up to $(r-1)$.

For instance, if we require that an equation of the seventh degree shall have three pairs of equal roots, we need only to call the seven roots respectively a, b, c, d, e, f, g, and then our type equation becomes

$$\Sigma\left\{\int_\mu (a\,b\,c)\times(\overline{d\,e\,f\,g})\times\left\{\begin{matrix}\binom{d\,e}{a\,b\,c}+\binom{d\,f}{a\,b\,c}+\binom{d\,g}{a\,b\,c}\\ +\binom{e\,f}{a\,b\,c}+\binom{e\,g}{a\,b\,c}+\binom{f\,g}{a\,b\,c}\end{matrix}\right\}\right\}=0.$$

From this it appears that the r distinctive equations for r pairs of equal roots are of different dimensions from the r general or overlying ones corresponding to the multiples r, anyhow distributed; the lowest of the latter being of $(n-r+1)(n-r)$, the lowest of the former of

$$(n-r)(n-r-1)+2r(n-2r+1),$$

that is, of $n(n-1)-3r(n-1)$ dimensions. In general we shall find that the more unequally distributed the multiplicity may be the lower are the dimensions of the distinctive equations, and are accordingly lowest when the multiplicity is absolutely undistributed*.

* It must not, however, be overlooked, that the equations above given, although decisive as to the existence of r pairs of equal roots *when* the multiplicity is known to be not greater than r, do not enable us to affirm with certainty their existence when this limitation is absent: for should the multiplicity exceed r, then inevitably (no matter how it may be distributed) $(\overline{e_{r+1}, e_{r+2} \ldots e_n})$ is always zero, and consequently nullifies each term of every one of the equations in question. In fact (repugnant as it may appear to be to the ordinary assumptions of analytical reasoning), it is not possible to express with *absolute* unambiguity the conditions of there being a multiplicity (r) distributed in any assigned manner by means of r *affirmative equations alone*.