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Applications of topological derivative for accelerating the Genetic Algorithm in shape optimization of coupled models

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Application of topological derivative for accelerating the Genetic Algorithm in shape optimization of coupled models

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Abstract

In the paper we consider a new variant of the genetic algorithm for finding the location and size of small holes in the domain, in which the coupled linear and non–linear boundary value problems are defined. The linear and non–linear parts are connected by the transmission condition on on the common boundary. The expansion of the shape functional for non-linear part and the expansion of Steklov-Poincaré operator for linear part are provided in order to determine the form of topological derivative for the coupled model. The value of topological derivative is then used for computing the probability density applied later in generating location of holes by the genetic algorithm.

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1 Introduction

In the paper we investigate the design problems for a coupled model, where one part is a structure modelled by a linear equation or system of equations, and the remaining part is a structure which is modelled by non-linear equations e.g. of the Navier-Stokes type. As an example we can consider a fluid-structure or a gas-structure interaction, another example constitutes the identification of inclusions in coupled models. It means that the computational domain is divided into two or more parts with some interface conditions (interaction conditions) on the common boundaries. The problem under investigations is the optimal design of the elastic or linear part. We employ the domain decomposition technique in order to split the coupled model into separate parts. The interaction of these parts in both domains is modelled by the appropriate Steklov-Poincaré non–local boundary operator, which can be defined for the linear part and includes the information about the design in its interior. It means that we can apply the asymptotic analysis for decoupled elliptic boundary value problems and determine topological derivatives for shape functionals of interest.

The Steklov-Poincaré operators appear in boundary conditions of the fluid or gas (non–linear) models in the second sub-domain. This transmission of the behavior of the linear part to the non–linear part allows us to apply the tools of shape and topology design to the coupled models. For example, if the non–linear part represents the compressible Navier-Stokes equation, we may use recent results concerning its shape optimization, see [25].

We start with a simple scalar model, for which the topological derivative is determined and numerically tested. The test consists in using the values of these derivative in order to accelerate the convergence of the genetic algorithm, which is used for finding the location of one or two holes inside the part of the domain described by the linear model, but on the basis of measurements conducted in the surrounding non–linear medium. Such combination of genetic algorithm and topological derivative seems to be new.

2 Problem formulation

Let D and ω be two bounded domains in $\mathbb{R}^2$ with the smooth boundaries $\partial\omega$ and $\Gamma = \partial D$. We suppose that $D = \Omega \cup \omega$ has a geometry presented in Fig.1, where $\Omega = D \setminus \overline{\omega}$, such that $\partial\Omega = \Gamma \cup \partial\omega$. In the domain D we consider the following non-linear boundary value problem for a fixed function φ:

$$-\Delta U(x) = F(x, U(x)), \quad x \in D,$$

$$U(x) = 0, \quad x \in \Gamma, \tag{1}$$

$$U(x) = \varphi(x),\ \partial_n U(x) = \partial_n \varphi(x), \quad x \in \partial\omega.$$

The function $F(x, U(x))$ is defined as follows

$$F(x, U(x)) = \begin{cases} -U^3(x) + f(x), & x \in \Omega, \\ 0, & x \in \omega, \end{cases} \tag{2}$$

where f is a constant or linear function. The boundary condition on the common boundary $\partial\omega$ constitutes the so-called transmission condition.

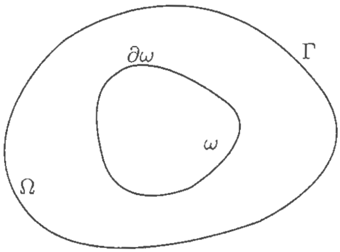

Figure 1: Domain $\Omega \cup \omega$.

Next we introduce a small perturbation in the domain ω by creating a small hole B_ε at the point $\mathcal{O}$, chosen, without loss of generality, at the origin, see Fig.2.

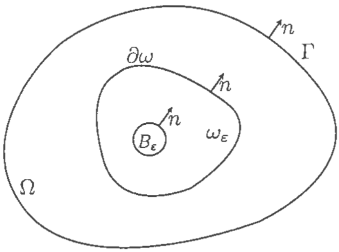

Figure 2: Domain Ω and $\omega_\varepsilon = \omega \setminus \overline{B}_\varepsilon$.

We denote

$$\omega_\varepsilon = \omega \setminus \overline{B}_\varepsilon \tag{3}$$

$$\partial\omega_\varepsilon = \partial\omega \cup \partial B_\varepsilon \tag{4}$$

The problem (1) can then be defined in the perturbed domain as follows:

$$\begin{gathered}
-\Delta U_\varepsilon(x) = F(x, U_\varepsilon(x)), \quad x \in D \setminus \overline{B}_\varepsilon, \\
U_\varepsilon(x) = 0, \quad x \in \Gamma, \\
U_\varepsilon(x) = \varphi(x),\, \partial_n U_\varepsilon(x) = \partial_n \varphi(x), \quad x \in \partial\omega, \\
\partial_n U_\varepsilon(x) = 0, \quad x \in \partial B_\varepsilon
\end{gathered} \tag{5}$$

where φ is a fixed function, $F(x, U_\varepsilon(x))$ is defined by:

$$F(x, U_\varepsilon(x)) = \begin{cases} -U_\varepsilon^3(x) + f(x), & x \in \Omega, \\ 0, & x \in \omega_\varepsilon. \end{cases} \tag{6}$$

and f is the same as in (6).

According to [7, 6] we can rewrite the condition in (5) using the Steklov-Poincaré operator $\mathcal{A}_\varepsilon$ defined below in the domain ω_ε. The operator $\mathcal{A}_\varepsilon$ is a mapping of $H^{1/2}(\partial\omega) \to H^{-1/2}(\partial\omega)$. It means that for each function $\varphi \in H^{1/2}(\partial\omega)$ we have

$$\mathcal{A}_\varepsilon : \varphi \in H^{1/2}(\partial\omega) \longrightarrow \partial_n U_\varepsilon \in H^{-1/2}(\partial\omega). \tag{7}$$

The problem (5) can then be rewritten as follows:

$$\begin{aligned} -\Delta U_\varepsilon(x) &= F(x, U_\varepsilon(x)), \quad x \in D \setminus \overline{B}_\varepsilon, \\ U_\varepsilon(x) &= 0, \quad x \in \Gamma, \\ \partial_n U_\varepsilon(x) &= \mathcal{A}_\varepsilon(U_\varepsilon(x)), \quad x \in \partial\omega, \\ \partial_n U_\varepsilon(x) &= 0, \quad x \in \partial B_\varepsilon \end{aligned} \tag{8}$$

with the function F defined as in (6).

3 Expansion of the Steklov-Poincaré operator

In order to find the form of the topological derivative, we have to determine the approximation of the operator

$$\mathcal{A}_\varepsilon : H^{1/2}(\partial\omega) \to H^{-1/2}(\partial\omega) \text{ on } \partial\omega.$$

To this goal we consider both linear and non-linear problems separately. In the domain Ω we have the following non-linear problem

$$\begin{aligned} -\Delta v_\varepsilon(x) + v_\varepsilon^3(x) = f(x), \quad x \in \Omega, \\ v_\varepsilon(x) = 0, \quad x \in \Gamma, \\ \partial_n v_\varepsilon(x) = \mathcal{A}_\varepsilon(v_\varepsilon(x)), \quad x \in \partial\omega. \end{aligned} \tag{9}$$

The Steklov-Poincaré operator is defined using the linear boundary value problem. If in the domain ω_ε we have

$$\begin{aligned} -\Delta u_\varepsilon(x) = 0, \quad x \in \omega_\varepsilon \\ u_\varepsilon(x) = \varphi(x), \quad x \in \partial\omega \\ \partial_n u_\varepsilon(x) = 0, \quad x \in \partial B_\varepsilon \end{aligned} \tag{10}$$

then $\mathcal{A}_\varepsilon(\varphi) = \partial_n u_\varepsilon$.

From (10) we may immediately compute the corresponding energy functional

$$
\begin{aligned}
0 = -\int_{\omega_\varepsilon} \Delta u_\varepsilon \cdot u_\varepsilon \, dx &= \int_{\omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx - \int_{\partial\omega} \partial_n u_\varepsilon \cdot u_\varepsilon \, dx \\
&= \int_{\omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx - \int_{\partial\omega} \mathcal{A}_\varepsilon(\varphi) \cdot \varphi \, dx.
\end{aligned} \tag{11}
$$

Thus the Steklov-Poincaré operator on the common boundary $\partial\omega$ allows us to compute the energy functional in the domain ω_ε.

$$
\int\limits_{\omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \int\limits_{\partial\omega} \mathcal{A}_\varepsilon(\varphi) \cdot \varphi \, dx \tag{12}
$$

for each function $\varphi \in H^{-1/2}(\partial\omega)$. Since the operator $\mathcal{A}_\varepsilon$ is symmetric, we can also write the energy as:

$$
\int\limits_{\omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \langle \mathcal{A}_\varepsilon(\varphi), \varphi \rangle_{(H^{-1/2} \times H^{1/2})(\partial\omega)} \tag{13}
$$

The topological asymptotic expansion for the energy functional can be written in the following way [8, 26]:

$$
\int\limits_{\omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \int\limits_{\omega} |\nabla u|^2 \, dx - 2\pi\varepsilon^2 |\nabla u(\mathcal{O})|^2 + o(\varepsilon^2) \tag{14}
$$

where u is the solution to the linear problem (10) for $\varepsilon = 0$ defined in the unperturbed domain ω. Thus, according to [7, 3] we have the following expansion of the Steklov-Poincaré operator:

$$
\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon^2 \mathcal{B} + o(\varepsilon^2) \tag{15}
$$

in the operator norm $\mathcal{L}(H^{1/2}(\partial\omega); H^{-1/2}(\partial\omega))$, and, by the symmetry of the operator this expansion can also be written in the following way:

$$\langle \mathcal{A}_\varepsilon(\varphi), \phi \rangle = \langle \mathcal{A}(\varphi), \phi \rangle + \varepsilon^2 \langle \mathcal{B}(\varphi), \phi \rangle + o(\varepsilon^2). \tag{16}$$

As a result, the first term in the asymptotic expansion of the energy functional has the form

$$\langle \mathcal{B}(\varphi), \phi \rangle = -2\pi \nabla u(\mathcal{O}) \cdot \nabla u(\mathcal{O}) \tag{17}$$

in the domain ω.

4 Topological Derivative

Let us consider the following shape functional

$$J(v_\varepsilon) = \frac{1}{2} \int\limits_\Omega (v_\varepsilon - z_d)^2 dx, \tag{18}$$

with v_ε the solution to the semi-linear problem (9) and z_d a fixed target function defined in the domain Ω. Let us introduce also the adjoint state, which is used usually in order to simplify the form of topological derivative:

$$\begin{aligned} -\Delta p + 3v^2 p &= (v - z_d), \quad \text{in } \Omega, \\ -\Delta p &= 0, \quad \text{in } \omega \\ p &= 0, \quad \text{on } \Gamma, \end{aligned} \tag{19}$$

where v is solution to (9) for $\varepsilon = 0$.

Theorem 4.1 *The topological derivative of the functional J has the following form:*

$$\mathcal{T}_\Omega(\mathcal{O}) = -\langle \mathcal{B}(v), p\rangle = 2\pi \nabla v(\mathcal{O}) \cdot \nabla p(\mathcal{O}). \tag{20}$$

Proof. Let us assume that the solution to the nonlinear problem (9) has the following expansion

$$v_\varepsilon(x) = v(x) + \varepsilon^2 w(x) + \tilde{v}_\varepsilon(x) \cdot o(\varepsilon^2). \tag{21}$$

Then from (9) we get

$$\begin{aligned}
&-\Delta(v+\varepsilon^2 w) + (v+\varepsilon^2 w)^3 = f, \quad \text{in } \Omega,\\
&v + \varepsilon^2 w = 0, \quad \text{on } \Gamma, \\
&\partial_n(v+\varepsilon^2 w) = \mathcal{A}(v+\varepsilon^2 w) + \varepsilon^2 \mathcal{B}(v+\varepsilon^2 w) + \dots, \quad \text{on } \partial\omega
\end{aligned} \tag{22}$$

Since

$$(v+\varepsilon^2 w)^3 = v^3 + 3v^2\varepsilon^2 w + 3v\varepsilon^4 w^2 + \varepsilon^6 w^3 \tag{23}$$

we get two non-linear boundary value problems. The first one comes from (22) taking into account the terms without ε:

$$\begin{aligned}
-\Delta v + v^3 &= f, \quad \text{in } \Omega,\\
v &= 0, \quad \text{on } \Gamma, \\
\partial_n v &= \mathcal{A}(v), \quad \text{on } \partial\omega.
\end{aligned} \tag{24}$$

where v is solution to (9) for $\varepsilon = 0$. The second one corresponds to the terms with ε^2:

$$\begin{aligned} -\Delta w + 3v^2 w &= 0, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \Gamma, \\ \partial_n w &= \mathcal{A}(w) + \mathcal{B}(v), \quad \text{on } \partial\omega. \end{aligned} \tag{25}$$

As a result the shape functional depending on the solution v_ε can be written in the following way:

$$\begin{aligned} J(v_\varepsilon) = \frac{1}{2}\int_\Omega (v_\varepsilon - z_d)^2\, dx &= \frac{1}{2}\int_\Omega (v + \varepsilon^2 w - z_d)^2\, dx \\ &= J(v) + \varepsilon^2 \int_\Omega w(v - z_d)\, dx + \ldots \end{aligned} \tag{26}$$

Next we apply the adjoint state to replace the last integral and obtain:

$$
\begin{aligned}
\int_\Omega w(v - z_d)\,dx &= \int_\Omega w(-\Delta p + 3v^2 p)\,dx \\
&= \int_\Omega p(-\Delta w + 3v^2 w)\,dx + \int_\Gamma (\partial_n w \cdot p - \partial_n p \cdot w\,dx \\
&\qquad - \int_{\partial\omega} (\partial_n w \cdot p - \partial_n p \cdot w)\,dx \\
&= \int_{\partial\omega} (\partial_n p \cdot w - \partial_n w \cdot p)\,dx \\
&= \int_\omega \Delta p \cdot w\,dx + \int_\omega \nabla w \cdot \nabla p\,dx - \int_{\partial\omega} \partial_n w \cdot p\,dx \\
&= \int_\omega \nabla w \cdot \nabla p\,dx - \int_{\partial\omega} (\mathcal{A}(w) + \mathcal{B}(v))p\,dx \\
&= \int_\omega \nabla w \cdot \nabla p\,dx - \int_{\partial\omega} \mathcal{A}(w)p\,dx - \int_{\partial\omega} \mathcal{B}(v)p\,dx \\
&= -\langle \mathcal{B}(v), p\rangle = 2\pi \nabla v(\mathcal{O}) \cdot \nabla p(\mathcal{O}).
\end{aligned}
\tag{27}
$$

5 Numerical Approach

The shape optimization problem considered in this section consists in finding locations and size of finite number of ball-shaped holes in the domain which minimize a certain integral functional. The standard approach would use the values of topological derivative for initial location of these holes, and then shape derivative would be applied for fine tuning their sizes and positions. Such a method is fast, but there is a danger of landing in a local optimum.

The main idea of the approach proposed in this paper is to combine the genetic algorithm (in order to avoid local optima) and continuous method using topological derivative for speeding up computations. This is illustrated by the examples based on boundary value problem described in preceding sections and cases of:

- one hole of fixed size;
- two holes of fixed sizes;

The topological derivative is used here for constructing the probability density defined on the domain in which the holes are allowed to appear. Then this density is used in the random selection of locations for the initial population of single holes or their pairs. The same probability is used to supplement the population in consecutive generations.

The approach exploiting the information supplied by topological derivative is compared to the method using exactly the same genetic algorithm, but with uniform distribution of holes locations. The performance is measured in both cases statistically over several runs.

5.1 Algorithm

5.1.1 Definition of the domain.

We define the domain D as a square $\langle -1,1\rangle \times \langle -1,1\rangle$ in $\mathbb{R}^2$. In D we define a sub-domain ω as a circle of the center at the point $\mathcal{O} = (0,0)$ and the radius $r = 0.5$. Thus D consists of two open sub-domains ω and $\Omega = D \setminus \overline{\omega}$, see Fig.3, with the boundaries $\Gamma = \partial D$ and $\partial\omega$.

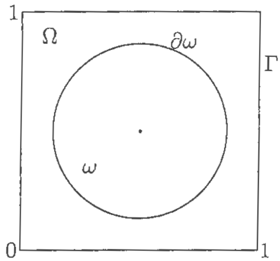

Figure 3: Domain $D = \Omega \cup \omega$.

In the interior domain we define a linear elliptic boundary value problem as follows:

$$\begin{aligned} -\Delta u(x) &= 0, \quad x \in \omega \\ u(x) &= v(x), \quad x \in \partial\omega \end{aligned} \tag{28}$$

Outside the circle ω, in the domain Ω, we define the following semi-linear boundary value problem with the Dirichlet condition on the exterior boundary Γ of the square and the transmission condition on the common boundary $\partial\omega$ of the circle:

$$\begin{aligned} -\Delta v(x) + v^3(x) &= f(x), \quad x \in \Omega, \\ v(x) &= 0, \quad x \in \Gamma, \\ \partial_n v(x) &= \mathcal{A}(v(x)), \quad x \in \partial\omega. \end{aligned} \tag{29}$$

The operator $\mathcal{A}$ is the Steklov-Poincaré operator and the function $f(x)$ are given.

5.1.2 Solution to the non-linear and linear problems.

In order to solve numerically the coupled problem described in (28) and (29), we introduce a characteristic function χ defined as follows:

$$\chi(\Omega) = \begin{cases} 1 & x \in \Omega, \\ 0 & x \in \omega. \end{cases} \tag{30}$$

Thus, the coupled problem under consideration can be rewritten in the following way:

$$\begin{aligned} -\Delta w(x) + \chi(\Omega) w^3(x) &= \chi(\Omega) f(x), \quad x \in D, \\ w(x) &= 0, \quad x \in \Gamma. \end{aligned} \tag{31}$$

Let $\mathcal{V}(D) = \{v \in H_0^1(D) : v = 0 \text{ on } \Gamma\}$. Multiplying (31) by a function $\varphi \in \mathcal{V}$ and integrating by parts we get the following weak formulation:

$$
\begin{aligned}
&\int\limits_{\Omega} \nabla w(x) \nabla \varphi(x) dx - \int\limits_{\partial\omega} \frac{\partial w(x)}{\partial n_1} \varphi(x) dS + \int\limits_{\omega} \nabla w(x) \nabla \varphi(x) dx \\
&\quad - \int\limits_{\partial\omega} \frac{\partial w(x)}{\partial n_2} \varphi(x) dS + \int\limits_{\Omega} w^3(x) \varphi(x) dx = \int\limits_{\Omega} f(x) \varphi(x) dx, \quad \forall \varphi \in \mathcal{V}(D).
\end{aligned}
\tag{32}
$$

Here we denote by n_1 the outward normal vector to ω, and by n_2 the outward normal vector to Ω. If we suppose that $w|_{\Omega} = v$, $w|_{\omega} = u$, then we get the following variational formulations of the nonlinear problem in the domain Ω:

$$
\begin{cases}
\text{Find } v(x) \text{ such, that} \\
\int\limits_{\Omega} \nabla v(x) \nabla \varphi(x) dx + \int\limits_{\Omega} v^3(x) \varphi(x) dx = \int\limits_{\Omega} f(x) \varphi(x) dx \quad \forall \varphi \in \mathcal{V}(D),
\end{cases}
\tag{33}
$$

with the transmission condition on the common boundary ensured by:

$$
\int\limits_{\partial\omega} \frac{\partial u(x)}{\partial n_1} \varphi(x) dS + \int\limits_{\partial\omega} \frac{\partial v(x)}{\partial n_2} \varphi(x) dS = 0. \tag{34}
$$

Similarly, for $u = v$ on $\partial\omega$ we obtain the variational formulation in the domain ω:

$$
\begin{cases}
\text{Find } u(x) \text{ such, that } u = v \text{ on } \partial\omega \text{ and} \\
\int\limits_{\omega} \nabla u(x) \nabla \varphi(x) dx = 0, \quad \forall \varphi \in \mathcal{V}(D).
\end{cases}
\tag{35}
$$

5.1.3 Minimization of the shape functional.

For numerical experiments we use the tracking type shape functional with a known element z_d, so an optimal value is 0.

In the first case we consider the instance of the single hole. Let us set $\omega_\varepsilon = \omega \setminus \overline{B}_\varepsilon(x_O)$, where $B_\varepsilon(x_O)$ is a small hole created in the interior circular domain ω at certain point x_O and of fixed radius ε. Thus, $D_\varepsilon = \Omega \cup \omega_\varepsilon$ and in such domain we define a target function z_d as a solution to the following boundary value problem :

$$\begin{aligned} -\Delta z_d(x) + \chi(\Omega) z_d^3(x) = \chi(\Omega) f(x), \quad & x \in D_\varepsilon, \\ z_d(x) = 0, \quad & x \in \Gamma, \\ \frac{\partial z_d}{\partial n} = 0, \quad & x \in \partial B_\varepsilon. \end{aligned} \tag{36}$$

The cost functional that we want to minimize is of the tracking type, as mentioned above, and depends on the location x_O of the hole:

$$J(v_\varepsilon) = \frac{1}{2} \int\limits_\Omega (v_\varepsilon(x) - z_d(x))^2 dx, \tag{37}$$

where v_ε is the solution to the semi-linear problem in perturbed domain

$$\begin{aligned} -\Delta v_\varepsilon(x) + v_\varepsilon^3(x) = f(x), \quad & x \in \Omega, \\ v_\varepsilon(x) = 0, \quad & x \in \Gamma, \\ \partial_n v_\varepsilon(x) = \mathcal{A}_\varepsilon(v_\varepsilon(x)), \quad & x \in \partial\omega. \end{aligned} \tag{38}$$

In order to minimize this shape functional we are looking for the optimal perforation of the domain ω. To this end, we apply the genetic algorithm, which uses the values of topological derivative as the probability density to determine the best location of an optimal hole. For this shape functional, due to the theorem 4.1, the topological derivative is given by the following formula:

$$\mathcal{T}_\Omega(\mathcal{O}) = 2\pi \nabla v(\mathcal{O}) \cdot \nabla p(\mathcal{O}). \tag{39}$$

where p is the so-called adjoint state being a solution to the following linear boundary value problem:

$$\begin{aligned} -\Delta p(x) + 3v^2 p(x) &= (v(x) - z_d(x)), \quad x \in \Omega, \\ -\Delta p(x) &= 0, \quad x \in \omega \\ p(x) &= 0, \quad x \in \Gamma, \end{aligned} \tag{40}$$

All numerical computations were performed using Matlab and its PDE toolbox. In particular, the triangulation of intact and perforated domains was done using the built-in procedures. The domain is divided into M triangles represented by the matrix $T = [t_{ij}]$, $i = 1, \ldots, M$, $j = 1, 2, 3$, where t_{i*} denote labels of points constituting vertices of the i-th triangle. The number of vertices is denoted by N. The boundary value problems were solved using linear finite elements. If we suppose that $\mathcal{T}_n$ is the value of the topological derivative at the node n, $n = 1, \ldots, N$, and v_n, p_n are the values of solutions to the boundary value problems (38), (39) respectively, then formula (39) can be rewritten in the discrete domain as follows

$$\mathcal{T}_n = 2\pi \nabla v_n \cdot \nabla p_n. \tag{41}$$

The probability density function associated with the topological derivative $\mathcal{T}_n$ is defined in the discrete domain in the following way. Let $\mathcal{T}_{\max} = \max\limits_{n=1,\ldots,N} \mathcal{T}_n$ and let $K_n = \mathcal{T}_{\max} - \mathcal{T}_n$, for $n = 1, \ldots, N$. Then the value $\mathcal{P}_n$ of the probability density function at the node n is given by the following formula

$$\mathcal{P}_n = \frac{K_n}{\sum\limits_{n=1}^{N} K_n}. \tag{42}$$

5.1.4 Application of genetic algorithm.

Initial population. The initial population consists of circular holes represented by the coordinates of their centres (the radius of the hole is fixed and is the same for each hole). Thus, each element of population represents a point in a space for which the dimension is determined by the number of coordinates of the holes centers. In case of one hole, the element of the population constitutes a vector $[x_0, y_0]$ of two coordinates of the center of the circular hole. In case of two holes, the individual is given by a vector of four coordinates defining centres $[x_1, y_1, x_2, y_2]$ of two holes. The initial population is randomly initialized with a constant number $\mathcal{S}$ of individuals. For the initialization, the probability density function defined in (42) is used. Since the values of this function are defined only in the nodes of the discretized domain, we need the following procedure to draw a random point, not necessarily the node, in the domain.

Let $t_{i*} = [t_{ij}]$, $i = 1, \ldots, M$ be a i-th triangle with vertices $[t_{i1}, t_{i2}, t_{i3}]$. The values $\mathcal{P}_{t_{ij}}$, $i = 1 \ldots N$, $j = 1 \ldots 3$ are the probabilities connected with these vertices. The probability $P(t_{i*})$ of the i-th triangle t_{i*} is calculated through the formula

$$P(t_{i*}) = \frac{\sum\limits_{j=1}^{3} \mathcal{P}_{t_{ij}}}{\sum\limits_{i=1}^{M}\sum\limits_{j=1}^{3} \mathcal{P}_{t_{ij}}}, \quad i = 1, \ldots, M \tag{43}$$

For each triangle t_{i*} we define the vector $\lambda_i = [\lambda_{ij}]$, $j = 1, 2, 3$, such that

$$\lambda_{ij} = \mathcal{P}_{t_{ij}} \cdot r_j, \quad i = 1, \ldots, M, \quad j = 1, 2, 3, \tag{44}$$

Here r_j are three independent random numbers generated using uniform distribution on $[0, 1]$ interval. The point $\hat{p} \in t_{i*}$ is selected using these λ_{ij}

and in dependence of probability density in the corners

$$\hat{p} = \frac{1}{\sum_{j=1}^{3} \lambda_{ij}} \cdot \sum_{j=1}^{3} \lambda_{ij} x(t_{ij}). \tag{45}$$

This procedure may be summarized as follows:

- first we select the triangle according to its average probability;
- next we select the point inside this triangle taking into account the values of probability density in the corners.

In case of two holes, selection of the initial population consists in drawing the pairs of holes. A pair of holes is retained only if the distance between their centers is greater that the fixed value 2ε, where ε is the radius of the hole. If the condition is not verified, the pair is eliminated and the drawing continues until the required number or elements is reached.

Evaluation. The fitness value of every individual is evaluated using the cost functional $\mathcal{J}$ such that the minimization of $\mathcal{J}$ is equivalent to the maximization of the fitness value. Since the population is considered as a vector of individuals sorted according to their fitness values, the lower value of the cost functional means the higher position in the population vector.

Crossover. Once the initialization and evaluation are complete, the crossover operations perform genetic code exchange between pairs of individuals. These pairs are chosen in following way. First we define, as a parameter, a number of dominating elements which are always at the beginning of the population vector. Then every dominating element undergoes crossover with every subordinate element. Let α be a random number with uniform distribution on $[0,1]$, different in every formula. In case of one hole, the crossover of two individuals $x = [x_1, x_2]^\top$ and $y = [y_1, y_2]^\top$ results in $z = [z_1, z_2]^\top$ given by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{46}$$

In case of two holes, the crossover of $x = [x_1, x_2, x_3, x_4]^\top$ and $y = [y_1, y_2, y_3, y_4]^\top$ gives two individuals $z = [z_1, z_2, z_3, z_4]^\top$ and $z' = [z'_1, z'_2, z'_3, z'_4]^\top$ according to formulae

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + (1-\alpha) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \tag{47}$$

and

$$\begin{bmatrix} z_1' \\ z_2' \\ z_3' \\ z_4' \end{bmatrix} = \alpha \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \tag{48}$$

The elements obtained in this way are added to the initial population increasing its size.

Mutation. Mutation is a mechanism for extending the search on the new areas of the design space and increasing the variability of the population. Each element produced during the crossover is perturbed with a given probability (in our case 0.2) within the fixed range δ (in our case $\delta = \varepsilon$) according to the formula

$$z^* = z + \begin{bmatrix} 1-2\alpha_1 \\ 1-2\alpha_2 \end{bmatrix} \cdot \delta, \tag{49}$$

in case of single hole, and for pair of holes

$$z^* = z + \begin{bmatrix} 1-2\alpha_1 \\ 1-2\alpha_2 \\ 1-2\alpha_3 \\ 1-2\alpha_4 \end{bmatrix} \cdot \delta. \tag{50}$$

where α_i are random numbers from the interval $[0,1]$.

New generation. In the next step the population consisting of old individuals and those produced in the crossover stage is pruned by removing the elements violating the constraints (distance between centres greater then 2ε in case of two holes) and sorted according to fitness value.

The next generation contains $\frac{2}{3}S$ of the best elements. In order to prevent locking in local optima, $\frac{1}{3}S$ individuals are again drawn randomly using appropriate probabilities, i.e. the one based on topological derivative or uniform on ω.

5.2 Numerical results

In the first example we consider a case with one hole. As a target domain we take a square with one hole inside the interior circular sub-domain. The center of this hole is at point $(-0.2; 0.2)$ and the radius is 0.08. The goal functional is computed using z_d corresponding to the function $f(x) = x_1$. The size of the initial population for GA is 21 genes, with 3 dominants, and the number of generations is 10. The optimization process is repeated 20 times which gives the average results presented in Fig.4. As we see, the algorithm using topological derivative gives on average faster convergence. For comparison, we give in Fig. 5 the history of true distances of the best

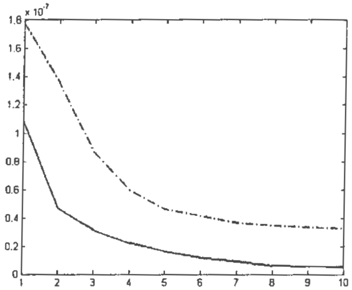

Figure 4: Value of shape functional due to the topological derivative density (solid line) and the uniform distribution density (dashed line) in case of one hole.

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individuals to the reference hole; however, this value was not subject to optimization. The value of density probability is shown in Fig. 6.

In the second example the reference shape contained two holes with radius 0.05 located at $(-0.1, 0.2,)$ and $(0.2, -0.1)$. The goal functional consisted of the sum of two parts with different reference functions $z_{d1},\ z_{d2}$ corresponding to right–hand sides $f_1 = x_1 + x_2$ and $f_2 = x_1 - x_2$. As a result, the topological derivative was also computed twice, and the value of the final $\mathcal{T}$ was the sum of these two parts. It is shown in Fig. 7 while the Fig. 8 presents the average

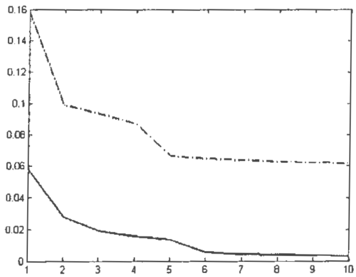

Figure 5: History of true distances of the best individuals to the reference hole.

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performance of both types of genetic algorithms.

In Fig. 9 we see again the average true distances of the best individual to the reference pair, and Fig. 10 shows the typical last population of the genetic algorithm using topological derivative. Nearly all members of this population overlap.

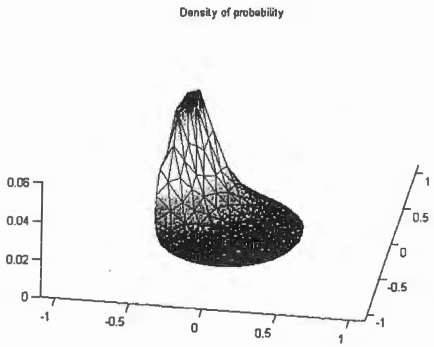

Figure 6: Value of probability density for one hole.

6 Conclusions

The problem studied in this paper belongs to a class of domain optimization problems which may be characterized by the following goal: find location of a fixed number of holes, with possibly variable sizes, in the definition area of the system of PDE's. These locations and sizes should minimize certain functionals and/or satisfy some constraints. Such problems are quite common in structural design. Because they can have many local optima, genetic algorithms are often suggested as suitable tools. at least in the initial stage of the design. Such algorithms are computationally costly, therefore speeding them up may be worthwhile.

The topological derivative is an ideal tool to this purpose. The cost of its computing is negligible in comparison to genetic calculations and it can be easily transformed into probability density for drawing elements from populations of all possible designs. As we see from examples, the effect is quite meaningful and justifies such usage. The approach proposed here is, to the knowledge of the authors, new.

References

[1] P. Fulmanski, A. Lauraine, J.-F. Scheid, J. Sokołowski, *A level set method in shape and topology optimization for variational inequalities*, Int. J. Appl. Math. Comput. Sci., 2007, Vol.17, No 3, p.413-430.

[2] M. Iguernane, S.A. Nazarov, J.-R. Roche, J. Sokolowski, K. Szulc, *Topological derivatives for semilinear elliptic equations*, Int. J. Appl. Math. Comput. Sci., 2009, Vol.19, No.2, p.191-205.

[3] A. Kowalewski, I. Lasiecka, J. Sokolowski, *Sensitivity analysis of hiperbolic optimal control problems*, Comput. Optim. Appl., 2010.

[4] V.G. Mazja, S.A. Nazarov, B.A. Plomenevskii, *Asymptotische Theorie elliptischer Randzertaufgaben in singulär gestörten*, Gebieten. 1, Berlin: Akademie-Verlag, [in Russian], 1991, [English translasion: V.G. MAZJA, S.A. NAZAROV, B.A. PLOMENEVSKII, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains*, Vol. 1, Basel: Birkhäuser Verlag, 2000].

[5] G. Pólya, G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951.

[6] J. Sokolowski, A. Zochowski, *Asymptotic analysis and topological derivatives for shape and topology optimization of elasticity problems in two spatial dimensions* Engineering Analysis with Boundary Elements. 32(2008) 533-544.

[7] J. Sokolowski, A. Zochowski, *Modelling of topological derivatives for contact problems* Numerische Mathematik, 102(2005), No. 1, 145 - 179.

[8] J. Sokołowski, J.-P. Zolésio, *Introduction to shape optimization. Shape sensitivity analysis.* Springer-Verlag, 1992, New York.

[9] K. Szulc, *Quelques méthode numérique en optimisation de formes*, Ph.D. Thesis, 2010.

[10] G. ALLAIRE, *Shape optimization by the homogeneization method*, Springer, New York, 2002.

[11] M.P. BENDSOE, O. SIGMUND, *Topology optimization. Theory, methods and applications*, Springer, New York, 2003.

[12] I.N. DEMKOV, V.N. OSTROVSKY, *A method of zero radius potential in atomic physics* (Russian), Leningrad: Leningrad University, 1975.

[13] S. GARREAU, PH. GUILLAUME, M. MASMOUDI, *The topological asymptotic for PDE systems: the elasticity case*, SIAM J. Control Optim., vol. 39, No 6, 2001, pp. 1756-1778.

[14] W. HACKBUSCH, *Elliptic differential equations*, vol. 18 of Springer Series in Computationnal Mathematics, Springer Verlag, Berlin, 1992.

[15] A. HENROT, M. PIERRE , *Variation et optimisation de formes: une analyse geometrique*, No 48 de Mathematiques et Applications, Springer , 2005.

[16] I.V. KAMOTSKII, S.A. NAZAROV, *Spectral problems in singularly perturbed domains and selfadjoint extensions of differential operators.*, Amer. Math. Soc. Transl. (2) Vol. 199, 2000.

[17] S.K. KANAUN, V.M. LEVIN, *The effective field method in the mechanics of composite materials* (Russian), Izdatel'stvo Petrozavodskogo Universiteta, Petrozavodsk, 1993.

[18] A. LAURAIN, *Singularly perturbed domains in shape optimization*, phD Thesis, Université Henri Poincaré, Nancy 1, 2006.

[19] S.A. NAZAROV, *Asymptotic conditions at a point, self adjoint extensions of operators, and the method of matched asymptotic expansions*, American Mathematical Society Translations (2), Vol. 198, 1999, pp. 77–125.

[20] S.A. NAZAROV, J. SOKOŁOWSKI, *Asymptotic analysis of shape functionals*, Journal de Mathématiques pures et appliquées. 82(2003), 125-196.

[21] S.A. NAZAROV, J. SOKOLOWSKI, *Self adjoint extensions of differential operators in application to shape optimization*, Comptes Rendus Mecanique, Volume 331, Issue 10, October 2003, 667-672.

[22] S.A. NAZAROV, J. SOKOLOWSKI, *Selfadjoint extensions for elasticity system in application to shape optimization*, to appear in Bulletin of the Polish Academy of Sciences – Mathematics.

[23] S.A. NAZAROV, J. SOKOLOWSKI, *Self adjoint extensions for the Neumann laplacian in application to Shape Optimization*, Les prepublications de l'institut Elie Cartan 9/2003.

[24] B.S. PAVLOV, *The theory of extension and explicitely soluble models*, Uspehi Mat. Nauk 42(6)(1987) 99-131. (Engl. Transl. in Soviet Math. Surveys 42(6)(1987) 127-168)

[25] P.I. Plotnikov, J. Sokolowski, *Inhomogeneous boundary value problems for compressible Navier-Stokes equations, well-posedness and sensitivity analysis*, SIAM Journal on Mathematical Analysis 40 (2008), pp. 1152- 1200.

[26] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *On the topological derivative in shape optimization*, SIAM Journal on Control and Optimization. 37, Number 4 (1999), pp. 1251–1272.

[27] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *Optimality conditions for simultaneous topology and shape optimization*, SIAM Journal on Control and Optimization, Vol. 42, no. 4 , 2003, pp. 1198–1221.

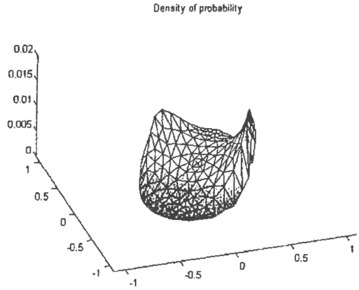

Figure 7: Value of probability density for two holes.

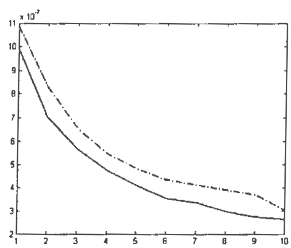

Figure 8: Value of shape functional due to the topological derivative density (solid line) and the uniform distribution density (dashed line) in case of two holes.

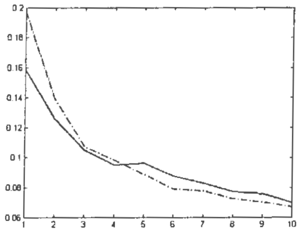

Figure 9: History of average true distances of the best individuals to the reference pair of holes.

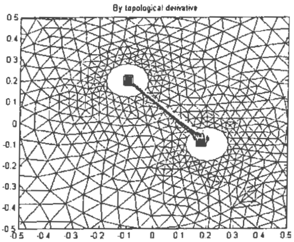

Figure 10: The best individuals given by the topological derivative.