13.

INTRODUCTION TO AN ESSAY ON THE AMOUNT AND DISTRIBUTION OF THE MULTIPLICITY OF THE ROOTS OF AN ALGEBRAIC EQUATION.

[*Philosophical Magazine*, XVIII. (1841), pp. 136—139.]

I USE the word *multiplicity* to denote a number, and distinguish between the total and partial multiplicities of the roots of an algebraic equation.

There may be r different roots repeated respectively $h_1, h_2 \dots h_r$ times.

r is the index of distribution.

$h_1, h_2 \dots h_r$ are the partial multiplicities, and if $h = h_1 + h_2 + \dots + h_r$

h is the *total* multiplicity.

The total multiplicity it is clear may be defined as the difference between the index of the equation and the number of its roots distinguishable from one another.

In this Introduction, I propose merely to consider how existing methods may be applied to determine the amount and distribution of multiplicity in a given equation, and conversely, how equations of condition can be formed which shall imply a *given* distribution and amount.

Let the greatest common factor between fx (the argument of the proposed equation) and $\frac{dfx}{dx}$ be called f_1x.

And in like manner, let the greatest common factor of f_1x and $\frac{df_1x}{dx}$ be called f_2x and so on, till in the end we come to f_rx, which has no common factor with $\frac{df_rx}{dx}$.

Let $k_1, k_2 \dots k_r$ denote the degrees in x of $fx, f_1x \dots f_rx$ respectively.

It is easy to see that

$k_1 - k_2$, partial multiplicities, are less than 2, that is, are each units.

$k_2 - k_3$, partial multiplicities, will be less than 3, and therefore either 1 or 2 in value respectively, and so on till we come to

$k_{r-1} - k_r$ which will severally be between zero and $r - 1$, and

$k_r - 0$ of values intermediate between zero and r.

Hence there will be

$k_1 - 2k_2 + k_3$ multiplicities each of the value 1,

$k_2 - 2k_3 + k_4$ „ „ „ 2,

....

$k_{r-1} - 2k_r$ ... of the value $r-1$,

and k_r .... of the value r.

In place of fx with $\frac{dfx}{dx}$ we might employ $\frac{dfx}{dx}$ with $\frac{d^2fx}{dx^2}$ and so on for the rest; the values of $k_2, k_3 \ldots k_r$ will remain unaffected by this change; but the former method would be more expeditious in practice.

The total multiplicity is, of course, $= k_1$.

Suppose now that we propose to ourselves the converse problem to determine the conditions that an algebraic equation may have a given amount of multiplicity distributed in a given manner.

If $h_1, h_2, h_3 \ldots h_r$ be used to denote the given number of partial multiplicities which are respectively of the values $1, 2, 3 \ldots r$, it is easy to see that the quantities derived above by $k_1, k_2 \ldots k_r$ are respectively equal to

$$h_1 + 2h_2 + \ldots\ldots + rh_r,$$
$$h_2 + 2h_3 + \ldots\ldots + rh_{r-1},$$
$$h_3 + 2h_4 + \ldots\ldots + rh_{r-2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_r.$$

Now from $\frac{dfx}{dx}$ having a factor of the degree k_1 common with fx we obtain k_1 conditions, from $\frac{df_1x}{dx}$ having a factor of the degree k_2 common with f_1x we obtain k_2 more, and so on. So that altogether we obtain in this way

$$k_1 + k_2 + \ldots\ldots + k_r \text{ conditions.}$$

But it may easily be seen that the total multiplicity being k_1, the number of conditions *need* never to exceed k_1 in number, no matter what its distribution may be. Hence, besides the enormous labour of the process, and the extreme complexity of the results, we obtain by this method more equations by far than are necessary, and it requires some caution to know which to reject.

In my forthcoming paper (to appear in *Philosophical Magazine* of next month) I shall show, by a most simple means, how without the use of derived or other subsidiary functions, to obtain the simplest equations of condition which correspond to a given distribution of a given amount of multiplicity.

The total multiplicity, say m, being given in as many ways as that number can be broken into parts, so many different systems of m equations can be formed differing each from the other in the dimensions of the terms.

These systems may be arranged in order so that each in the series shall imply all those that follow it, and be implied in all those that go before, without the converse being satisfied.

The subject of the unreciprocal implication of systems of equations is a very curious one, upon which the limits assigned to me prevent me from enlarging at present. It is closely connected with a part of the theory of elimination, which, as far as I am aware, has either been overlooked, or has not met with the attention which it deserves; I mean the theory of *Special Factors.*

An *example* may make what I mean by these clear.

Let C be a function (if my reader please) void of x, which equivalent to zero implies two given equations in x having a common root.

Let C be rid of all irrelevant factors, that is, let C be the simplest form of the determinant, when the coefficients of the two equations are perfectly independent qualities. Now suppose, as is *quite possible in a variety of ways,* that such relations are instituted between the coefficients alluded to as make C split up into factors, so that $C = L \times M \times N = 0$.

Only one of the factors L, M, N will satisfy the condition of the co-existence of the two given equations: the others are clearly, however, not to be confounded with factors of solution, or irrelevant factors, as they are termed, but are of quite a different nature, and enjoy remarkable properties, which point to an enlarged theory of elimination, and constitute what I call special or singular factors.

I shall feel much obliged to any of the readers of your widely circulated Journal, interested in the subject of this paper, who would do me the honour of communicating with me upon it, and especially if they would (between now and the next coming out of the *Magazine*) inform me whether anything, and if so how much, different from what is here stated has been done in the matter of determining the relations between the coefficients of an equation corresponding to a given amount and distribution of multiplicity in its roots.

I ought to add, that my method enables me not merely to determine the conditions of multiplicity, but also to decompose the equations containing multiple roots into others free of multiplicity, that is, to find, *à priori*, the values of the several quantities

$$\frac{fx f_2 x}{(f_1 x)^2}, \quad \frac{f_1 x f_3 x}{(f_2 x)^2}, \quad \ldots\ldots, \quad \frac{f_{r-1} x}{(f_r x)^2}, \; f_r x.$$

Moreover, other decompositions, not necessary to be enlarged upon in this place, may be obtained with equal facility.