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**Recovering gradients from surface  
integrals and applications**

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Recovering gradients from surface integrals  
and applications

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# 1 Introduction

## 1.1 Domain decomposition with Steklov-Poincaré operators for linear problems

We start with a simple model problem on applications of the Steklov-Poincaré operators to the topological sensitivity analysis of linear variational problems. The shape functional is given by the associated energy functional to the boundary value problem. Given domains  $\Omega, \Omega(\varepsilon) = \Omega \setminus \overline{\mathbb{B}_\varepsilon} \subset \mathbb{R}^d, d \geq 2$ , with a small hole  $\mathbb{B}_\varepsilon = \{x \in \mathbb{R}^d \mid \|x\| < \varepsilon\}$  of radius  $\varepsilon \rightarrow 0$ , the associated energy functional to the elliptic boundary value problem under considerations is introduced for the singularly perturbed equation :

Find  $u_\varepsilon = u(\Omega(\varepsilon))$  such that

$$-\Delta u_\varepsilon = f \quad \text{in } \Omega(\varepsilon), \quad (1)$$

$$u_\varepsilon = 0 \quad \text{on } \Gamma = \partial\Omega, \quad (2)$$

$$\frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_\varepsilon = \partial\mathbb{B}_\varepsilon, \quad (3)$$

where  $f \in L^2(\mathbb{R}^d)$  is a given element which vanishes in the vicinity of the origin  $\mathcal{O} \in \omega$ .

The boundary value problem corresponds to the minimization of the

quadratic functional

$$I_\varepsilon(\varphi) = \frac{1}{2} \int_{\Omega(\varepsilon)} |\nabla \varphi|^2 - \int_{\Omega(\varepsilon)} f \varphi \quad (4)$$

over the linear subspace  $V \subset H^1(\Omega(\varepsilon))$  of the form

$$V = \{\varphi \in H^1(\Omega(\varepsilon)) \mid \varphi = 0 \text{ on } \Gamma\} \quad (5)$$

The shape functional

$$J(\Omega(\varepsilon)) := \frac{1}{2} \int_{\Omega(\varepsilon)} |\nabla u_\varepsilon|^2 - \int_{\Omega(\varepsilon)} f u_\varepsilon = -\frac{1}{2} \int_{\Omega(\varepsilon)} f u_\varepsilon . \quad (6)$$

defined by the equality

$$J(\Omega(\varepsilon)) = I_\varepsilon(u_\varepsilon) \quad (7)$$

is the energy for the singularly perturbed domain  $\Omega(\varepsilon)$ . We know already that the energy admits the expansion with respect to the small parameter  $\varepsilon \rightarrow 0$  of the following form

$$J(\Omega(\varepsilon)) = J(\Omega) - C(d)\varepsilon^d e_u(\mathcal{O}) + o(\varepsilon^d) , \quad (8)$$

where  $e_u(\mathcal{O})$  is the bulk energy density at the origin  $\mathcal{O}$ , in general the bulk energy density at a point  $x_0$ ,  $x_0 = (x_{1,0}, \dots, x_{d,0}) \in \mathbb{R}^d$  is given by the formula

$$e_u(x_0) = \|\nabla u(x_0)\|^2 .$$

## 1.2 Two spatial dimensions

The results for the Laplacian in two spatial dimensions are obtained explicitly. We have

$$J(\Omega(\varepsilon)) = J(\Omega) - \frac{1}{2}\pi\varepsilon^2 e_{\mathbf{u}}(\mathcal{O}) + o(\varepsilon^2), \quad (9)$$

where  $e_{\mathbf{u}}(\mathcal{O})$  is the *bulk energy density* at the origin  $\mathcal{O}$ , in general the bulk energy density at a point  $x_0$ ,  $x_0 = (x_{1,0}, x_{2,0}) \in \mathbb{R}^2$  is given by the formula

$$e_{\mathbf{u}}(x_0) = \|\nabla u(x_0)\|^2.$$

If the function  $u$  is harmonic in a ball  $\mathbb{B}_R \subset \mathbb{R}^2$ , of radius  $R > 0$  and the centre  $x_0$  then the expressions for the first order derivatives of  $u$  in the following form

$$u_{/1}(x_0) = \frac{1}{\pi R^3} \int_{\Gamma_R(x_0)} u \cdot (x_1 - x_{1,0}) ds, \quad u_{/2}(x_0) = \frac{1}{\pi R^3} \int_{\Gamma_R(x_0)} u \cdot (x_2 - x_{2,0}) ds. \quad (10)$$

are exact.

Here we use the notation

$$u_{/i}(x_0) = \frac{\partial u}{\partial x_i}, \quad i = 1, \dots, d.$$

In view of this, expansion (9) can be rewritten in the equivalent form

$$J(\Omega(\varepsilon)) = J(\Omega) - \frac{\varepsilon^2}{2\pi R^6} \left[ \left( \int_{\Gamma_R} u_{\Omega} x_1 ds \right)^2 + \left( \int_{\Gamma_R} u_{\Omega} x_2 ds \right)^2 \right] + o(\varepsilon^2), \quad (11)$$

which is interesting on its own, as it is observed in [2], [3], since (11) can be rewritten as follows

$$J(\Omega(\varepsilon)) = J(\Omega) + \varepsilon^2 \langle Bu, u \rangle_{\Gamma_R} + o(\varepsilon^2). \quad (12)$$

with a certain integral boundary operator  $B$ . The operator  $B$  is selfadjoint since it is defined by the symmetric and positive bilinear form

$$\langle Bu, u \rangle = b(\Gamma_R; u, u) = -\frac{1}{2\pi R^6} \left[ \left( \int_{\Gamma_R} u x_1 ds \right)^2 + \left( \int_{\Gamma_R} u x_2 ds \right)^2 \right] \quad (13)$$

From the above representation, since the line integrals on  $\Gamma_R$  are well defined for functions in  $L_2(\Gamma_R)$ , or even in  $L_1(\Gamma_R)$ , it follows that the operator  $B$  can be extended to the bounded operator on  $L_2(\Gamma_R)$ ,

$$B \in \mathcal{L}(L_2(\Gamma_R) \rightarrow L_2(\Gamma_R)) \quad (14)$$

since the symmetric bilinear form of the operator, given by the equality

$$\begin{aligned} \langle Bu, v \rangle &= b(\Gamma_R; u, v) = \\ &= -\frac{1}{2\pi R^6} \left[ \left( \int_{\Gamma_R} u x_1 ds \right) \left( \int_{\Gamma_R} v x_1 ds \right) + \left( \int_{\Gamma_R} u x_2 ds \right) \left( \int_{\Gamma_R} v x_2 ds \right) \right] \end{aligned} \quad (15)$$

is continuous for all  $u, v \in L_2(\Gamma_R)$ . In fact, the bilinear form

$$L_2(\Gamma_R) \times L_2(\Gamma_R) \ni (u, v) \mapsto b(\Gamma_R; u, v) \in \mathbb{R}$$

is continuous with respect to the weak convergence since it has the simple structure

$$b(\Gamma_R; u, v) = L_1(u)L_1(v) + L_2(u)L_2(v) \quad u, v \in L_1(\Gamma_R)$$

with two linear forms  $v \rightarrow L_i(v), i = 1, 2$ , given by the line integrals on  $\Gamma_R$ . This gives us an additional regularity for approximation of the singular perturbation of geometrical domain by the regular non-local perturbation  $B$  of the pseudo-differential Steklov-Poincaré boundary operator  $A_\varepsilon$ . The Steklov-Poincaré boundary operator  $A_\varepsilon$  is defined in the following way. For given element  $v \in H^{1/2}(\Gamma_R)$  solve the boundary value problem

$$-\Delta w = 0 \quad \text{in } C(R, \varepsilon), \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_\varepsilon, \quad w = v \quad \text{on } \Gamma_R, \quad (16)$$

and set

$$A_\varepsilon v = \frac{\partial w}{\partial \nu} \quad \text{on } \Gamma_R, \quad (17)$$

where  $\nu$  is the unit exterior normal vector on  $\partial C(R, \varepsilon)$ , note that the unit exterior normal vector  $n$  on  $\Gamma_R \subset \partial\Omega_R$  is  $n = -\nu$ . We denote by  $C(R, \varepsilon)$  the annulus  $\mathbb{B}_R \setminus \overline{\mathbb{B}_\varepsilon}$ , in our application we also introduce the truncated domain



$\Omega_R = \Omega \setminus \overline{\mathbb{B}_R}$  and use the domain decomposition technique with the Steklov-Poincaré operator.

Since the energy functional in  $\Omega(\varepsilon)$  takes the form

$$J(\Omega(\varepsilon)) = \frac{1}{2} \int_{\Omega_R} |\nabla u_\varepsilon|^2 - \int_{\Omega_R} f u_\varepsilon + \frac{1}{2} \langle A_\varepsilon u_\varepsilon, u_\varepsilon \rangle_{\Gamma_R}, \quad (18)$$

provided the source term  $f$  vanishes in the small ball  $\mathbb{B}_R$  around the origin, and the Steklov-Poincaré operator admits the expansion

$$A_\varepsilon = A + \varepsilon^2 B + R_\varepsilon, \quad (19)$$

where the remainder  $R_\varepsilon$  is of order  $o(\varepsilon^2)$  in the operator norm  $\mathcal{L}(\mathcal{H}^{1/2}(\Gamma_R) \mapsto \mathcal{H}^{-1/2}(\Gamma_R))$ . We obtain another approach to the evaluation of the topological derivative for the energy type functional. The approach is based on the representation of the energy functional in the form of a minimization procedure for regularly perturbed quadratic functional in view of (18),

$$J(\Omega(\varepsilon)) = \inf_{\varphi \in H_\Gamma^1(\Omega_R)} \left\{ \frac{1}{2} \int_{\Omega_R} |\nabla \varphi|^2 - \int_{\Omega_R} f \varphi + \frac{1}{2} \langle A_\varepsilon \varphi, \varphi \rangle_{\Gamma_R} \right\}, \quad (20)$$

where  $H_\Gamma^1(\Omega_R)$  is a subset of  $H^1(\Omega_R)$  with the functions which vanish on  $\Gamma$ . This approach is of some importance for the variational inequalities since allow us to derive the formulae for the topological derivatives which coincide with the formulae obtained for the corresponding liner boundary value problems.

## 2 Energy functionals and Steklov-Poincaré operators for variational inequalities

Let us assume that  $\Omega \subset \mathbb{R}^d$  is a given domain. If  $\mathbb{B}_\varrho$  denotes the sphere with the centre at the origin

$$\mathbb{B}_\varrho = \{x \in \mathbb{R}^d \mid |x| \leq \varrho\}$$

then we consider the domain  $\Omega(\varepsilon) = \Omega \setminus \mathbb{B}_\varepsilon$  and the truncated domain  $\Omega_R = \Omega \setminus \mathbb{B}_R$ .

The same mathematical model (20) can serve us in order to derive the topological derivatives of the energy shape functional for variational inequalities. It means that we replace in (20) the linear space  $H_1^1(\Omega_R)$  by the convex and closed subset  $K \subset H_1^1(\Omega_R)$ , and consider

$$I_\varepsilon^R(u_\varepsilon^R) = \inf_{\varphi \in K \subset H_1^1(\Omega_R)} \left\{ \frac{1}{2} \int_{\Omega_R} |\nabla \varphi|^2 - \int_{\Omega_R} f \varphi + \frac{1}{2} (A_\varepsilon \varphi, \varphi)_{\Gamma_R} \right\}. \quad (21)$$

The assumption we need now is simple, that the minimizer  $u_\varepsilon^R$  in (21) coincides with the restriction to  $\Omega_R$  of the minimizer  $u_\varepsilon := u(\Omega(\varepsilon))$  of the corresponding quadratic functional defined in the whole singularly perturbed domain  $\Omega(\varepsilon)$ ,

$$J(\Omega(\varepsilon)) = \inf_{\varphi \in K \subset H_1^1(\Omega(\varepsilon))} \frac{1}{2} \int_{\Omega(\varepsilon)} |\nabla \varphi|^2 - \int_{\Omega(\varepsilon)} f \varphi. \quad (22)$$

In this way the equality

$$\begin{aligned} J(\Omega(\varepsilon)) &= \frac{1}{2} \int_{\Omega(\varepsilon)} |\nabla u_\varepsilon|^2 - \int_{\Omega(\varepsilon)} f u_\varepsilon = I_\varepsilon^R(u_\varepsilon^R) = \\ &= \frac{1}{2} \int_{\Omega_R} |\nabla u_\varepsilon|^2 - \int_{\Omega_R} f u_\varepsilon + \frac{1}{2} \langle A_\varepsilon u_\varepsilon, u_\varepsilon \rangle_{\Gamma_R} \end{aligned} \quad (23)$$

holds for  $\varepsilon \rightarrow 0$  and we can determine the topological derivative of  $J(\Omega)$  by using the expansion of  $I_\varepsilon^R(u_\varepsilon^R)$ . The assumption we need to perform the derivation of  $I_\varepsilon^R(u_\varepsilon^R)$  with respect to the parameter  $\varepsilon$  at  $\varepsilon = 0^+$  is the strong convergence for  $R > 0$ ,

$$u_\varepsilon^R \rightarrow u^R \quad \text{strongly in } H^1(\Omega_R), \quad (24)$$

i.e., there is no need to require any differentiability properties of the minimizer  $u_\varepsilon^R \in H^1(\Omega_R)$  with respect to  $\varepsilon$ . In fact, we establish the existence of the conical differential for the mapping

$$[0, \varepsilon_0) \ni \varepsilon \mapsto u_\varepsilon^R \in H^1(\Omega_R), \quad (25)$$

and of the expansion

$$u_\varepsilon^R = u^R + \varepsilon^d q^R + o^R(\varepsilon) \quad \text{in } H^1(\Omega_R). \quad (26)$$

The element  $q^R \in H^1(\Omega_R)$  is uniquely determined by a solution of an associated variational problem with the constraints defined in function of the

solution  $u^R$  and its coincidence set for the unilateral constraints imposed by the cone  $K$ .

We describe the framework in details. The abstract model of an energy functional for the non-linear boundary value problem in the form of variational inequality considered in this paper can be described in the following way in three spatial dimensions.

Given a domain  $\Omega(\varepsilon) = \Omega \setminus \overline{\mathbb{B}_\varepsilon} \subset \mathbb{R}^3$ , with a small cavity  $\mathbb{B}_\varepsilon \subset \mathbb{B}_R$  of radius  $\varepsilon \rightarrow 0$ , denote by  $\Omega_R = \Omega \setminus \overline{\mathbb{B}_R}$  the domain without the cavity, and by  $C(R, \varepsilon) = \mathbb{B}_R \setminus \overline{\mathbb{B}_\varepsilon}$  the annulus with the small cavity. It means that the domain  $\Omega(\varepsilon)$  is decomposed into two subdomains, the truncated domain  $\Omega_R$  and the annulus  $C(R, \varepsilon)$ . The main idea which is employed here is to restrict the asymptotic analysis to the annulus  $C(R, \varepsilon)$ , and apply the obtained result to the variational inequality considered only in the truncated domain  $\Omega_R$ . In this way the singular domain perturbation in the annulus influences the variational inequality by a pseudodifferential operator of Steklov-Poincaré type, which is localised on the exterior boundary  $\Gamma_R$  of the annulus, since at the same time  $\Gamma_R$  is the interior boundary of the truncated domain  $\Omega_R$ . In other words, the variational inequality in the truncated domain takes the following form :

Find  $u_\varepsilon \in K \subset V^R$  which is the unique minimizer of the quadratic energy functional defined in the Sobolev space  $V^R = \mathcal{H}^1(\Omega_R)$ ,

$$I_\varepsilon^R(\varphi) = \frac{1}{2}a^R(\varphi, \varphi) - L^R(\varphi) + \frac{1}{2}\langle A_\varepsilon(\gamma^R\varphi), \gamma^R\varphi \rangle_R, \quad (27)$$

where  $A_\varepsilon$  stands for the Steklov-Poincaré operator for the ring  $C(R, \varepsilon)$ . The linear mapping  $\gamma^R : \mathcal{H}^1(\Omega_R) \mapsto \mathcal{H}^{1/2}(\Gamma_R)$  denotes the trace operator on the interface  $\Gamma_R$  created by the domain decomposition, and  $\langle \cdot, \cdot \rangle_R$  is the duality pairing defined for the fractional Sobolev spaces  $\mathcal{H}^{-1/2}(\Gamma_R) \times \mathcal{H}^{1/2}(\Gamma_R)$  on the interface  $\Gamma_R$ , associated with the corresponding Steklov-Poincaré operator  $A_\varepsilon : \mathcal{H}^{1/2}(\Gamma_R) \mapsto \mathcal{H}^{-1/2}(\Gamma_R)$ . The Steklov-Poincaré operator is evaluated in the ring  $C(R, \varepsilon)$  for  $\varepsilon \geq 0$ ,  $\varepsilon$  small enough, and it admits the expansion

$$A_\varepsilon = A + \varepsilon^3 B + R_\varepsilon, \quad (28)$$

where the remainder  $R_\varepsilon$  is of order  $o(\varepsilon^3)$  in the operator norm  $\mathcal{L}(\mathcal{H}^{1/2}(\Gamma_R) \mapsto \mathcal{H}^{-1/2}(\Gamma_R))$ .

We want to replace the original variational inequality defined in the domain  $\Omega(\varepsilon)$  by the variational inequality defined in the truncated domain  $\Omega_R$ , i.e. we want to replace for the purposes of asymptotic analysis the original quadratic functional defined in the domain of integration  $\Omega(\varepsilon)$

$$I_\varepsilon(\psi) = \frac{1}{2}a_\varepsilon(\psi, \psi) - L_\varepsilon(\psi), \quad \psi \in \mathcal{H}^1(\Omega(\varepsilon)), \quad (29)$$

by the functional  $I_\varepsilon^R(\varphi)$  defined in the truncated domain without any hole. To this end, we require the following property for the minimizers  $u_\varepsilon$  and  $u_\varepsilon^R$  of  $I_\varepsilon(\psi)$  and  $I_\varepsilon^R(\varphi)$ , respectively. For  $\varepsilon > 0$  small enough the minimizer  $u_\varepsilon^R$  in the truncated domain is given the restriction to the truncated domain  $\Omega_R$  of the minimizer  $u_\varepsilon$  in the singularly perturbed domain  $\Omega(\varepsilon)$ . If it is the case, we can determine the topological derivative of the energy functional

$$J(\Omega(\varepsilon)) = \frac{1}{2}a_\varepsilon(u_\varepsilon, u_\varepsilon) - L_\varepsilon(u_\varepsilon) \quad (30)$$

from the expansion of the energy functional in the truncated domain

$$I_\varepsilon^R(u_\varepsilon^R) = \frac{1}{2}a^R(u_\varepsilon^R, u_\varepsilon^R) - L^R(u_\varepsilon^R) + \frac{1}{2}\langle A_\varepsilon(\gamma^R u_\varepsilon^R), \gamma^R u_\varepsilon^R \rangle_R \quad (31)$$

which takes the form

$$I_\varepsilon^R(u_\varepsilon^R) = J(\Omega) + \varepsilon^d \langle B(\gamma^R u), \gamma^R u \rangle_R + o(\varepsilon^d), \quad (32)$$

where  $u^R = u(\Omega_R)$  stands for the restriction to the truncated domain of the solution  $u = u(\Omega)$  of the original variational inequality in the unperturbed domain  $\Omega$ .

This result is based on the equality

$$J(\Omega(\varepsilon)) = I_\varepsilon^R(u_\varepsilon^R), \quad (33)$$

and the following characterisation of the energy functional for the specific case

$$I_\varepsilon^R(u_\varepsilon^R) = \inf_{\varphi \in K} \left\{ \frac{1}{2} a^R(\varphi, \varphi) - L^R(\varphi) + \frac{1}{2} \langle A_\varepsilon(\gamma^R \varphi), \gamma^R \varphi \rangle_R \right\}. \quad (34)$$

The domain decomposition method can be applied under the assumption that the quadratic term  $\varepsilon \mapsto \langle A_\varepsilon(\gamma^R u_\varepsilon), \gamma^R u_\varepsilon \rangle_R$  is a regular perturbation of the bilinear form.

**Proposition 2.1** *Assume that (28) holds in the operator norm and that the strong convergence takes place*

$$u_\varepsilon^R \rightarrow u^R \quad (35)$$

in the energy norm for the functional (34). Then we have

$$I_\varepsilon^R(u_\varepsilon^R) = I^R(u^R) + \varepsilon^3 \langle B(u^R), u^R \rangle_R + o(\varepsilon^3), \quad (36)$$

where  $o(\varepsilon^3)/\varepsilon^3 \rightarrow 0$  with  $\varepsilon^3 \rightarrow 0$  in the same energy norm.

The proof of Proposition 2.1 is based on the evident inequalities

$$\frac{I_\varepsilon^R(u_\varepsilon^R) - I^R(u^R)}{\varepsilon^3} \leq \frac{I_\varepsilon^R(u_\varepsilon^R) - I^R(u_\varepsilon^R)}{\varepsilon^3} \leq \frac{I_\varepsilon^R(u^R) - I^R(u^R)}{\varepsilon^3}, \quad (37)$$

which imply the existence of the limit

$$\begin{aligned} \limsup_{\varepsilon^3 \rightarrow 0} \frac{I_\varepsilon^R(u_\varepsilon^R) - I^R(u_\varepsilon^R)}{\varepsilon^3} &= \lim_{\varepsilon^3 \rightarrow 0} \frac{I_\varepsilon^R(u_\varepsilon^R) - I^R(u^R)}{\varepsilon^3} = \\ \liminf_{\varepsilon^3 \rightarrow 0} \frac{I_\varepsilon^R(u^R) - I^R(u^R)}{\varepsilon^3} &= \langle B(u^R), u^R \rangle_R. \end{aligned} \quad (38)$$

In the subsequent sections the explicit form of the bilinear form which gives rise to the operator  $B$  is presented for the laplacian and for the linear elasticity in three spatial dimensions.

### 3 Case of Laplace equation in $\mathbb{R}^3$

Let us consider the equation

$$\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^3 \quad (39)$$

with some, unspecified for the moment, boundary conditions. We assume also that  $0 \in \text{int } \Omega$ , so that we may surround it with the sphere of some radius  $R$ ,  $S(R) \subset \text{int } \Omega$ . Our goal is to express  $\mathbf{grad } u = (u_{/1}, u_{/2}, u_{/3})^T$  in terms of integrals of  $u$  on  $S(R)$ . As it is known, the solution to (39) is analytic in the interior of domain, so we may use for this derivation the formal series method, as mentioned already in [2] for the 2D case.

To this end we shall assume that in the neighbourhood of 0 the solution has the form

$$u(\mathbf{x}) = a_{000} + \sum_{\alpha=1}^{\infty} \sum_{k_1+k_2+k_3=\alpha} \frac{1}{k_1!k_2!k_3!} a_{k_1 k_2 k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3} \quad (40)$$

where  $k_1, k_2, k_3$  are positive integers. In spherical coordinate system around



0 we have also

$$x_1(\varphi, \theta) = R \cos \theta \cos \varphi, \quad x_2(\varphi, \theta) = R \cos \theta \sin \varphi, \quad x_3(\varphi, \theta) = R \sin \theta$$

and we may define  $A(\varphi, \theta) = R^2 \cos \theta$ , where  $\varphi \in (-\pi, \pi]$ ,  $\theta \in [-\pi/2, \pi/2]$ .

We shall define also functions

$$I(k_1, k_2, k_3) = \int_{S(R)} x_1^{k_1} x_2^{k_2} x_3^{k_3} ds = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} x_1^{k_1}(\varphi, \theta) x_2^{k_2}(\varphi, \theta) x_3^{k_3}(\varphi, \theta) A(\varphi, \theta) d\varphi d\theta. \quad (41)$$

For example,

$$I(2, 0, 0) = \frac{4}{3}\pi R^4, \quad I(4, 0, 0) = \frac{4}{5}\pi R^6, \quad I(2, 2, 0) = I(2, 0, 2) = \frac{4}{15}\pi R^6. \quad (42)$$

From representation of the solution (40) results the following expression for the surface integral

$$\begin{aligned} \int_{S(R)} u x_1 ds = & I(2, 0, 0) a_{100} + \\ & \frac{1}{6} I(4, 0, 0) a_{300} + \frac{1}{2} I(2, 2, 0) a_{120} + \frac{1}{2} I(2, 0, 2) a_{102} + \\ & \frac{30}{120} I(2, 2, 2) a_{122} + \frac{10}{120} I(4, 2, 0) a_{320} + \frac{10}{120} I(4, 0, 2) a_{302} + \\ & \frac{5}{120} I(2, 4, 0) a_{140} + \frac{5}{120} I(2, 0, 4) a_{104} + \frac{1}{120} I(6, 0, 0) a_{500} + \\ & \text{terms corresponding to } \alpha = 7, 9, \dots \end{aligned} \quad (43)$$

The terms corresponding to even values of  $\alpha$  disappear for antisymmetry reasons.

Exploiting once more the representation (40), we may explicitly compute that

$$\Delta u_{/1} |_{x=0} = a_{300} + a_{120} + a_{102} = 0 \quad (44)$$

what means, in view of (42), that

$$\frac{1}{6}I(4, 0, 0)a_{300} + \frac{1}{2}I(2, 2, 0)a_{120} + \frac{1}{2}I(2, 0, 2)a_{102} = 0.$$

This takes care of the part corresponding to  $\alpha = 3$ . For  $\alpha = 5$  we shall differentiate (39) more times, obtaining

$$\Delta \Delta u_{/1} |_{x=0} = a_{500} + a_{140} + a_{104} + 2(a_{320} + a_{302} + a_{122}) = 0 \quad (45)$$

Taking into account that

$$I(2, 2, 2) = \frac{4}{105}\pi R^8, \quad I(4, 2, 0) = I(4, 0, 2) = I(2, 4, 0) = I(2, 0, 4) = \frac{12}{105}\pi R^8,$$

$$I(6, 0, 0) = \frac{60}{105}\pi R^8,$$

the appropriate part of (42) also vanishes

$$\begin{aligned} & \frac{30}{120}I(2, 2, 2)a_{122} + \frac{10}{120}I(4, 2, 0)a_{320} + \frac{10}{120}I(4, 0, 2)a_{302} + \\ & \frac{5}{120}I(2, 4, 0)a_{140} + \frac{5}{120}I(2, 0, 4)a_{104} + \frac{1}{120}I(6, 0, 0)a_{500} = 0. \end{aligned}$$

In the same way we may cancel further terms.

As a result we get

$$a_{100} = u_{/1}(0) = \frac{3}{4\pi R^4} \int_{S(R)} ux_1 ds$$

or, in general

$$\begin{aligned} u_{/1}(0) &= \frac{3}{4\pi R^4} \int_{S(R)} ux_1 ds, \\ u_{/2}(0) &= \frac{3}{4\pi R^4} \int_{S(R)} ux_2 ds, \\ u_{/3}(0) &= \frac{3}{4\pi R^4} \int_{S(R)} ux_3 ds. \end{aligned} \tag{46}$$

To test the formula, let us take the function  $u(\mathbf{x}) = 1/|\mathbf{x} - \mathbf{p}|$ , where  $\mathbf{p} = (2, 2, 2)^\top$ , and  $R = 1$ . Then the exact value of  $u_{/1}(0)$  is  $\sqrt{3}/36 = 0.04811252243$  and numerical integration using Maple gives

$$u_{/1}(0) = \frac{3}{4\pi R^4} \int_{S(R)} ux_1 ds = 0.04811252242.$$

Using (46) one can easily write down the bilinear form

$$\begin{aligned} b_R(u, v) &= \left(\frac{3}{4\pi R^4}\right)^2 \left( \int_{S(R)} ux_1 ds \cdot \int_{S(R)} vx_1 ds \right. \\ &\quad \left. + \int_{S(R)} ux_2 ds \cdot \int_{S(R)} vx_2 ds + \int_{S(R)} ux_3 ds \cdot \int_{S(R)} vx_3 ds \right) \end{aligned}$$

such, that energy density at 0 is  $\frac{1}{2}b_R(u, u)$ . From the computational point of view, the effort in comparison to 2D case [2] grows similarly as the difficulty of computing integrals over circle versus integrals over sphere.

## 4 Case of elasticity system in $\mathbb{R}^3$

Let us now consider the same domain as in the last section, but a different problem, namely the elasticity system

$$(1 - 2\nu)\Delta\mathbf{u} + \mathbf{grad} \operatorname{div} \mathbf{u} = 0, \quad (47)$$

where  $\mathbf{u} = (u_1, u_2, u_3)^T$ ,  $\nu = \lambda/(2(\lambda + \mu))$  - Poisson ratio and  $\lambda, \mu$  are Lamé coefficients. Our goal is to express elements of strain tensor  $\varepsilon_{ij}$  at  $\mathbf{x} = 0$  in terms of integrals of  $u_i$  over  $S(R)$ . In principle it could be done in the same way as in case of Laplace equation in the last section or 2-D elasticity in [2]. However, the calculations would be extremely complicated, as we shall see later. Therefore we will use the knowledge of the form of the solution for Dirichlet problem in the ball, as proposed by Trefftz,

$$\mathbf{u} = \mathbf{U} + (r^2 - R^2)\mathbf{grad} \Psi, \quad (48)$$

where  $\mathbf{U}$  is a harmonic vector,  $\Psi$  - harmonic scalar and  $r = \sqrt{\mathbf{x}^T \mathbf{x}}$ . In the sequel we shall follow partially the derivation from [1], chapter 5. We introduce standard polar coordinates

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta.$$

with  $\theta \in [0, \pi]$ ,  $\varphi \in (-\pi, \pi]$ . It is well known that a harmonic vector  $\mathbf{U}$  has the form

$$\mathbf{U} = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \mathbf{Y}_n(\theta, \varphi), \quad r < R, \quad (49)$$

where  $\mathbf{Y}_n(\theta, \varphi)$  consists of Laplace spherical vectors

$$\mathbf{Y}_n = \mathbf{a}_{n0} P_n(\tau) + \sum_{m=1}^n [\mathbf{a}_{nm} \cos(m\varphi) + \mathbf{b}_{nm} \sin(m\varphi)] P_n^m(\tau). \quad (50)$$

Here  $\tau = \cos \theta$ ,  $\mathbf{a}_{nm} = (a_{nm}^1, a_{nm}^2, a_{nm}^3)^\top$ ,  $\mathbf{b}_{nm} = (b_{nm}^1, b_{nm}^2, b_{nm}^3)^\top$  and

$$P_n(\tau) = \frac{1}{2^n n!} \frac{d^n}{d\tau^n} (\tau^2 - 1)^n, \quad P_n^m(\tau) = (1 - \tau^2)^{m/2} \frac{d^m}{d\tau^m} P_n(\tau).$$

The explicit forms of these polynomials for values of  $n, m$  which are needed in our calculations are given in Appendix.

We introduce the scalar product and the corresponding norm on  $S(R)$ ,

$$\langle f, g \rangle_R = \int_{S(R)} f g \, ds, \quad \|f\|_R = \sqrt{\langle f, f \rangle_R}. \quad (51)$$

Furthermore we denote

$$\mathbf{U}_n = \frac{r^n}{R^n} \mathbf{Y}_n(\theta, \varphi) = \frac{1}{R^n} \hat{\mathbf{Y}}_n(\mathbf{x}) \quad (52)$$

where

$$\hat{\mathbf{Y}}_n(\mathbf{x}) = \mathbf{a}_{n0} \hat{P}_n(\mathbf{x}) + \sum_{m=1}^n [\mathbf{a}_{nm} \hat{P}_n^{m,c}(\mathbf{x}) + \mathbf{b}_{nm} \hat{P}_n^{m,s}(\mathbf{x})]. \quad (53)$$

The polynomials  $\hat{P}$  are given by identities

$$\begin{aligned}\hat{P}_n(\mathbf{x}) &= r^n P_n(\cos \theta), \\ \hat{P}_n^{m,c}(\mathbf{x}) &= r^n P_n^m(\cos \theta) \cos(m\varphi), \\ \hat{P}_n^{m,s}(\mathbf{x}) &= r^n P_n^m(\cos \theta) \sin(m\varphi).\end{aligned}\tag{54}$$

These are homogeneous polynomials of  $n$ -th order in terms of variables  $x_1, x_2, x_3$ . Their normalized versions are defined as

$$\begin{aligned}d_n(\mathbf{x}) &= \frac{1}{\|\hat{P}_n\|_R} \hat{P}_n(\mathbf{x}), \\ c_n^m(\mathbf{x}) &= \frac{1}{\|\hat{P}_n^{m,c}\|_R} \hat{P}_n^{m,c}(\mathbf{x}), \\ s_n^m(\mathbf{x}) &= \frac{1}{\|\hat{P}_n^{m,s}\|_R} \hat{P}_n^{m,s}(\mathbf{x}).\end{aligned}\tag{55}$$

The set of functions

$$\{d_0; d_1, c_1^1, s_1^1; d_2, c_2^1, s_2^1, c_2^2, s_2^2; d_3, c_3^1, s_3^1, c_3^2, s_3^2, c_3^3, s_3^3; \dots\}$$

constitutes a system of harmonic polynomials orthonormal on  $S(R)$ . In these terms

$$\mathbf{U}_n = \frac{1}{R^n} [a_{n0} d_n(\mathbf{x}) + \sum_{m=1}^n (a_{nm} c_n^m(\mathbf{x}) + b_{nm} s_n^m(\mathbf{x}))].\tag{56}$$

Now from (48) it follows that

$$\mathbf{u}_{|r=R} = \mathbf{U}_{|r=R} = \sum_{n=0}^{\infty} \mathbf{U}_n|_{r=R}.\tag{57}$$

Using (56) and orthonormality we easily compute constant coefficients for  $n \geq 0, m = 1..n, i = 1, 2, 3$ :

$$\begin{aligned} a_{n0}^i &= R^n \langle u_i, d_n(\mathbf{x}) \rangle_R, \\ \alpha_{nm}^i &= R^n \langle u_i, c_n^m(\mathbf{x}) \rangle_R, \\ b_{nm}^i &= R^n \langle u_i, s_n^m(\mathbf{x}) \rangle_R. \end{aligned} \quad (58)$$

The harmonic scalar  $\Psi$  in (48) also has a representation

$$\Psi = \sum_{n=0}^{\infty} \Psi_n = \sum_{n=0}^{\infty} \frac{r^n}{R^n} Z_n(\theta, \varphi). \quad (59)$$

Using (47) one can obtain [1] the relation

$$\Psi_{n-1} = -\frac{1}{2[(3-2\nu)n-2(1-\nu)]} \operatorname{div} \mathbf{U}_n. \quad (60)$$

After denoting  $k_n(\nu) = 1/2[(3-2\nu)n-2(1-\nu)]$  the solution of the elasticity system in the ball takes on the form

$$\mathbf{u} = \sum_{n=0}^{\infty} [\mathbf{U}_n + (R^2 - r^2)k_n(\nu)\mathbf{g} \operatorname{grad} \operatorname{div} \mathbf{U}_n]. \quad (61)$$

Since we are looking for  $\varepsilon_{ij}(0)$ , only the part of  $\mathbf{u}$  which is linear in  $\mathbf{x}$  is relevant. It contains only two terms:

$$\hat{\mathbf{u}} = \mathbf{U}_1 + R^2 k_3(\nu)\mathbf{g} \operatorname{grad} \operatorname{div} \mathbf{U}_3. \quad (62)$$

Since, for any  $f(\mathbf{x})$ ,  $\text{grad div}(\mathbf{a}f) = H(f) \cdot \mathbf{a}$ , where  $H(f)$  is the Hessian matrix of  $f$ , we obtain

$$\hat{\mathbf{u}} = \frac{1}{R} [\mathbf{a}_{10}d_1(\mathbf{x}) + \mathbf{a}_{11}c_1^1(\mathbf{x}) + \mathbf{b}_{11}s_1^1(\mathbf{x})] + R^2 k_3(\nu) \frac{1}{R^3} \left[ H(d_3)(\mathbf{x})\mathbf{a}_{30} + \sum_{m=1}^3 (H(c_3^m)(\mathbf{x})\mathbf{a}_{3m} + H(s_3^m)(\mathbf{x})\mathbf{b}_{3m}) \right] \quad (63)$$

From the above we may single out the coefficients standing at  $x_1, x_2, x_3$  in  $u_1, u_2, u_3$ . In particular, see Appendix,

$$\begin{aligned} \varepsilon_{11}(0) &= u_{1/1}(0) = \frac{1}{R^3} \sqrt{\frac{3}{4\pi}} a_{11}^1 + \frac{1}{R^5} k_3(\nu) \left[ -3\sqrt{\frac{7}{4\pi}} a_{30}^3 - 9\sqrt{\frac{7}{24\pi}} a_{31}^1 \right. \\ &\quad \left. - 3\sqrt{\frac{7}{24\pi}} b_{31}^2 + 30\sqrt{\frac{7}{240\pi}} a_{32}^3 + 90\sqrt{\frac{7}{1440\pi}} a_{33}^1 + 90\sqrt{\frac{7}{1440\pi}} b_{33}^2 \right], \\ \varepsilon_{22}(0) &= u_{2/2}(0) = \frac{1}{R^3} \sqrt{\frac{3}{4\pi}} b_{11}^2 + \frac{1}{R^5} k_3(\nu) \left[ -3\sqrt{\frac{7}{4\pi}} a_{30}^3 - 3\sqrt{\frac{7}{24\pi}} a_{31}^1 \right. \\ &\quad \left. - 9\sqrt{\frac{7}{24\pi}} b_{31}^2 - 30\sqrt{\frac{7}{240\pi}} a_{32}^3 - 90\sqrt{\frac{7}{1440\pi}} a_{33}^1 - 90\sqrt{\frac{7}{1440\pi}} b_{33}^2 \right], \\ \varepsilon_{33}(0) &= u_{3/3}(0) = \frac{1}{R^3} \sqrt{\frac{3}{4\pi}} a_{10}^3 + \frac{1}{R^5} k_3(\nu) \left[ 6\sqrt{\frac{7}{4\pi}} a_{30}^3 + 12\sqrt{\frac{7}{24\pi}} a_{31}^1 \right. \\ &\quad \left. + 12\sqrt{\frac{7}{24\pi}} b_{31}^2 \right], \end{aligned}$$

Observe that

$$\varepsilon_{11}(0) + \varepsilon_{22}(0) + \varepsilon_{33}(0) = \frac{1}{R^3} \sqrt{\frac{3}{4\pi}} (R\langle u_1, c_1^1 \rangle_R + R\langle u_2, s_1^1 \rangle_R + R\langle u_3, d_1 \rangle_R)$$

and  $c_1^1 = \frac{1}{R^2} \sqrt{\frac{3}{4\pi}} x_1$ ,  $s_1^1 = \frac{1}{R^2} \sqrt{\frac{3}{4\pi}} x_2$ ,  $d_1 = \frac{1}{R^2} \sqrt{\frac{3}{4\pi}} x_3$ , exactly the same as for the case of Laplace equation. This should be expected, since  $\text{tr} \varepsilon$  is a



harmonic function.

$$\begin{aligned}\varepsilon_{12}(0) &= \frac{1}{2}(u_{1/2}(0) + u_{2/1}(0)) = \frac{1}{R^3} \sqrt{\frac{3}{4\pi}} (b_{11}^1 + a_{11}^2) \\ &+ \frac{1}{R^5} k_3(\nu) \left[ -3\sqrt{\frac{7}{24\pi}} a_{31}^2 - \sqrt{\frac{7}{24\pi}} b_{31}^1 + 15\sqrt{\frac{7}{60\pi}} b_{32}^3 \right. \\ &\left. - 90\sqrt{\frac{7}{1440\pi}} a_{33}^2 + 90\sqrt{\frac{7}{1440\pi}} b_{33}^1 \right],\end{aligned}$$

and similarly for  $\varepsilon_{13}(0), \varepsilon_{23}(0)$  In order to test these formulas, they were applied to one of the fundamental solutions of the elasticity system. Let  $\mathbf{p} = (2, 2, 2)^\top$ ,  $r = |\mathbf{x} - \mathbf{p}|$ ; then the displacement corresponding to the unit concentrated force in the direction  $x_1$  has the form

$$\mathbf{V}_1 = ((3-4\nu)/r + (x_1 - p_1)^2/r^3, (x_1 - p_1)(x_2 - p_2)/r^3, (x_1 - p_1)(x_3 - p_3)/r^3)^\top.$$

Hence for  $R = 1$  and  $\nu = 0.25$  we have  $\varepsilon_{11}(0) = 1/12\sqrt{3} = 0.04811252245$ .

The value obtained from the above expression, after numerical integration for computing  $a_{nm}^i, b_{nm}^i$  (Maple), was  $\varepsilon_{11}(0) = 0.04811252241$ .

Again, as in the case of Laplace equation in the last section, one can construct the bilinear form representing the energy density at 0. In 2D case [2] it was necessary to compute to this goal 8 integrals over circle using 1-st  $(x_1, x_2)$  and 3-rd order  $(x_1^3, x_2^3)$  polynomials. Here we must compute 24 such integrals over sphere, and polynomials, even if more complicated, are still explicitly given. The computational effort grows only slightly more as in the case of Laplace equation

## 5 Appendix

The Laplace spherical polynomials have the following explicit form.

For  $n = 1$ :

$$\begin{aligned}\hat{P}_1(\mathbf{x}) &= x_3, & \hat{P}_1^{1,c}(\mathbf{x}) &= x_1, & \hat{P}_1^{1,s}(\mathbf{x}) &= x_2, \\ \|\hat{P}_1\|_R &= \|\hat{P}_1^{1,c}\|_R = \|\hat{P}_1^{1,s}\|_R &= R^2 \sqrt{\frac{4\pi}{3}},\end{aligned}$$

and for  $n = 3$ :

$$\begin{aligned}\hat{P}_3(\mathbf{x}) &= x_3^3 - \frac{3}{2}x_2^2x_3 - \frac{3}{2}x_1^2x_3, & \|\hat{P}_3\|_R &= R^4 \sqrt{\frac{4\pi}{7}}, \\ \hat{P}_3^{1,c}(\mathbf{x}) &= 6x_1x_3^2 - \frac{3}{2}x_1^3 - \frac{3}{2}x_1x_2^2, & \|\hat{P}_3^{1,c}\|_R &= R^4 \sqrt{\frac{24\pi}{7}}, \\ \hat{P}_3^{1,s}(\mathbf{x}) &= 6x_2x_3^2 - \frac{3}{2}x_2^3 - \frac{3}{2}x_1^2x_2, & \|\hat{P}_3^{1,s}\|_R &= R^4 \sqrt{\frac{24\pi}{7}}, \\ \hat{P}_3^{2,c}(\mathbf{x}) &= 15x_1^2x_3 - 15x_2^2x_3, & \|\hat{P}_3^{2,c}\|_R &= R^4 \sqrt{\frac{240\pi}{7}}, \\ \hat{P}_3^{2,s}(\mathbf{x}) &= 15x_1x_2x_3, & \|\hat{P}_3^{2,s}\|_R &= R^4 \sqrt{\frac{60\pi}{7}}, \\ \hat{P}_3^{3,c}(\mathbf{x}) &= 15x_1^3 - 45x_1x_2^2, & \|\hat{P}_3^{3,c}\|_R &= R^4 \sqrt{\frac{1440\pi}{7}}, \\ \hat{P}_3^{3,s}(\mathbf{x}) &= 45x_1^2x_2 - 15x_2^3, & \|\hat{P}_3^{3,s}\|_R &= R^4 \sqrt{\frac{1440\pi}{7}},\end{aligned}$$

The Hessian matrices of  $d_3$ ,  $c_3^i$ ,  $s_3^i$  are

$$H(d_3) = \frac{1}{R^4} \sqrt{\frac{7}{4\pi}} \begin{bmatrix} -3x_3 & , & 0 & , & -3x_1 \\ 0 & , & -3x_3 & , & -3x_2 \\ -3x_1 & , & -3x_2 & , & 6x_3 \end{bmatrix}$$

$$H(c_3^1) = \frac{1}{R^4} \sqrt{\frac{7}{24\pi}} \begin{bmatrix} -9x_1 & , & -3x_2 & , & 12x_3 \\ -3x_2 & , & -3x_1 & , & 0 \\ 12x_3 & , & 0 & , & 12x_1 \end{bmatrix}$$

$$H(s_3^1) = \frac{1}{R^4} \sqrt{\frac{7}{24\pi}} \begin{bmatrix} -3x_2 & , & -3x_1 & , & 0 \\ -3x_1 & , & -9x_2 & , & 12x_3 \\ 0 & , & 12x_3 & , & 12x_2 \end{bmatrix}$$

$$H(c_3^2) = \frac{1}{R^4} \sqrt{\frac{7}{240\pi}} \begin{bmatrix} 30x_3 & , & 0 & , & 30x_1 \\ 0 & , & -30x_3 & , & -30x_2 \\ 30x_1 & , & -30x_2 & , & 0 \end{bmatrix}$$

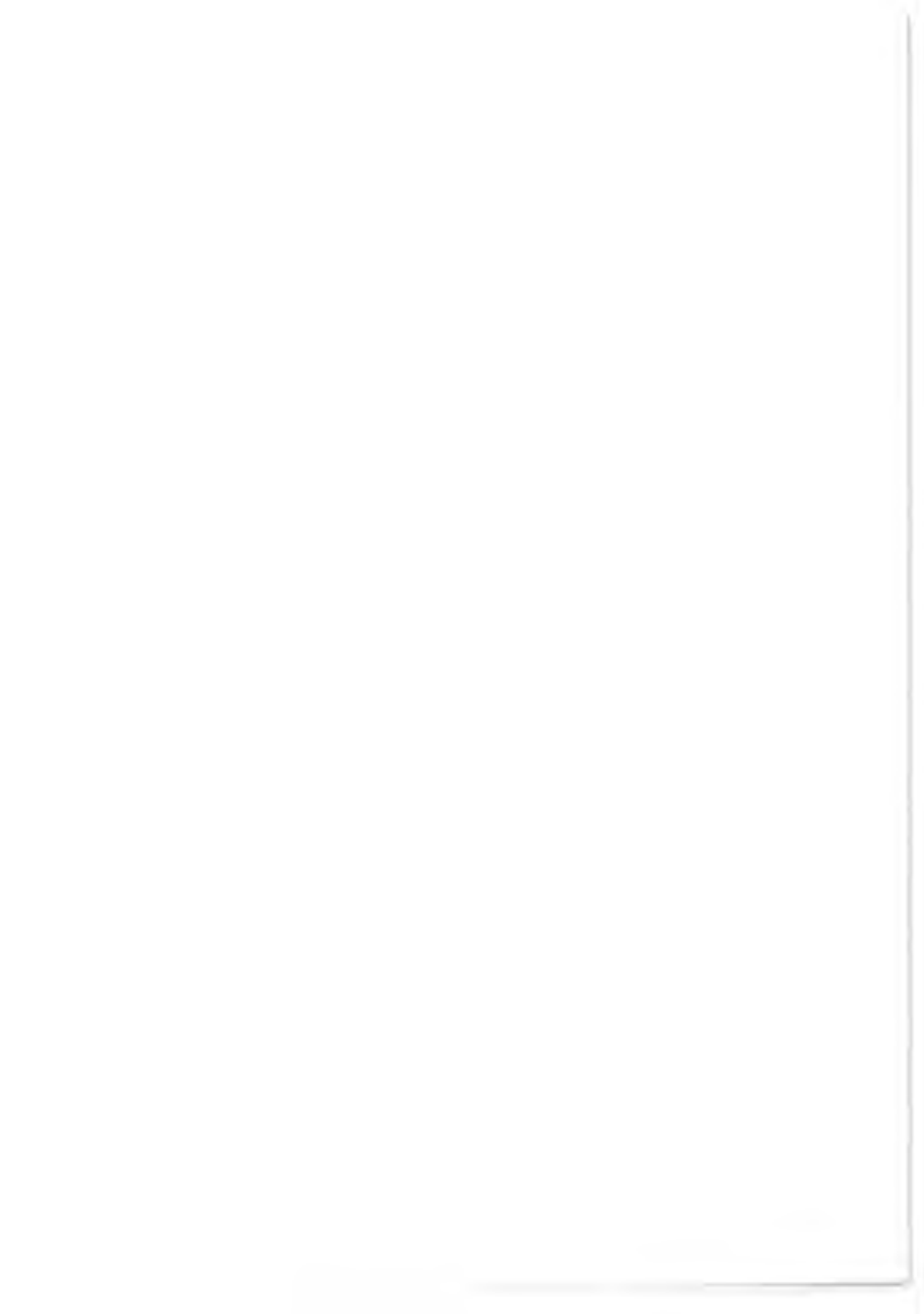
$$H(s_3^2) = \frac{1}{R^4} \sqrt{\frac{7}{60\pi}} \begin{bmatrix} 0 & , & 15x_3 & , & 15x_2 \\ 15x_3 & , & 0 & , & 15x_1 \\ 15x_2 & , & 15x_1 & , & 0 \end{bmatrix}$$

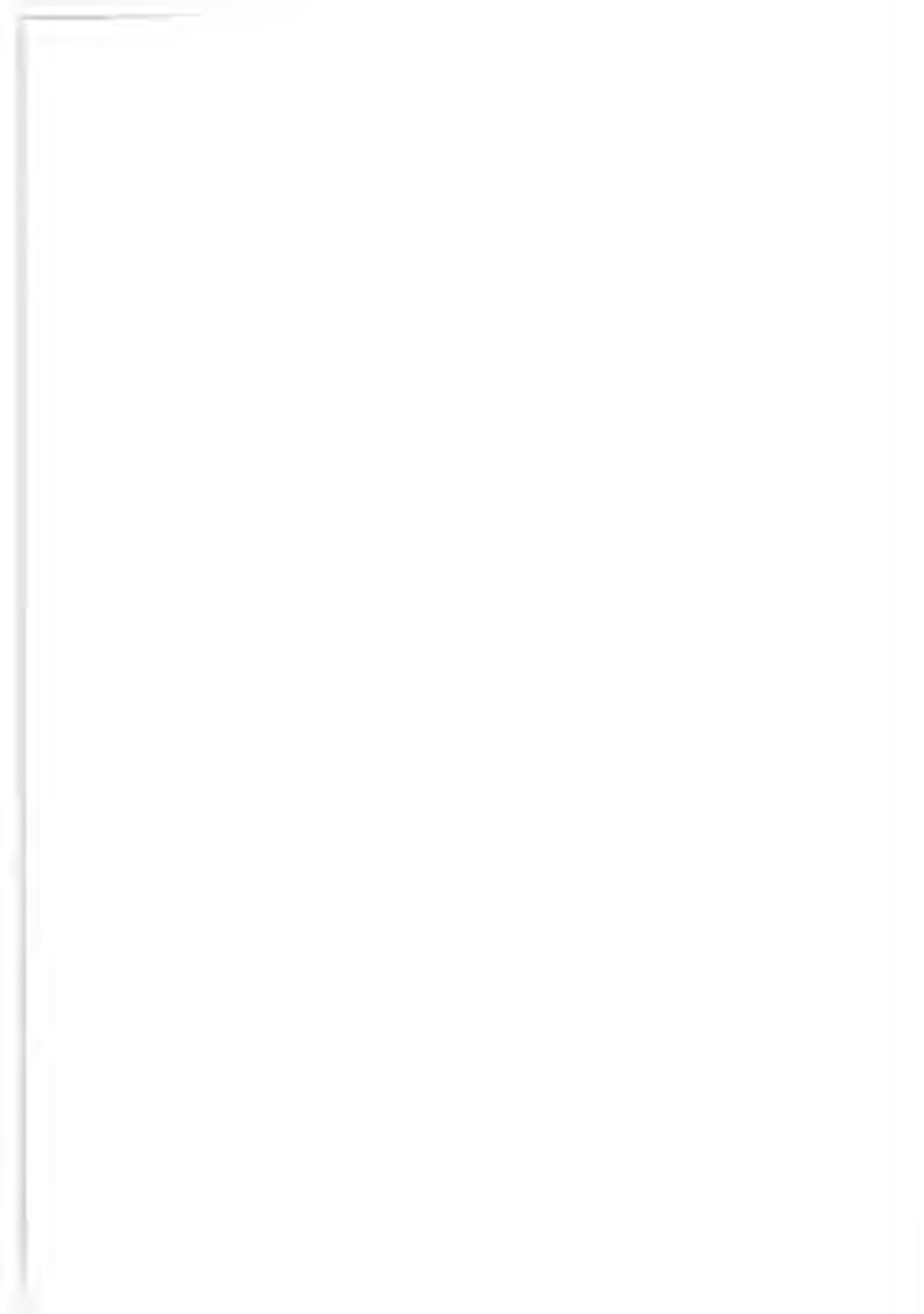
$$H(c_3^3) = \frac{1}{R^4} \sqrt{\frac{7}{1440\pi}} \begin{bmatrix} 90x_1 & , & -90x_2 & , & 0 \\ -90x_2 & , & -90x_1 & , & 0 \\ 0 & , & 0 & , & 0 \end{bmatrix}$$

$$H(s_3^3) = \frac{1}{R^4} \sqrt{\frac{7}{1440\pi}} \begin{bmatrix} 90x_2 & , & 90x_1 & , & 0 \\ 90x_1 & , & -90x_2 & , & 0 \\ 0 & , & 0 & , & 0 \end{bmatrix}$$

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the 1990s, the number of people in the world who are undernourished has increased from 600 million to 800 million (FAO 2001).

There are a number of reasons for this increase. One of the main reasons is the increase in the world population. The world population is expected to increase from 6 billion in 1999 to 9 billion by 2050 (UN 2000).

Another reason is the increase in the number of people who are living in poverty. The number of people living on less than \$1 per day has increased from 1.2 billion in 1990 to 1.6 billion in 2001 (World Bank 2002).

A third reason is the increase in the number of people who are living in rural areas. The number of people living in rural areas has increased from 3 billion in 1990 to 4 billion in 2001 (World Bank 2002).

There are a number of factors that contribute to the increase in the number of people who are undernourished. These factors include:

• The increase in the number of people who are living in poverty.

• The increase in the number of people who are living in rural areas.

• The increase in the number of people who are living in areas that are prone to drought and other natural disasters.

• The increase in the number of people who are living in areas that are prone to conflict and political instability.

• The increase in the number of people who are living in areas that are prone to environmental degradation.

• The increase in the number of people who are living in areas that are prone to food insecurity.

• The increase in the number of people who are living in areas that are prone to malnutrition.

• The increase in the number of people who are living in areas that are prone to obesity.

• The increase in the number of people who are living in areas that are prone to chronic diseases.

• The increase in the number of people who are living in areas that are prone to mental health problems.

• The increase in the number of people who are living in areas that are prone to social inequality.

• The increase in the number of people who are living in areas that are prone to environmental pollution.

the 1990s, the number of people aged 65 and over in the United States is projected to increase from 20 million in 1990 to 35 million in 2010, and the number of people aged 75 and over from 10 million to 18 million (U.S. Census Bureau 1996).

As the number of people aged 65 and over increases, the number of people aged 75 and over will increase at a faster rate. The number of people aged 75 and over is projected to increase from 10 million in 1990 to 18 million in 2010, an increase of 80% (U.S. Census Bureau 1996). The number of people aged 85 and over is projected to increase from 2 million in 1990 to 5 million in 2010, an increase of 150% (U.S. Census Bureau 1996).

As the number of people aged 75 and over increases, the number of people aged 85 and over will increase at a faster rate. The number of people aged 85 and over is projected to increase from 2 million in 1990 to 5 million in 2010, an increase of 150% (U.S. Census Bureau 1996). The number of people aged 95 and over is projected to increase from 0.5 million in 1990 to 1.5 million in 2010, an increase of 200% (U.S. Census Bureau 1996).

As the number of people aged 95 and over increases, the number of people aged 100 and over will increase at a faster rate. The number of people aged 100 and over is projected to increase from 0.1 million in 1990 to 0.3 million in 2010, an increase of 200% (U.S. Census Bureau 1996). The number of people aged 105 and over is projected to increase from 0.05 million in 1990 to 0.15 million in 2010, an increase of 200% (U.S. Census Bureau 1996).

As the number of people aged 105 and over increases, the number of people aged 110 and over will increase at a faster rate. The number of people aged 110 and over is projected to increase from 0.01 million in 1990 to 0.03 million in 2010, an increase of 200% (U.S. Census Bureau 1996). The number of people aged 115 and over is projected to increase from 0.005 million in 1990 to 0.015 million in 2010, an increase of 200% (U.S. Census Bureau 1996).

As the number of people aged 115 and over increases, the number of people aged 120 and over will increase at a faster rate. The number of people aged 120 and over is projected to increase from 0.001 million in 1990 to 0.003 million in 2010, an increase of 200% (U.S. Census Bureau 1996). The number of people aged 125 and over is projected to increase from 0.0005 million in 1990 to 0.0015 million in 2010, an increase of 200% (U.S. Census Bureau 1996).

As the number of people aged 125 and over increases, the number of people aged 130 and over will increase at a faster rate. The number of people aged 130 and over is projected to increase from 0.0001 million in 1990 to 0.0003 million in 2010, an increase of 200% (U.S. Census Bureau 1996). The number of people aged 135 and over is projected to increase from 0.00005 million in 1990 to 0.00015 million in 2010, an increase of 200% (U.S. Census Bureau 1996).

As the number of people aged 135 and over increases, the number of people aged 140 and over will increase at a faster rate. The number of people aged 140 and over is projected to increase from 0.00001 million in 1990 to 0.00003 million in 2010, an increase of 200% (U.S. Census Bureau 1996). The number of people aged 145 and over is projected to increase from 0.000005 million in 1990 to 0.000015 million in 2010, an increase of 200% (U.S. Census Bureau 1996).

As the number of people aged 145 and over increases, the number of people aged 150 and over will increase at a faster rate. The number of people aged 150 and over is projected to increase from 0.000001 million in 1990 to 0.000003 million in 2010, an increase of 200% (U.S. Census Bureau 1996). The number of people aged 155 and over is projected to increase from 0.0000005 million in 1990 to 0.0000015 million in 2010, an increase of 200% (U.S. Census Bureau 1996).