Raport Badawczy

Research Report

RB/38/2016

Topological derivative and neural network for inverse problems of coupled models

M. Lipnicka, K. Szulc,
A. Żochowski

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Zakładu zgłaszający pracę:
Prof. dr hab. inż. Antoni Żochowski

Warszawa 2016

SYSTEMS RESEARCH INSTITUTE
POLISH ACADEMY OF SCIENCES

Topological derivative and Neural Network for Inverse Problems of Coupled Models

Marta Lipnicka

University of Lodz, Faculty of Mathematics and Computer Science,
Banacha 22, 90-238 Lodz, Poland; marta@math.uni.lodz.pl

Katarzyna Szulc, Antoni Żochowski

Systems Research Institute of the Polish Academy of Sciences,
Newelska 6, 01-447 Warsaw, Poland e-mail: szulc@ibspan.waw.pl,
zochowsk@ibspan.waw.pl

Acknowledgement: This work is also partially supported by the National Science Centre under contract no. UMO-2014/15/B/ST1/00067

Warszawa 2016

Contents

Chapter 1

Introduction

We consider a coupled model described by the domain bounded in $\mathbb{R}^2$ and decomposed into two sub-domains Ω and ω in such way that the interior sub-domain ω is surrounded by the exterior sub-domain Ω. In the interior sub-domain the physical phenomenon are described by the linear partial differential equation (PDE), and in the exterior domain the processes are governed by nonlinear PDEs subject to some external function. An example of such a system constitutes a gravity flow around an elastic obstacle. Here the Navier-Stokes equation, which is nonlinear, is coupled through transmission conditions with the linear static or dynamic elasticity system. Such situations have numerous physical interpretations. Let us mention here only two: the water flow around submarine or gas flow inside the jet engine. For real life models the coupling conditions are still a subject of research [6]. In the paper we assume that the number and the radii of hollow voids are given, and we determine only the locations. The topological derivative, introduced in [9], and developed in the papers [2, 3, 8, 10, 11, 12, 13], was created specifically in order to asses the influence of voids or inclusions on the solutions of PDE's.

Our goal in this paper consists in proposing the combination of neural network and information given by the topological derivative for solving such difficult problems or at least providing the initial approximation of the solutions. For a fixed number of holes we consider the differential equation and then we solve them. We use the solution of the differential equation on $\partial\omega$ to generate Fourier series. We take fixed number of coefficients of Fourier series. After that we define new mapping f. The arguments of mapping f is a vector of locations of holes. The values of mapping f is a vector of coefficients of Fourier series. Next we consider the inverse mapping, which for a vector of coefficients of Fourier series calculates

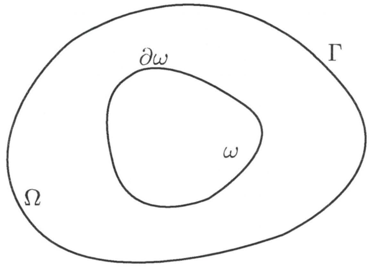

Figure 1.1: Domain $\Omega \cup \omega$.

a vector of locations of holes. We approximate this mapping by artificial neural network.

1.1 Problem formulation

Let D and ω be two bounded domains in $\mathbb{R}^2$ with the smooth boundaries $\partial\omega$ and $\Gamma = \partial D$. We suppose that $D = \Omega \cup \omega$ has a geometry presented in Fig.1.1, where $\Omega = D \setminus \overline{\omega}$, such that $\partial\Omega = \Gamma \cup \partial\omega$. In the domain D we consider the following nonlinear boundary value problem:

$$\begin{aligned} -\Delta U(x) = F(x, U(x)), \quad x \in D, \\ U(x) = 0, \quad x \in \Gamma, \end{aligned} \tag{1.1}$$

The function $F(x, U(x))$ is defined as follows

$$F(x, U(x)) = \begin{cases} -U^3(x) + f(x), & x \in \Omega, \\ 0, & x \in \omega, \end{cases} \tag{1.2}$$

where f is a constant or linear function. The boundary condition on the common boundary $\partial\omega$ constitutes the so-called transmission condition.

Next we introduce a small perturbation in the domain ω by creating a small hole B_ε at the point $\mathcal{O}$, chosen, without loss of generality, at the origin, see Fig.1.2.

We denote

$$\omega_\varepsilon = \omega \setminus \overline{B}_\varepsilon \tag{1.3}$$

$$\partial\omega_\varepsilon = \partial\omega \cup \partial B_\varepsilon \tag{1.4}$$

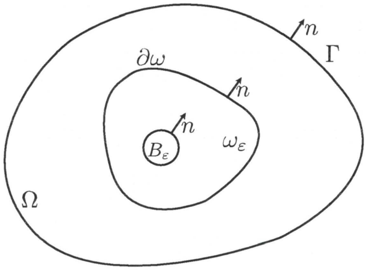

Figure 1.2: Domain Ω and $\omega_\varepsilon = \omega \setminus \overline{B}_\varepsilon$.

The problem (1.1) can then be redefined in the perturbed domain as follows:

$$\begin{aligned} -\Delta U_\varepsilon(x) &= F(x, U_\varepsilon(x)), \quad x \in D \setminus \overline{B}_\varepsilon, \\ U_\varepsilon(x) &= 0, \quad x \in \Gamma, \\ \partial_n U_\varepsilon(x) &= 0, \quad x \in \partial B_\varepsilon \end{aligned} \tag{1.5}$$

where again φ is a fixed function and $F(x, U_\varepsilon(x))$ is defined by:

$$F(x, U_\varepsilon(x)) = \begin{cases} -U_\varepsilon^3(x) + f(x), & x \in \Omega, \\ 0, & x \in \omega_\varepsilon. \end{cases} \tag{1.6}$$

with f the same as in (1.2).

According to [11, 12] we can rewrite the condition in (1.5) using the Steklov-Poincaré operator $\mathcal{A}_\varepsilon$ defined below in the domain ω_ε. The operator $\mathcal{A}_\varepsilon$ is a mapping of $H^{1/2}(\partial\omega) \to H^{-1/2}(\partial\omega)$. It means that for each function $\varphi \in H^{1/2}(\partial\omega)$ we have

$$\mathcal{A}_\varepsilon : \varphi \in H^{1/2}(\partial\omega) \longrightarrow \partial_n U_\varepsilon \in H^{-1/2}(\partial\omega). \tag{1.7}$$

The expantion of the Stieklov-Poincare operator was described in details in [14] The problem (1.5) can then be rewritten as follows:

$$\begin{aligned} -\Delta U_\varepsilon(x) &= F(x, U_\varepsilon(x)), \quad x \in D \setminus \overline{B}_\varepsilon, \\ U_\varepsilon(x) &= 0, \quad x \in \Gamma, \\ \partial_n U_\varepsilon(x) &= \mathcal{A}_\varepsilon(U_\varepsilon(x)), \quad x \in \partial\omega, \\ \partial_n U_\varepsilon(x) &= 0, \quad x \in \partial B_\varepsilon \end{aligned} \tag{1.8}$$

with the function F defined as in (1.6).

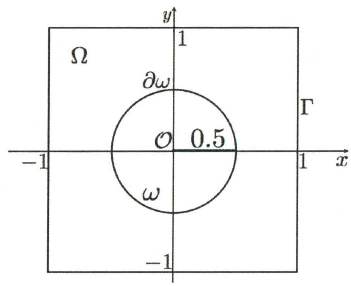

Figure 1.3: Domain $D = \Omega \cup \omega$.

The shape optimization problem considered in this section consists in finding locations of a finite number of ball-shaped holes in the domain which minimize a certain integral functional. The standard approach would use the values of topological derivative for initial location of these holes, and then shape derivative would be applied for fine tuning their sizes and positions [1, 7]. Such a method is fast, but there is a danger of landing in a local optimum, especially when the number of holes is bigger than one.

The main idea of the approach proposed in this paper is to combine the neural network and continuous method using topological derivative for speeding up computations.

For the sake of numerical experiments we assume that the domain D is a square $\langle -1, 1\rangle \times \langle -1, 1\rangle$ in $\mathbb{R}^2$. In D we define a sub-domain ω as a circle of the center at the point $\mathcal{O} = (0, 0)$ and the radius $r = 0.5$. Thus D consists of two open sub-domains ω and $\Omega = D \setminus \overline{\omega}$, see Fig.1.3, with the boundaries $\Gamma = \partial D$ and $\partial\omega$.

In the interior domain we define a linear elliptic boundary value problem as follows:

$$\begin{aligned} -\Delta u(x) &= 0, \quad x \in \omega \\ u(x) &= v(x), \quad x \in \partial\omega \end{aligned} \tag{1.9}$$

Outside the circle ω, in the domain Ω, we define the following semi-linear boundary value problem with the Dirichlet condition on the exterior boundary Γ of the square and the transmission condition on the common boundary $\partial\omega$ of the circle:

$$\begin{aligned} -\Delta v(x) + v^3(x) &= f(x), \quad x \in \Omega, \\ v(x) &= 0, \quad x \in \Gamma, \\ \partial_n v(x) &= \mathcal{A}(v(x)), \quad x \in \partial\omega. \end{aligned} \tag{1.10}$$

The operator $\mathcal{A}$ is the Steklov-Poincaré operator and the function $f(x)$ is given.

In order to solve numerically the coupled problem described in (1.9) and (1.10), we introduce a characteristic function χ of the domain Ω,

$$\chi(\Omega) = \begin{cases} 1 & x \in \Omega, \\ 0 & x \in \omega. \end{cases} \tag{1.11}$$

Then the coupled problem under consideration can be rewritten in the following way:

$$\begin{aligned} -\Delta w(x) + \chi(\Omega) w^3(x) &= \chi(\Omega) f(x), \quad x \in D, \\ w(x) &= 0, \quad x \in \Gamma. \end{aligned} \tag{1.12}$$

Let $\mathcal{V}(D) = \{v \in H_0^1(D) : v = 0 \text{ on } \Gamma\}$. Multiplying (1.12) by a function $\varphi \in \mathcal{V}$ and integrating by parts we get the following weak formulation:

$$\begin{aligned} &\int_\Omega \nabla w(x) \nabla \varphi(x) dx - \int_{\partial\omega} \frac{\partial w(x)}{\partial n_1} \varphi(x) dS \\ &+ \int_\omega \nabla w(x) \nabla \varphi(x) dx - \int_{\partial\omega} \frac{\partial w(x)}{\partial n_2} \varphi(x) dS \\ &+ \int_\Omega w^3(x) \varphi(x) dx = \int_\Omega f(x) \varphi(x) dx, \quad \forall \varphi \in \mathcal{V}(D). \end{aligned} \tag{1.13}$$

Here we denote by n_1 the outward normal vector to ω, and by n_2 the outward normal vector to Ω. If we suppose that $w|_\Omega = v$, $w|_\omega = u$, then we get the following variational formulations of the nonlinear problem in the domain Ω:

$$\begin{cases} \text{Find } v(x) \text{ such, that} \\ \int_\Omega \nabla v(x) \nabla \varphi(x) dx + \int_\Omega v^3(x) \varphi(x) dx \\ = \int_\Omega f(x) \varphi(x) dx \quad \forall \varphi \in \mathcal{V}(D), \end{cases} \tag{1.14}$$

with the transmission condition on the common boundary ensured by:

$$\int_{\partial\omega} \frac{\partial u(x)}{\partial n_1}\varphi(x)dS + \int_{\partial\omega} \frac{\partial v(x)}{\partial n_2}\varphi(x)dS = 0. \tag{1.15}$$

Similarly, for $u = v$ on $\partial\omega$ we obtain the variational formulation in the domain ω:

$$\begin{cases} \text{Find } u(x) \text{ such, that } u = v \text{ on } \partial\omega \text{ and} \\ \int_{\omega} \nabla u(x)\nabla\varphi(x)dx = 0, \quad \forall\varphi \in \mathcal{V}(D). \end{cases} \tag{1.16}$$

1.2 Optimization problem

In our numerical experiments we use the tracking type shape functional with a known element z_d, so an optimal value is 0.

Let us set $\omega_\varepsilon = \omega \setminus \overline{B}_\varepsilon(\mathcal{O})$, where $B_\varepsilon(\mathcal{O})$ is a small hole created in the interior circular domain ω at certain point $\mathcal{O}$ and of fixed radius ε. Thus, $D_\varepsilon = \Omega \cup \omega_\varepsilon$ and in such domain we define a target function z_d as a solution to the following boundary value problem :

$$\begin{aligned} -\Delta z_d(x) + \chi(\Omega)z_d^3(x) = \chi(\Omega)f(x), \quad & x \in D_\varepsilon, \\ z_d(x) = 0, \quad & x \in \Gamma, \\ \frac{\partial z_d}{\partial n} = 0, \quad & x \in \partial B_\varepsilon. \end{aligned} \tag{1.17}$$

The cost functional that we want to minimize is of the tracking type, as mentioned above, and depends on the location $\mathcal{O}$ of the hole:

$$J(v_\varepsilon) = \frac{1}{2}\int_{\Omega} (v_\varepsilon(x) - z_d(x))^2 dx, \tag{1.18}$$

where v_ε is the solution to the semi-linear problem in perturbed domain

$$\begin{aligned} -\Delta v_\varepsilon(x) + v_\varepsilon^3(x) = f(x), \quad & x \in \Omega, \\ v_\varepsilon(x) = 0, \quad & x \in \Gamma, \\ \partial_n v_\varepsilon(x) = \mathcal{A}_\varepsilon(v_\varepsilon(x)), \quad & x \in \partial\omega. \end{aligned} \tag{1.19}$$

In order to minimize this shape functional we are looking for the optimal perforation of the domain ω. For this shape functional, the topological derivative is given by the following formula:

$$\mathcal{T}_\Omega(\mathcal{O}) = 2\pi \nabla v(\mathcal{O}) \cdot \nabla p(\mathcal{O}). \tag{1.20}$$

where p is the so-called adjoint state being a solution to the linear boundary value problem

$$\begin{aligned} -\Delta p + 3v^2 p &= (v - z_d), \quad \text{in } \Omega, \\ -\Delta p &= 0, \quad \text{in } \omega \\ p &= 0, \quad \text{on } \Gamma. \end{aligned} \tag{1.21}$$

Bibliography

[1] P. Fulmanski, A. Lauraine, J.-F. Scheid, J. Sokołowski, A level set method in shape and topology optimization for variational inequalities, Int. J. Appl. Math. Comput. Sci., Vol.17, No 3, p.413-430 (2007).

[2] S. Garreau, Ph. Guillaume, M. Masmoudi, The topological asymptotic for PDE systems: the elasticity case, SIAM J. Control Optim., vol. 39, No 6, pp. 1756-1778 (2001).

[3] S. M. Giusti, A. A. Novotny, C. Padra, Topological sensitivity analysis of inclusion in two-dimensional linear elasticity, Engineering Analysis With Boundary Elements, vol. 32, no. 11, pp. 926-935 (2008).

[4] M. Grzanek, A. Nowakowski, J. Sokolowski, Topological derivatives of eigenvalues and neural networks in identification of imperfections, Journal of Physics: Conference Series 135 (2008) 012046.

[5] M. Grzanek, A. Laurain, K. Szulc, Numerical algorithms for an inverse problem in shape optimization, Journal of Physics: Conference Series 135 (2008) 012047.

[6] M. Ignatowa, I. Kukavica, I. Lasiecka, A. Tuffaha, On well-posedness for a free boundary fluid-structure model, Journal of Mathematical Physics, 53 (2013).

[7] M. Iguernane, S.A. Nazarov, J.-R. Roche, J. Sokolowski, K. Szulc, Topological derivatives for semilinear elliptic equations, Int. J. Appl. Math. Comput. Sci., Vol.19, No.2, p.191-205 (2009).

[8] S.A. Nazarov, J. Sokołowski, Asymptotic analysis of shape functionals, Journal de Mathématiques pures et appliquées, Vol. 82, pp.125-196 (2003).

[9] J. Sokołowski, A. Żochowski, On the topological derivative in shape optimization, SIAM Journal on Control and Optimization. Vol. 37, No. 4, pp.1251–1272 (1999).

[10] J. Sokołowski, A. Żochowski, Optimality conditions for simultaneous topology and shape optimization, SIAM Journal on Control and Optimization, Vol. 42, No. 4, pp.1198–1221 (2003).

[11] J. Sokolowski, A. Zochowski, Modelling of topological derivatives for contact problems, Numerische Mathematik, Vol. 102, No. 1, pp.145 - 179 (2005).

[12] J. Sokolowski, A. Zochowski, Asymptotic analysis and topological derivatives for shape and topology optimization of elasticity problems in two spatial dimensions, Engineering Analysis with Boundary Elements. Vol. 32, pp.533-544 (2008).

[13] J. Sokołowski, J.-P. Zolésio, Introduction to shape optimization. Shape sensitivity analysis, Springer-Verlag, New York (1992).

[14] K. Szulc, A. Żochowski, Application of topological derivative to accelerate genetic algorithm in shape optimization of coupled models. Structural and Multidisciplinary Optimization: Volume 51, Issue 1 (2015), Page 183-192.

[15] H. White, Connectionist Nonparametric Regression: Multilayer Feedforward Networks Can Learn Arbitrary Mappings, Neural Networks, vol. 3 (1990), pp. 535-549.