ON CERTAIN NUMERICAL PRODUCTS IN WHICH THE EXPONENTS DEPEND UPON THE NUMBERS.

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Ratios of products in which the exponent is the same as the number, §§ 1–17.

§ 1. IN Vol. VI. (pp. 71–76) of the *Messenger* an expression was obtained for the value of the quotient

$$\frac{3^3.7^7.11^{11}...(4n-1)^{4n-1}}{1^1.5^5.9^9...(4n-3)^{4n-3}},$$

when n is very large. It is only recently that I have noticed that a similar method enables us to assign the value of the product

$$\frac{2^2.5^5.8^8.11^{11}...(3n+2)^{3n+2}}{1^1.4^4.7^7.10^{10}...(3n+1)^{3n+1}}.*$$

§ 2. Proceeding as in Vol. VI., p. 189, let

$$u=\left(\frac{1+x}{1-x}\right)^1\left(\frac{2+x}{2-x}\right)^2\left(\frac{3+x}{3-x}\right)^3...\left(\frac{n+x}{n-x}\right)^n,$$

then

$$\begin{aligned}\tfrac{1}{2}\log u &= x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\tfrac{1}{7}x^7+\&c.\\ &\quad+2\left(\frac{x}{2}+\tfrac{1}{3}\frac{x^3}{2^3}+\tfrac{1}{5}\frac{x^5}{2^5}+\tfrac{1}{7}\frac{x^7}{2^7}+\&c.\right)\\ &\quad+3\left(\frac{x}{3}+\tfrac{1}{3}\frac{x^3}{3^3}+\tfrac{1}{5}\frac{x^5}{3^5}+\tfrac{1}{7}\frac{x^7}{3^7}+\&c.\right)\\ &\quad\dots\dots\dots\dots\dots\dots\\ &\quad+n\left(\frac{x}{n}+\tfrac{1}{3}\frac{x^3}{n^3}+\tfrac{1}{5}\frac{x^5}{n^5}+\tfrac{1}{7}\frac{x^7}{n^7}+\&c.\right)\\ &= n+\tfrac{1}{3}S_2x^3+\tfrac{1}{5}S_4x^5+\tfrac{1}{7}S_6x^7+\&c.,\end{aligned}$$

where $$S_r=1+\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}+\&c.$$

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* 'On a numerical continued product,' *Messenger*, Vol. VI., pp. 71–76, and 'Further note on certain numerical continued products,' Vol. VI., pp. 189–192. Other papers connected with the same subject, and which are referred to in the present paper, are 'On the numerical value of a certain series,' *Proc. Lond. Math. Soc.*, Vol. VIII., pp. 200–204, and 'Proof of Stirling's theorem $1.2.3...n=\sqrt{(2n\pi)}\,n^ne^{-n}$,' *Quart. Journ. Math.*, Vol. XV., pp. 57–64.

Thus

$$\left(\frac{1+x}{1-x}\right)^1 \left(\frac{2+x}{2-x}\right)^2 \left(\frac{3+x}{3-x}\right)^3 \cdots \left(\frac{n+x}{n-x}\right)^n$$

$$= e^{2nx} e^{\frac{2}{3}S_2x^3 + \frac{2}{5}S_4x^5 + \frac{2}{7}S_6x^7 + \&c.}$$

§ 3. Putting $x = \frac{1}{3}$ in this result, we find

$$\frac{4^1.7^2.10^3\ldots(3n+1)^n}{2^1.5^2.8^3\ldots(3n-1)^n} = e^{\frac{2}{3}n} e^{\frac{2}{3}S_2\frac{1}{3^3} + \frac{2}{5}S_4\frac{1}{3^5} + \&c.},$$

whence, cubing each side,

$$\frac{4^3.7^6.10^9\ldots(3n+1)^{3n}}{2^3.5^6.8^9\ldots(3n-1)^{3n}} = e^{2n} e^{\frac{2}{3}S_2\frac{1}{3^2} + \frac{2}{5}S_4\frac{1}{3^4} + \&c.},$$

Now $1.2.3.4\ldots(3n+1) = \sqrt{(2\pi)}\,(3n+1)^{3n+\frac{3}{2}}e^{-3n-1}$

$$= \sqrt{(2\pi)}\,(3n)^{3n+\frac{3}{2}}e^{-3n},$$

and $$3.6.9\ldots3n = 3^n\sqrt{(2\pi)}\,n^{n+\frac{1}{2}}e^{-n};$$

so that, by division,

$$1.2.4.5.7\ldots(3n+1) = e^{-2n}3^{2n+\frac{3}{2}}n^{2n+1}.$$

Multiplying the above quotient by this product, we have

$$\frac{1^1.4^4.7^7\ldots(3n+1)^{3n+1}}{2^2.5^5.8^8\ldots(3n-1)^{3n-1}} = \sqrt{3}\,(3n)^{2n+1}e^{\frac{2}{3}S_2\frac{1}{3^2} + \frac{2}{5}S_4\frac{1}{3^4} + \&c.}.$$

The values of S_2, S_4, S_6, ... have been tabulated*, so that the calculation of the series in the exponent presents no difficulty.

§ 4. The exponent in the general formula of § 2 is

$$\tfrac{2}{3}S_2x^3 + \tfrac{2}{5}S_4x^5 + \tfrac{2}{7}S_6x^7 + \&c.;$$

and this series may be readily expressed as an integral.

For, we have

$$\pi\cot\pi x - \frac{1}{x} = -2S_2x - 2S_4x^3 - 2S_6x^5 - \&c.,$$

* The values of S_n, from $n=1$ to $n=35$, were calculated, to sixteen places of decimals, by Legendre (*Traité des Fonctions Elliptiques*, Vol. II., p. 432). This table is reprinted De Morgan's *Diff. and Int. Calc.*, p. 554. The values of S_2, S_4, ..., S_{12} were given in Vol. VIII., p. 190, of the *Proc. Lond. Math. Soc.* to twenty-two places.

whence $\frac{2}{3}S_2x^3 + \frac{2}{5}S_4x^5 + \frac{2}{7}S_6x^7 + \&c.$

$$= x - \pi\int_0^x \frac{x\,dx}{\tan \pi x} = x - \frac{1}{\pi}\int_0^{\pi x}\frac{x\,dx}{\tan x}.$$

We may also write the value of the series in the form

$$x - x\log\sin\pi x + \int_0^x \log\sin\pi x\,dx,$$

which is readily obtained by integration by parts.

§ 5. The value of the first quotient in § 1 may also be deduced from the general theorem in § 2; for, by putting $x=\frac{1}{4}$, we find

$$\left(\frac{5}{3}\right)^1\left(\frac{9}{7}\right)^2\left(\frac{13}{11}\right)^3\cdots\left(\frac{4n+1}{4n-1}\right)^n = e^{\frac{1}{2}n}e^{\frac{2}{3}S_2\frac{1}{4^3}+\frac{2}{5}S_4\frac{1}{4^5}+\&c.},$$

whence, raising both sides to the fourth power,

$$\left(\frac{5}{3}\right)^4\left(\frac{9}{7}\right)^8\left(\frac{13}{11}\right)^{12}\cdots\left(\frac{4n+1}{4n-1}\right)^{4n} = e^{2n}e^{\frac{2}{3}S_2\frac{1}{4^2}+\frac{2}{5}S_4\frac{1}{4^4}+\&c.}.$$

Now $1.2.3\ldots(4n+1) = \sqrt{(2\pi)}\,(4n+1)^{4n+\frac{3}{2}}e^{-4n-1}$,

and $2.4.6\ldots4n = \sqrt{(2\pi)}\,2^{2n}\,(2n)^{2n+\frac{1}{2}}e^{-2n}$;

whence, by division,

$$1.3.5.7\ldots(4n+1) = e^{-2n}2^{4n+\frac{5}{2}}n^{2n+1}.$$

Multiplying the above result by this product, we find

$$\frac{5^5.9^9.13^{13}\ldots(4n+1)^{4n+1}}{3^3.7^7.11^{11}\ldots(4n-1)^{4n-1}} = \sqrt{2}\,(4n)^{2n+1}e^{\frac{2}{3}S_2\frac{1}{4^2}+\frac{2}{5}S_4\frac{1}{4^4}+\&c.}.$$

§ 6. The result obtained in the previous papers already referred to was*

$$\frac{3^3.7^7.11^{11}\ldots(4n-1)^{4n-1}}{1^1.5^5.9^9\ldots(4n-3)^{4n-3}} = (4n)^{2n}e^A,$$

where

$$A = \frac{2}{\pi}\left(1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \&c.\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{3}s_2\frac{1}{2^2} + \tfrac{1}{5}s_4\frac{1}{2^4} + \tfrac{1}{7}s_6\frac{1}{2^6} + \&c.,$$

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* *Messenger*, Vol. VI., p. 192.

s_r denoting the series

$$1-\frac{1}{2^r}+\frac{1}{3^r}-\frac{1}{4^r}+\frac{1}{5^r}-\&c.$$

From this result it follows that

$$\frac{1^1.5^5.9^9\ldots(4n+1)^{4n+1}}{3^3.7^7.11^{11}\ldots(4n-1)^{4n-1}}=(4n)^{2n+1}e^{1-A}.$$

§ 7. By equating this value of the product to that found in § 5, we obtain the relations

$$\tfrac{1}{2}\log 2+\tfrac{2}{3}S_2\frac{1}{4^2}+\tfrac{2}{5}S_4\frac{1}{4^4}+\&c.$$

$$=1-\frac{2}{\pi}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\&c.\right)$$

$$=\tfrac{1}{2}-\tfrac{1}{3}s_2\frac{1}{2^2}-\tfrac{1}{5}s_4\frac{1}{2^4}-\tfrac{1}{7}s_6\frac{1}{2^6}-\&c.$$

§ 8. The equality of the first two expressions may be readily verified: for, by § 4,

$$\tfrac{2}{3}S_2\frac{1}{4^2}+\tfrac{2}{5}S_4\frac{1}{4^4}+\&c.=1-\frac{4}{\pi}\int_0^{\frac{1}{4}\pi}\frac{x\,dx}{\tan x},$$

so that the equation becomes

$$\int_0^{\frac{1}{4}\pi}\frac{x\,dx}{\tan x}=\tfrac{1}{8}\pi\log 2+\tfrac{1}{2}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\&c.\right),$$

which is derivable from the known results

$$\int_0^{\frac{1}{4}\pi}x\tan x\,dx=-\tfrac{1}{8}\pi\log 2+\tfrac{1}{2}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\&c.\right)$$

$$\int_0^{\frac{1}{2}\pi}\frac{x}{\tan x}\,dx=\tfrac{1}{2}\pi\log 2.$$

§ 9. By equating the first and last expressions in § 7, we find

$$\log 2+\tfrac{1}{3}S_2\frac{1}{4}+\tfrac{1}{5}S_4\frac{1}{4^3}+\&c.=1-\tfrac{1}{3}s_2\frac{1}{2}-\tfrac{1}{5}s_4\frac{1}{2^3}-\&c.$$

Now $\qquad s_n=\left(1-\dfrac{1}{2^{n-1}}\right)S_n,$

so that the right-hand side when expressed in terms of S' becomes

$$1-\tfrac{1}{3}S_2\frac{1}{2}+\tfrac{1}{3}S_2\frac{1}{2^2}-\tfrac{1}{5}S_4\frac{1}{2^3}+\tfrac{1}{5}S_4\frac{1}{2^4}-\&c.$$

Thus the equation reduces to

$$\log 2=1-\tfrac{1}{3}S_2\frac{1}{2}-\tfrac{1}{5}S_4\frac{1}{2^3}-\tfrac{1}{7}S_6\frac{1}{2^5}-\&c.,$$

which may be easily verified: for by § 4,

$$\tfrac{1}{3}S_2\frac{1}{2}+\tfrac{1}{5}S_4\frac{1}{2^3}+\tfrac{1}{7}S_6\frac{1}{2^5}+\&c.=1-\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{x\,dx}{\tan x}$$
$$=1-\log 2.$$

§ 10. It will be noticed that the expression

$$\tfrac{2}{3}S_2\frac{1}{4^2}+\tfrac{2}{5}S_4\frac{1}{4^4}+\tfrac{2}{7}S_6\frac{1}{4^6}+\&c.$$

is even more convenient for calculation than

$$\tfrac{1}{3}s_2\,\frac{1}{2^2}+\tfrac{1}{5}s_4\,\frac{1}{2^4}+\tfrac{1}{7}s_6\,\frac{1}{2^6}+\&c.$$

from which the numerical value of the series $1-\frac{1}{9}+\frac{1}{25}-\&c.$ was obtained to 22 places of decimals in vol. VIII. of the *Proc. Lond. Math. Soc.* (p. 200).

§ 11. The process employed in §§ 2 and 3 affords also the value of the quotient

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1}\dots(na+1)^{na+1}}{(a-1)^{a-1}(2a-1)^{2a-1}\dots(na-1)^{na-1}},$$

the numerator containing the first n numbers which $\equiv 1$, mod. a, and the denominator the first n numbers which $\equiv -1$, mod. a*, each number being raised to a power equal to itself.

* These numbers are of interest in connexion with several arithmetical enquiries, and I have suggested that they may be conveniently called the supereven and subeven numbers to modulus a. (*Quar. Jour. Math.* vol. XXVI., p. 64). Thus the numbers occurring in the numerators of the quotients in § 1 are the subeven numbers to mod. 4 and mod. 3 respectively, and the numbers in the denominators are the supereven numbers to the same moduli.

For, putting $x=\frac{1}{a}$ in the theorem of § 2, and raising both sides to the power a, we find

$$\left(\frac{a+1}{a-1}\right)^a \left(\frac{2a+1}{2a-1}\right)^{2a} \left(\frac{3a+1}{3a-1}\right)^{3a} \cdots \left(\frac{na+1}{na-1}\right)^{na}$$

$$= e^{2n} e^{\frac{2}{3}S_2\frac{1}{a^2}+\frac{2}{5}S_4\frac{1}{a^4}+\&c.}$$

Now

$$(1^2-x^2)(2^2-x^2)\ldots(n^2-x^2)$$

$$=1^2.2^2.3^2\ldots n^2\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots\left(1-\frac{x^2}{n^2}\right)$$

$$=2\pi n^{2n+1}e^{-2n}\frac{\sin\pi x}{\pi x}.$$

Putting $n=\frac{1}{a}$, this result becomes

$$(a^2-1)(2^2a^2-1)\ldots(n^2a^2-1)=2\sin\frac{\pi}{a}(na)^{2n+1}e^{-2n}.$$

Multiplying the products as before, we find

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1}\ldots(na+1)^{na+1}}{(a-1)^{a-1}(2a-1)^{2a-1}\ldots(na-1)^{na-1}}$$

$$=2\sin\frac{\pi}{a}(na)^{2n+1}e^{\frac{2}{3}S_2\frac{1}{a^2}+\frac{2}{5}S_4\frac{1}{a^4}+\&c.}.$$

§ 12. If $a=3$, the factor multiplying the exponential

$$=\sqrt{3}\,(3n)^{2n+1},$$

and, if $a=4$, it

$$=\sqrt{2}\,(4n)^{2n+1},$$

agreeing with §§ 3 and 5.

If $a=6$, the factor $=(6n)^{2n+1}$.

§ 13. It is easy to obtain a corresponding general formula in which only uneven multiples of a occur.

For, proceeding as in §§ 2 and 11, we find

$$\left(\frac{a+1}{a-1}\right)^a \left(\frac{3a+1}{3a-1}\right)^{3a} \left(\frac{5a+1}{5a-1}\right)^{5a} \cdots \left\{\frac{(2n-1)a+1}{(2n-1)a-1}\right\}^{(2n-1)a}$$

$$=e^{2n}e^{\frac{2}{3}U_2\frac{1}{a^2}+\frac{2}{5}U_4\frac{1}{a^4}+\&c.},$$

where $$U_r = 1 + \frac{1}{3^r} + \frac{1}{5^r} + \frac{1}{7^r} + \&c.$$

Also

$$(1^2 - x^2)(3^2 - x^2)\ldots\{(2n-1)^2 - x^2\}$$
$$= 1^2.3^2\ldots(2n-1)^2\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{3^2}\right)\ldots\left\{1 - \frac{x^2}{(2n-1)^2}\right\}$$
$$= 2^{2n+1} n^{2n} e^{-2n} \cos\tfrac{1}{2}\pi x,$$

so that

$$(a^2 - 1)(3^2a^2 - 1)\ldots\{(2n-1)^2 a^2 - 1\} = 2\cos\frac{\pi}{2a}(2an)^{2n} e^{-2n}.$$

Thus $$\frac{(a+1)^{a+1}(3a+1)^{3a+1}\ldots\{(2n-1)a+1\}^{(2n-1)a+1}}{(a-1)^{a-1}(3a-1)^{3a-1}\ldots\{(2n-1)a-1\}^{(2n-1)a-1}}$$
$$= 2\cos\frac{\pi}{2a}(2an)^{2n} e^{\frac{2}{3}U_2\frac{1}{a^2} + \frac{2}{5}U_4\frac{1}{a^4} + \&c.}$$

§ 14. Putting $a = 2$, we find

$$\frac{3^3.7^7\ldots(4n-1)^{4n-1}}{1^1.5^5\ldots(4n-3)^{4n-3}} = \sqrt{2}\,(4n)^{2n} e^{\frac{2}{3}U_2\frac{1}{2^2} + \frac{2}{5}U_4\frac{1}{2^4} + \&c.}$$

Dividing by $(4n+1)^{4n+1}$ and inverting the quotient, we obtain

$$\frac{1^1.5^5\ldots(4n+1)^{4n+1}}{3^3.7^7\ldots(4n-1)^{4n-1}} = \frac{1}{\sqrt{2}}(4n)^{2n+1} e^{1-B},$$

where $$B = \tfrac{2}{3}U_2\frac{1}{2^2} + \tfrac{2}{5}U_4\frac{1}{2^4} + \&c.$$

§ 15. Comparing the result with that found in § 5, we find

$$-\log 2 + 1 - \tfrac{2}{3}U_2\frac{1}{2^2} - \tfrac{2}{5}U_4\frac{1}{2^4} - \&c. = \tfrac{2}{3}S_2\frac{1}{4^2} + \tfrac{2}{5}S_4\frac{1}{4^4} + \&c.$$

Since $$U_n = \left(1 - \frac{1}{2^n}\right)S_n,$$

the left-hand side becomes

$$-\log 2 + 1 - \tfrac{2}{3}S_2\frac{1}{2^2} + \tfrac{2}{3}S_2\frac{1}{2^4} - \tfrac{2}{5}S_4\frac{1}{2^4} + \tfrac{2}{5}S_4\frac{1}{2^8} - \&c.$$

Thus the equation reduces to

$$\log 2 = 1 - \tfrac{1}{3} S_2 \frac{1}{2} - \tfrac{1}{5} S_4 \frac{1}{2^3} - \&c.,$$

which is the same as that found in § 9.

§ 16. We may also obtain without difficulty the value of the product

$$\frac{(a+1)^{a+1}(2a-1)^{2a-1}(3a+1)^{3a+1}\ldots(2na-1)^{2na-1}}{(a-1)^{a-1}(2a+1)^{2a+1}(3a-1)^{3a-1}\ldots(2na+1)^{2na+1}}$$

in which the signs are alternatively positive and negative in the numerator and in the denominator.

For, proceeding as in § 2, if

$$u = \left(\frac{1+x}{1-x}\right)^1 \left(\frac{2-x}{2+x}\right)^2 \left(\frac{3+x}{3-x}\right)^3 \ldots \left(\frac{2n-x}{2n+x}\right)^{2n},$$

then $$\tfrac{1}{2}\log u = \tfrac{1}{3} s_2 x^3 + \tfrac{1}{5} s_4 x^5 + \tfrac{1}{7} s_6 x^7 + \&c.,$$

where, as before, $$s_r = 1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \&c.$$

Thus

$$\left(\frac{a+1}{a-1}\right)^a \left(\frac{2a-1}{2a+1}\right)^{2a} \left(\frac{3a+1}{3a-1}\right)^{3a} \ldots \left(\frac{2na-1}{2na+1}\right)^{2na}$$
$$= e^{\frac{2}{3}s_2 \frac{1}{a^2} + \frac{2}{5}s_4 \frac{1}{a^4} + \&c.}.$$

Also

$$\frac{(a^2-1)(3^2a^2-1)\ldots\{(2n-1)^2a^2-1\}}{(2^2a^2-1)(4^2a^2-1)\ldots(4n^2a^2-1)}$$

$$= \frac{1^2.3^2\ldots(2n-1)^2}{2^2.4^2\ldots 4n^2} \cdot \frac{\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{3^2a^2}\right)\ldots\left\{1-\frac{1}{(2n-1)^2a^2}\right\}}{\left(1-\frac{1}{2^2a^2}\right)\left(1-\frac{1}{4^2a^2}\right)\ldots\left(1-\frac{1}{4n^2a^2}\right)}$$

$$= \frac{1}{\pi n} \frac{\cos\frac{\pi}{2a}}{\frac{2a}{\pi}\sin\frac{\pi}{2a}} = \frac{1}{2na}\cot\frac{\pi}{2a}.$$

Thus, by multiplication, we find

$$\frac{(a+1)^{a+1}(2a-1)^{2a-1}\dots(2na-1)^{2na-1}}{(a-1)^{a-1}(2a+1)^{2a+1}\dots(2na+1)^{2na+1}}=\frac{1}{2na}\cot\frac{\pi}{2a}e^{C},$$

where $$C=\tfrac{2}{3}s_2\frac{1}{a^2}+\tfrac{2}{5}s_4\frac{1}{a^4}+\tfrac{2}{7}s_6\frac{1}{a^6}+\&c.$$

§ 17. Putting $2n$ for n in the result of § 11, it becomes

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1}\dots(2na+1)^{2na+1}}{(a-1)^{a-1}(2a-1)^{2a-1}\dots(2na-1)^{2na-1}}$$
$$=2\sin\frac{\pi}{a}(2na)^{4na+1}e^{A},$$

where $$A=\tfrac{2}{3}S_2\frac{1}{a^2}+\tfrac{2}{5}S_4\frac{1}{a^4}+\tfrac{2}{7}S_6\frac{1}{a^6}+\&c.$$

Multiplying together this formula and that found in the preceding section, and taking the square root of each side of the equation, we find

$$\frac{(a+1)^{a+1}(3a+1)^{3a+1}\dots\{(2n-1)a+1\}^{(2n-1)a+1}}{(a-1)^{a-1}(3a-1)^{3a-1}\dots\{(2n-1)a-1\}^{(2n-2)a-1}}$$
$$=\sqrt{\left\{2\sin\frac{\pi}{a}\cot\frac{\pi}{2a}(2na)^{4na}\right\}}e^{\frac{1}{2}(A+C)},$$

which agrees with the formula in § 13, for the quantity under the square root sign

$$=4\cos^2\frac{\pi}{2a}(2na)^{4na},$$

and, obviously,

$$\tfrac{1}{2}(A+C)=\tfrac{2}{3}U_2\frac{1}{a^2}+\tfrac{2}{5}U_4\frac{1}{a^4}+\tfrac{2}{7}U_6\frac{1}{a^6}+\&c.$$

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§ 18. We may readily obtain similar formulæ in which the exponents are the squares of the numbers.

Thus, let

$$u=\left(\frac{1+x}{1-x}\right)^{1}\left(\frac{2+x}{2-x}\right)^{4}\left(\frac{3+x}{3-x}\right)^{9}\dots\left(\frac{n+x}{n-x}\right)^{n^2},$$

then

$$\tfrac{1}{2}\log u = \tfrac{1}{2}n(n+1)x + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)x^3$$
$$+ \tfrac{1}{5}S_3x^5 + \tfrac{1}{7}S_5x^7 + \text{\&c.},$$

and, since

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \gamma + \log n,$$

we have

$$u = n^{\frac{2}{3}x^3} e^{n(n+1)x + \frac{2}{3}\gamma x^3 + \frac{2}{5}S_3x^5 + \frac{2}{7}S_5x^7 + \text{\&c.}},$$

so that

$$\left(\frac{a+1}{a-1}\right)^{a^2}\left(\frac{2a+1}{2a-1}\right)^{4a^2}\left(\frac{3a+1}{3a-1}\right)^{9a^2}\ldots\left(\frac{na+1}{na-1}\right)^{n^2a^2}$$
$$= n^{\frac{2}{3a}} e^{n(n+1)a + \frac{2}{3}\frac{\gamma}{a} + \frac{2}{5}S_3\frac{1}{a^3} + \frac{2}{7}S_5\frac{1}{a^5} + \text{\&c.}}.$$

§ 19. Let

$$v = (1^2 - x^2)(2^2 - x^2)^2(3^2 - x^2)^3\ldots(n^2 - x^2)^n,$$

then

$$\log v = \log(1^2.2^4.3^6\ldots n^{2n})$$
$$- \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)x^2$$
$$- \tfrac{1}{2}S_3x^4 - \tfrac{1}{3}S_5x^6 - \text{\&c.};$$

whence

$$v = 1^2.2^4.3^6\ldots n^{2n}\, n^{-x^2} e^{-\gamma x^2 - \frac{1}{2}S_3x^4 - \frac{1}{3}S_5x^6 - \text{\&c.}}.$$

Now it was shown in Vol. VII. of the *Messenger* (p. 43) that

$$1^1.2^2.3^3\ldots n^n = An^{\frac{1}{2}n^2 + \frac{1}{2}n + \frac{1}{12}} e^{-\frac{1}{4}n^2},$$

where A is a constant whose value is there assigned.

Therefore

$$v = A^2 n^{n^2 + n + \frac{1}{6} - x^2} e^{-\frac{1}{2}n^2 - \gamma x^2 - \frac{1}{2}S_3x^4 - \frac{1}{3}S_5x^6 - \text{\&c.}},$$

and, putting $x = \dfrac{1}{a}$, we find

$$(a^2-1)^{2a}(2^2a^2-1)^{4a}(3^2a^2-1)^{6a}\ldots(n^2a^2-1)^{2na}$$
$$= A^{4a} a^{2(n^2+n)a} n^{2(n^2+n+\frac{1}{6})a - \frac{2}{a}} e^{-2a - \frac{2\gamma}{a} - S_3\frac{1}{a^3} - \frac{2}{3}S_5\frac{1}{a^5} - \text{\&c.}}.$$

§ 20. Let

$$w=\left(\frac{1+x}{1-x}\right)\left(\frac{2+x}{2-x}\right)\left(\frac{3+x}{3-x}\right)\cdots\left(\frac{n+x}{n-x}\right),$$

then $\tfrac{1}{2}\log w=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)x$

$$+\tfrac{1}{3}S_3x^3+\tfrac{1}{5}S_5x^5+\&c.,$$

so that $w=n^{2n}e^{2\gamma x+\frac{2}{3}S_3x^3+\frac{2}{5}S_5x^5+\&c.}$

and therefore $\dfrac{(a+1)(2a+1)(3a+1)\ldots(na+1)}{(a-1)(2a-1)(3a-1)\ldots(na-1)}$

$$=n^{\frac{2}{a}}e^{\frac{2\gamma}{a}+\frac{2}{3}S_3\frac{1}{a^3}+\frac{2}{5}S_5\frac{1}{a^5}+\&c.}.$$

§ 21. Multiplying together the results contained in the three preceding sections, we find

$$\frac{(a+1)^{(a+1)^2}(2a+1)^{(2a+1)^2}\ldots(na+1)^{(na+1)^2}}{(a-1)^{(a-1)^2}(2a-1)^{(2a-1)^2}\ldots(na-1)^{(na-1)^2}}$$

$$=A^{4a}(na)^{2(n^2+n)a}n^{\frac{a}{3}+\frac{2}{3a}}e^{na+\frac{2}{3}\frac{\gamma}{a}+4P},$$

where $P=\dfrac{S_3}{3.4.5}\dfrac{1}{a^3}+\dfrac{S_5}{5.6.7}\dfrac{1}{a^5}+\dfrac{S_7}{7.8.9}\dfrac{1}{a^7}+\&c.$

§ 22. The value of the constant A was expressed in Vol. VII. of the *Messenger*, (p. 46) in the form

$$A=2^{\frac{1}{36}}\pi^{\frac{1}{6}}e^{\frac{1}{3}(-\frac{1}{4}\gamma+\frac{1}{3}s_2-\frac{1}{4}s_3+\frac{1}{5}s_4-\&c.)},$$

where s_r has the same meaning as in § 6; and its numerical value was there found to be

$$A=1.28242\ 7130.$$

We can also express A in terms of S's (instead of s's) as follows.

§ 23. In the paper just quoted it was shown that

$$A=2^{\frac{7}{36}}\pi^{-\frac{1}{6}}e^{\frac{2}{3}\int_0^{\frac{1}{2}}\log\Gamma(1+x)\,dx}.$$

Now

$$\log \Gamma(1+x) = \tfrac{1}{2}\log \pi + \tfrac{1}{2}\log x - \tfrac{1}{2}\log \sin \pi x$$
$$- \gamma x - \tfrac{1}{3}S_3 x^3 - \tfrac{1}{5}S_5 x^5 - \&c.;$$

whence

$$\int_0^{\frac{1}{2}} \log \Gamma(1+x)\,dx = \tfrac{1}{4}\log \pi + \tfrac{1}{2}\Big[x\log x - x\Big]_0^{\frac{1}{2}} - \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi} \log \sin x\,dx$$
$$- \gamma\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} - \tfrac{1}{3}S_3\left[\frac{x^4}{4}\right]_0^{\frac{1}{2}} - \tfrac{1}{5}S_5\left[\frac{x^6}{6}\right]_0^{\frac{1}{2}} - \&c.$$

$$= \tfrac{1}{4}\log \pi + \tfrac{1}{4}\log \tfrac{1}{2} - \tfrac{1}{4} - \frac{1}{2\pi}\cdot \tfrac{1}{2}\pi \log \frac{1}{2} - \tfrac{1}{8}\gamma$$
$$- \frac{S_3}{3.4}\,\frac{1}{2^4} - \frac{S_5}{5.6}\,\frac{1}{2^6} - \frac{S_7}{7.8}\cdot\frac{1}{2^8} - \&c.$$

Thus

$$1 + 2\int_0^{\frac{1}{2}} \log \Gamma(1+x)\,dx = \tfrac{1}{2}\log \pi + \tfrac{1}{2} - \tfrac{1}{4}\gamma - \frac{S_3}{3.4}\,\frac{1}{2^3} - \frac{S_5}{5.6}\,\frac{1}{2^5} - \&c.,$$

and therefore

$$A = 2^{\frac{7}{36}} e^{\frac{1}{6} - \frac{1}{12}\gamma - \frac{1}{3}Q},$$

where $Q = \dfrac{S_3}{3.4}\,\dfrac{1}{2^3} + \dfrac{S_5}{5.6}\,\dfrac{1}{2^5} + \dfrac{S_7}{7.8}\,\dfrac{1}{2^7} + \&c.$

§ 24. Substituting this value of A in the formula of § 21 it becomes

$$\frac{(a+1)^{(a+1)^2}\,(2a+1)^{(2a+1)^2} \ldots (na+1)^{(na+1)^2}}{(a-1)^{(a-1)^2}\,(2a-1)^{(2a-1)^2} \ldots (na-1)^{(na-1)^2}}$$
$$= 2^{\frac{7}{9}a}\,(na)^{2(n^2+n)\frac{a}{n^3}+\frac{2}{3a}}\,e^{na + \frac{2}{3}a - \frac{1}{3}\gamma a + \frac{2}{3}\frac{\gamma}{a} - 4R},$$

where

$$R = \frac{a}{3}\left(\frac{S_3}{3.4}\,\frac{1}{2^3} + \frac{S_5}{5.6}\,\frac{1}{2^5} + \frac{S_7}{7.8}\,\frac{1}{2^7} + \&c.\right)$$
$$- \left(\frac{S_3}{3.4.5}\,\frac{1}{a^3} + \frac{S_5}{5.6.7}\,\frac{1}{a^5} + \frac{S_7}{7.8.9}\,\frac{1}{a^7} + \&c.\right).$$

§ 25. By putting $a = 2$ in the formula of § 21, it becomes

$$\frac{3^9 . 5^{25} \dots (2n+1)^{4n^2+4n+1}}{1^1 . 3^9 \dots (2n-1)^{4n^2-4n+1}} = A^8 (2n)^{4n^2+4n} n e^{2n+\frac{1}{3}\gamma+4P},$$

where

$$P = \frac{S_3}{3.4.5} \frac{1}{2^3} + \frac{S_5}{5.6.7} \frac{1}{2^5} + \frac{S_7}{7.8.9} \frac{1}{2^7} + \&c.$$

The left-hand side

$$= (2n+1)^{4n^2+4n+1} = (2n)^{4n^2+4n+1} \left(1 + \frac{1}{2n}\right)^{4n^2+4n+1},$$

and, denoting the second factor by u,

$$\log u = (4n^2 + 4n + 1) \log \left(1 + \frac{1}{2n}\right)$$

$$= (4n^2 + 4n + 1) \left(\frac{1}{2n} - \frac{1}{2(2n)^2} + \&c.\right)$$

$$= 2n + 2 - \tfrac{1}{2} = 2n + \tfrac{3}{2}.$$

Thus, the left-hand side

$$= (2n)^{4n^2+4n+1} e^{2n+\frac{3}{2}}.$$

The right-hand side

$$= \tfrac{1}{2} A^8 (2n)^{4n^2+4n+1} e^{2n+\frac{1}{3}\gamma+4P},$$

so that the equation becomes

$$\tfrac{3}{2} = -\log 2 + 8 \log A + \tfrac{1}{3}\gamma + 4P,$$

that is

$$4P = \tfrac{3}{2} + \log 2 - 8 \log A - \tfrac{1}{3}\gamma.$$

§ 26. It is easy to deduce from *Messenger*, VII., p. 46, that

$$\log A = 0{\cdot}24875\ 44770\ 3,$$

the last figure being uncertain.

We also have

$$\log 2 = 0{\cdot}69314\ 71805\ 6, \quad \gamma = 0{\cdot}57721\ 56649\ 0,$$

and the right-hand side is thus found to be

$$= 0{\cdot}01070\ 61426\ 9.$$

§ 27. In order to calculate the value of the series $4P$, viz.

$$\frac{S_3}{3.4.5}\frac{1}{2}+\frac{S_5}{5.6.7}\frac{1}{2^3}+\frac{S_7}{7.8.9}\frac{1}{2^5}+\&c.,$$

it is convenient to consider the more general series

$$\frac{S_3}{3.4.5}\frac{1}{a^3}+\frac{S_5}{5.6.7}\frac{1}{a^5}+\frac{S_7}{7.8.9}\frac{1}{a^7}+\&c.$$

The value of this series can obviously be obtained much more rapidly by transforming it into one in which the terms depend upon S_3', S_5', &c., where

$$S_3'=S_3-1,\ S_5'=S_5-1,\ \&c.$$

This transformation may be readily effected for

$$\frac{1}{3.4.5}\frac{1}{a^3}+\frac{1}{5.6.7}\frac{1}{a^5}+\frac{1}{7.8.9}\frac{1}{a^7}+\&c.$$

$$=\frac{1}{2}\left[\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right)\frac{1}{a^3}+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{3}\right)\frac{1}{a^5}+\&c.\right]$$

$$=\frac{1}{2}\left[\tfrac{1}{2}\log\frac{1+a^{-1}}{1-a^{-1}}-\frac{1}{a}\right.$$

$$+\tfrac{1}{2}a^2\log\frac{1+a^{-1}}{1-a^{-1}}-a-\frac{1}{3a}$$

$$\left.+a\log\left(1-\frac{1}{a^2}\right)+\frac{1}{a}\right]$$

$$=\frac{1+a^2}{4}\log\frac{a+1}{a-1}+\frac{a}{2}\log(a^2-1)-a\log a-\frac{a}{2}-\frac{1}{6a}.$$

Thus the series in question

$$=\left(\frac{a+1}{2}\right)^2\log(a+1)-\left(\frac{a-1}{2}\right)^2\log(a-1)-a\log a-\frac{a}{2}-\frac{1}{6a}$$

$$+\frac{S_3'}{3.4.5}\frac{1}{a^3}+\frac{S_5'}{5.6.7}\frac{1}{a^5}+\frac{S_7'}{7.8.9}\frac{1}{a^7}+\&c.$$

§ 28. The series $4P$ of § 28 is therefore equal to

$$9\log 3-8\log 2-\tfrac{13}{3}$$

$$+\frac{S_3'}{3.4.5}\frac{1}{2}+\frac{S_5'}{5.6.7}\frac{1}{2^3}+\frac{S_7}{7.8.9}\frac{1}{2^5}+\&c.$$

The first line is found to be

$$=0{\cdot}00899\ 98202\ 0,$$

and the terms of the series in the second line are respectively

$$\begin{array}{r} 0{\cdot}00168\ 38075\ 3, \\ 2\ 19808\ 1, \\ 5176\ 9, \\ 158\ 5, \\ 5\ 6, \\ 2, \end{array}$$

giving as the value of $4P$

$$0{\cdot}01070\ 61426\ 6,$$

which agrees, except in the last figure, with the value obtained in § 26.

§ 29. Putting $a=1$ in the formula of § 21 (and noticing that the limit of x^{x^2} when x is zero is unity) the left-hand member

$$= \frac{2^4.3^9\ldots(n+1)^{n^2+2n+1}}{1^1.2^4\ldots(n-1)^{n^2-2n+1}} = n^{n^2}(n+1)^{n^2+2n+1}$$

$$= n^{n^2} n^{n^2+2n+1}\left(1+\frac{1}{n}\right)^{n^2+2n+1}$$

$$= n^{2n^2+2n+1} e^{n+\frac{3}{2}}$$

The right-hand member

$$= A^4 n^{2n^2+2n} n e^{n+\frac{2}{3}\gamma+4P},$$

where $$P = \frac{S_3}{3.4.5} + \frac{S_5}{5.6.7} + \frac{S_7}{7.8.9} + \&c.$$

Thus the equation becomes

$$\tfrac{3}{2} = 4 \log A + \tfrac{2}{3}\gamma + 4P,$$

that is $$P = \tfrac{3}{8} - \log A - \tfrac{1}{6}\gamma.$$

§ 30. The right-hand side of the equation is easily found to be

$$=0{\cdot}03004\ 29121\ 5.$$

Expressing, as in § 27, the series

$$\frac{S_3}{3.4.5} + \frac{S_5}{5.6.7} + \frac{S_7}{7.8.9} + \&c.$$

as the sum of the two series

$$\frac{1}{3.4.5} + \frac{1}{5.6.7} + \frac{1}{7.8.9} + \&c.,$$

and
$$\frac{S_3'}{3.4.5} + \frac{S_5'}{5.6.7} + \frac{S_7'}{7.8.9} + \&c.,$$

we notice that the first series

$$= \frac{1}{2}\left[2\left\{1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right\} - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n} - 1 - \frac{1}{3} + \frac{1}{2n+1}\right],$$

n being infinite,

$$= \frac{1}{2}\left(\gamma + 2\log 2 + \log n - \gamma - \log n - 1 - \frac{1}{3} + \frac{1}{2n+1}\right)$$

$$= \log 2 - \tfrac{2}{3}$$

$$= 0{\cdot}02648\ 05138\ 9.$$

The terms in the second series are

0·00336 76150 5,
17 58464 5,
1 65660 3,
20286 8,
2879 9,
449 5,
75 0,
13 1,
2 4,
4,

giving as the value of P,

0·03004 29121 3,

which agrees with the value found for the right-hand side of the equation, except in the last figure.

§ 31. By putting $a=4$ in the formula of §§ 21 and 24, we find

$$\frac{1^1.5^{25}.9^{81}\ldots(4n+1)^{(4n+1)^2}}{3^9.7^{49}.11^{121}\ldots(4n-1)^{(4n-1)^2}}$$

$$=A^{16}(4n)^{8n^2+8n}n^{\frac{3}{2}}e^{4n+\frac{1}{6}\gamma+4P},$$

where $$P=\frac{S_3}{3.4.5}\frac{1}{4^3}+\frac{S_5}{5.6.7}\frac{1}{4^5}+\frac{S_7}{7.8.9}\frac{1}{4^7}+\&c.,$$

or, substituting for A its value (§ 23)

$$=2^{\frac{28}{9}}(4n)^{8n^2+8n}n^{\frac{3}{2}}e^{4n+\frac{8}{3}-\frac{7}{6}\gamma-4R},$$

where $$R=\frac{1}{3}\left(\frac{S_3}{3.4}\frac{1}{2}+\frac{S_5}{5.6}\frac{1}{2^3}+\frac{S_7}{7.8}\frac{1}{2^5}+\&c.\right)$$
$$-\left(\frac{S_3}{3.4.5}\frac{1}{4^3}+\frac{S_5}{5.6.7}\frac{1}{4^5}+\frac{S_7}{7.8.9}\frac{1}{4^7}+\&c.\right).$$

§ 32. On p. 191 of Vol. VI. of the *Messenger* it was shown that, n being even,

$$\frac{3^9.7^{49}.11^{121}\ldots(2n-1)^{(2n-1)^2}}{1^1.5^{25}.9^{81}\ldots(2n-3)^{(2n-3)^2}}=(2n)^{2n^2-\frac{1}{2}}2^{-\frac{1}{3}}e^{-\frac{1}{4}-T},$$

where $$T=\frac{s_3}{3.5}\frac{1}{2^2}+\frac{s_5}{5.7}\frac{1}{2^4}+\frac{s_7}{7.9}\frac{1}{2^6}+\&c.,$$

s_r denoting, as before, the series

$$1-\frac{1}{2^r}+\frac{1}{3^r}-\frac{1}{5^r}+\&c.$$

§ 33. Writing $2n$ for n in this result, inverting the quotient and multiplying by

$$(4n+1)^{(4n+1)^2}=(4n)^{(4n+1)^2}\left(1+\frac{1}{4n}\right)^{16n^2+8n+1}$$

$$=(4n)^{(4n+1)^2}e^{4n+\frac{3}{2}},$$

we find

$$\frac{1^1.5^{25}.9^{81}\ldots(4n+1)^{(4n+1)^2}}{3^9.7^{69}.11^{121}\ldots(4n-1)^{(4n-1)^2}}=2^{\frac{1}{3}}(4n)^{8n^2+8n+\frac{3}{2}}e^{4n+\frac{7}{4}+T}.*$$

* The formula given in the bottom line but two of p. 191 of Vol. VI. is correct, but the expression derived from it, on the bottom line of the page, is erroneous, the factor $e^{-\frac{1}{4}}$ being omittted.

§ 34. The value found in § 31 may be written

$$2^{\frac{1}{9}}(4n)^{8n^2+8n+\frac{3}{2}}\,e^{4n+\frac{8}{3}-\frac{7}{6}\gamma-4R},$$

so that, by comparing the two values, we have

$$-\tfrac{2}{9}\log 2+\tfrac{7}{4}+T=\tfrac{8}{3}-\tfrac{7}{6}\gamma-4R,$$

that is, $$4R+T=\tfrac{11}{12}-\tfrac{7}{6}\gamma-\tfrac{2}{9}\log 2.$$

This result I have verified to five places of decimals, each side of the equation being $=0{\cdot}08921\ldots.$

§ 35. By putting $a=3$ in §§ 21 and 24, we find

$$\frac{1^1.4^{16}.7^{49}\ldots(3a+1)^{(3a+1)^2}}{2^4.5^{25}.8^{64}\ldots(3a-1)^{(3a-1)^2}}=A^{12}(3a)^{6(n^2+n)}\,n^{\frac{11}{9}}\,e^{3n+\frac{2}{3}\gamma+4P},$$

where $$P=\frac{S_3}{3.4.5}\,\frac{1}{3^3}+\frac{S_5}{5.6.7}\,\frac{1}{3^5}+\frac{S_7}{7.8.9}\,\frac{1}{3^7}+\&c.;$$

or $$=2^{\frac{7}{3}}(3a)^{6(n^2+n)}\,n^{\frac{11}{9}}\,e^{3n+2-\frac{7}{9}\gamma-4R},$$

where $$R=\frac{S_3}{3.4}\,\frac{1}{2^3}+\frac{S_5}{5.6}\,\frac{1}{2^5}+\frac{S_7}{7.8}\,\frac{1}{2^7}+\&c.$$

$$-\frac{S_3}{3.4.5}\,\frac{1}{3^3}-\frac{S_5}{5.6.7}\,\frac{1}{3^5}-\frac{S_7}{7.8.9}\,\frac{1}{3^7}-\&c.$$

§ 36. Considering now the general product in which uneven numbers only occur, and proceeding as in § 18, we find that

$$\left(\frac{a+1}{a-1}\right)^{a^2}\left(\frac{3a+1}{3a-1}\right)^{7a^2}\cdots\left\{\frac{(2n-1)\,a+1}{(2n-1)\,a+1}\right\}^{(2n-1)^2a^2}$$

$$=e^{2an^2+\frac{2}{3}U_1\frac{1}{a}+\frac{2}{5}U_3\frac{1}{a^3}+\frac{2}{7}U_5\frac{1}{a^5}+\&c.},$$

where, if $r>1$,

$$U_r=1+\frac{1}{3^r}+\frac{1}{5^2}+\frac{1}{7^r}+\&c.,$$

and $$U_1=1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}.$$

Thus $$U_1=\tfrac{1}{2}\gamma+\log 2+\tfrac{1}{2}\log n;$$

and therefore $$e^{U_1}=2n^{\frac{1}{2}}e^{\frac{1}{2}\gamma}.$$

§ 37. As in § 19, we can show that

$$(1^2 - x^2)^1 (3^2 - x^2)^9 \dots \{(2n-1)^2 - x^2\}^{(2n-1)^2}$$
$$= 1^2 . 3^6 . 5^{10} \dots (2n-1)^{2(2n-1)^2} e^{-N},$$

where $\qquad N = U_1 x^2 + \tfrac{1}{2} U_3 x^4 + \tfrac{1}{3} U_5 x^6 + \&c.$

Now, putting $2n$ for n in the formula quoted in § 19,

$$1^1 . 2^2 . 3^3 \dots (2n)^{2n} = A\,(2n)^{2n^2+n+\frac{1}{12}} e^{-n^2};$$

and, multiplying the same formula by $2^{2+4+\dots+2n}$,

$$2^2 . 4^2 . 6^2 \dots (2n)^{2n} = A^2\, 2^{n^2+n}\, n^{n^2+n+\frac{1}{6}} e^{-\frac{1}{2}n^2};$$

whence, by division,

$$1^1 . 3^3 \dots (2n-1)^{m-1} (2n-1)^{2n-1} = A^{-1}\, 2^{n^2+\frac{1}{12}}\, n^{n^2-\frac{1}{12}} e^{-\frac{1}{2}n^2}.$$

Substituting this value, we find that

$$(a^2-1)^{2a} (3^2a^2-1)^{6a} \dots \{(2n-1)^2 a^2 - 1\}^{2(2n-1)a}$$
$$= A^{-4a} (2an)^{4an^2}\, 2^{\frac{1}{3}a}\, n^{-\frac{1}{3}a}\, e^{-2an^2 - 2U_1 \frac{1}{a} - U_3 \frac{1}{a^3} - \frac{2}{3} U_5 \frac{1}{a_5} - \&c.}$$

§ 38. We also have

$$\frac{(a+1)(3a+1)\dots\{(2n-1)a+1\}}{(a-1)(3a-1)\dots\{(2n-1)a-1\}} = e^{2U_1 \frac{1}{a} + \frac{2}{3} U_3 \frac{1}{a^3} + \frac{2}{5} U_5 \frac{1}{a^5} + \&c.}.$$

§ 39. Multiplying these results, we have finally

$$\frac{(a+1)^{(a+1)^2} (3a+1)^{(3a+1)^2} \dots \{(2n-1)a+1\}^{\{(2n-1)a+1\}^2}}{(a-1)^{(a-1)^2} (3a-1)^{(3a-1)^2} \dots \{(2n-1)a-1\}^{\{(2n-1)a-1\}^2}}$$
$$= A^{-4a} (2an)^{4an^2}\, 2^{\frac{1}{3}a}\, n^{-\frac{1}{3}a}\, e^{\frac{2}{3} U_1 \frac{1}{a} + 4V},$$

where $\qquad V = \dfrac{U_3}{3.4.5} \dfrac{1}{a^3} + \dfrac{U_5}{5.6.7} \dfrac{1}{a^5} + \dfrac{U_7}{7.8.9} \dfrac{1}{a^7} + \&c.;$

or, substituting for U_1 its value, the quotient

$$= A^{-4a}\, 2^{\frac{a}{3} + \frac{2}{3a}} (2an)^{4an^2}\, n^{-\frac{a}{3} + \frac{1}{3a}}\, e^{\frac{1}{3}\frac{\gamma}{a} + 4V}.$$

§ 40. Putting $a=2$,

$$\frac{3^9.7^{49}\ldots(4n-1)^{(4n-1)^2}}{1^1.5^{25}\ldots(4n-3)^{(4n-3)^2}}=A^{-8}\,2\,(4n)^{8n^2}\,n^{-\frac{1}{2}}\,e^{\frac{1}{6}\gamma+4V},$$

where $$V=\frac{U_3}{3.4.5}\,\frac{1}{2^3}+\frac{U_5}{5.6.7}\,\frac{1}{2^5}+\frac{U_7}{7.8.9}\,\frac{1}{2^7}+\&c.$$

§ 41. Inverting the quotient and multiplying by

$$(4n+1)^{(4n+1)^2},$$

as in § 33, the formula becomes

$$\frac{1^1.5^{25}\ldots(4n+1)^{(4n+1)^2}}{3^9.7^{49}\ldots(4n-1)^{(4n-1)^2}}=2A^8\,(4n)^{8n^2+8n}\,n^{\frac{3}{2}}\,e^{4n+\frac{3}{2}-\frac{1}{6}\gamma-4V}.$$

§ 42. Comparing this result with that found in § 31, we have

$$\log 2+8\log A+\tfrac{3}{2}-\tfrac{1}{6}\gamma-4V=16\log A+\tfrac{1}{6}\gamma+4P,$$

that is, $$4P+4V=\tfrac{3}{2}+\log 2-8\log A-\tfrac{1}{3}\gamma.$$

Now $$P=\frac{S_3}{3.4.5}\,\frac{1}{4^3}+\frac{S_5}{5.6.7}\,\frac{1}{4^5}+\frac{S_7}{7.8.9}\,\frac{1}{4^7}+\&c.;$$

and $$V=\frac{U_3}{3.4.5}\,\frac{1}{2^3}+\frac{U_5}{5.6.7}\,\frac{1}{2^5}+\frac{U_7}{7.8.9}\,\frac{1}{2^7}+\&c.$$

Also $$U_r=\left(1-\frac{1}{2^r}\right)S_r,$$

so that $$P+V=\frac{S_3}{3.4.5}\,\frac{1}{2^3}+\frac{S_5}{5.6.7}\,\frac{1}{2^5}+\frac{S_7}{7.8.9}\,\frac{1}{2^7}+\&c.$$

Thus $P+V$ is the same as the P of § 25, and the above equation is the same as the relation found in that section, and verified in the three following sections.

§ 43. As another verification let $a=1$ in § 39. The formula then gives

$$(2n)^{4n^2}=A^{-4}\,2\,(2n)^{4n^2}\,e^{\frac{1}{3}\gamma+4V},$$

where $$V=\frac{U_3}{3.4.5}+\frac{U_5}{5.6.7}+\frac{U_7}{7.8.9}+\&c.,$$

that is, $$4V=4\log A-\log 2-\tfrac{1}{3}\gamma.$$

Now $$V=\frac{S_3}{3.4.5}+\frac{S_5}{5.6.7}+\frac{S_7}{7.8.9}+\&c.$$

$$-\frac{S_3}{3.4.5}\frac{1}{2^3}-\frac{S_5}{5.6.7}\frac{1}{2^5}-\frac{S_7}{7.8.9}\frac{1}{2^7}-\&c.;$$

the former series

$$=-4\log A+\tfrac{3}{2}-\tfrac{2}{3}\gamma \quad (\S\,29),$$

and the latter

$$=-8\log A+\tfrac{3}{2}+\log 2-\tfrac{1}{3}\gamma \quad (\S\,25),$$

so that the equation is verified.

Calculation of $\log A$, §§ 44—4 .

§44. The constant A was calculated in the *Messenger*, Vol. VII. p. 46, by means of the series

$$3\log A=\tfrac{1}{12}\log 2+\tfrac{1}{2}\log\pi-\tfrac{1}{4}\gamma+\tfrac{1}{3}s_2-\tfrac{1}{4}s_3+\tfrac{1}{5}s_4-\&c.$$

It may, however, be calculated more readily by means of the formulæ obtained in §§ 25 and 29.

§45. Thus, from § 25 we have

$$8\log A=\tfrac{3}{2}+\log 2-\tfrac{1}{3}\gamma-4P,$$

where $$P=\frac{S_3}{3.4.5}\frac{1}{2^3}+\frac{S_5}{5.6.7}\frac{1}{2^5}+\frac{S_7}{7.8.9}\frac{1}{2^7}+\&c.,$$

and, taking the expression for $4P$ given in § 28, we find

$$8\log A=\tfrac{35}{6}-9\log\tfrac{3}{2}-\tfrac{1}{3}\gamma$$

$$-\frac{S_3'}{3.4.5}\frac{1}{2}-\frac{S_5'}{5.6.7}\frac{1}{2^3}-\frac{S_7'}{7.8.9}\frac{1}{2^5}-\&c.,$$

from which $\log A$ may be readily calculated to as many figures as the values of S_3', S_5',... permit.

§46. Similarly from § 29

$$4\log A=\tfrac{3}{2}-\tfrac{2}{3}\gamma-4P,$$

where $$P=\frac{S_3}{3.4.5}+\frac{S_5}{5.6.7}+\frac{S_7}{7.8.9}+\&c.$$

In § 30 it was shown that

$$P=\log 2-\tfrac{2}{3}+\frac{S_3'}{3.4.5}+\frac{S_5'}{5.6.7}+\&c.,$$

so that

$$4\log A = \tfrac{25}{6} - \tfrac{2}{3}\gamma - 4\log 2$$
$$- 4\left(\frac{S_3'}{3.4.5} + \frac{S_5'}{5.6.7} + \frac{S_7'}{7.8.9} + \&c.\right),$$

but this formula does not converge so rapidly as that given in the preceding section.

§47. We can obtain another equation for $\log A$ by comparing the results in §§ 31 and 32, but the formula so obtained requires the calculation of two series.

§48. When I wrote the paper upon $1^1.2^2.3^3 \ldots. n^n$ in Vol. VI. of the *Messenger* (1877) I was not aware that this product had been considered before. I have recently found, however, that on page 97 of Vol. V. of the *Quarterly Journal** (1862) the late Mr. H. M. Jeffery gave the formula

$$1^1.2^2.3^3 \ldots. x^x = e^{C - \frac{1}{4}x^2} x^{\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{12}},$$

and determined C as 0·24875. This C is the same as $\log A$ of this paper (§26). Apparently Jeffery did not seek to obtain converging series for the calculation of C.

Values of some other products, §§ 49—55.

§49. By integrating between the limits x and 0 the equation

$$\log(1+x) + \log(2+x) + \ldots + \log(n+x)$$
$$= \log(1.2.3\ldots n) + S_1 x - \tfrac{1}{2} S_2 x^2 + \tfrac{1}{3} S_3 x^3 - \&c.,$$

where $$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

it is easy to show that

$$\frac{(1+x)^{1+x}(2+x)^{2+x}\ldots(n+x)^{n+x}}{1^1.2^2.3^3\ldots n^n} = 2^{\frac{1}{2}x}\pi^{\frac{1}{2}x}n^{(n+\frac{1}{2})x}e^{R},$$

where $$R = \frac{S_1}{1.2}x^2 - \frac{S_2}{2.3}x^3 + \frac{S_3}{3.4}x^4 - \&c.$$

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* 'On the expansion of powers of the trigonometrical ratios in terms of series of ascending powers of the variables,' pp. 91—108.

Putting $x=\frac{1}{a}$, and writing for S_1 its value $\gamma+\log n$, the equation becomes, when raised to the a^{th} power,

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1}\dots(na+1)^{na+1}}{1^a.2^{2a}.3^{3a}\dots n^{na}}$$

$$=2^{\frac{1}{2}}\pi^{\frac{1}{2}}a^{\frac{1}{2}(n^2+n)a+n}n^{n+\frac{1}{2}+\frac{1}{2a}}e^{\frac{1}{2}\frac{\gamma}{a}-T},$$

where $$T=\frac{S_2}{2.3}\frac{1}{a^2}-\frac{S_3}{3.4}\frac{1}{a^3}+\frac{S_4}{4.5}\frac{1}{a^4}-\&c.,$$

whence $$(a+1)^{a+1}(2a+1)^{2a+1}\dots(na+1)^{na+1}$$

$$=A^a 2^{\frac{1}{2}}\pi^{\frac{1}{2}}a^{\frac{1}{2}(n^2+n)a+n}n^{\frac{1}{2}an^2+\frac{1}{2}an+n+\frac{a}{12}+\frac{1}{2}+\frac{1}{2a}}e^{-\frac{1}{4}an^2+\frac{1}{2}\frac{\gamma}{a}-T}.$$

§ 50. By putting $a=1$ in this result we do not obtain a series for $\log A$, for the left-hand member

$$=2^2.3^3\dots(n+1)^{n+1}=A.n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}e^{-\frac{1}{4}n^2}n^{n+1}e,$$

and, equating this value to the right-hand member, we find

$$\frac{S_2}{2.3}-\frac{S_3}{3.4}+\frac{S_5}{5.6}-\&c.=\tfrac{1}{2}(\log 2+\log\pi+\gamma)-1.$$

This equation can be easily verified; for, by integrating

$$\log\Gamma(1+x)=-\gamma x+\frac{S_2}{2}x^2-\frac{S_3}{3}x^3+\frac{S_4}{4}x^4-\&c.,$$

we find

$$\int_0^1\log\Gamma(1+x)\,dx=-\tfrac{1}{2}\gamma+\frac{S_2}{2.3}-\frac{S_3}{3.4}+\frac{S_4}{4.5}-\&c.;$$

and

$$\int_0^1\log\Gamma(1+x)\,dx=\int_0^1\log x\,dx+\int_0^1\log\Gamma(x)\,dx$$

the former integral being zero and the latter $=\frac{1}{2}\log(2\pi)$.

§ 51. If we put $a=2$, the right-hand member of the equation becomes

$$A^2\pi^{\frac{1}{2}}2^{n^2+2n+\frac{1}{2}}n^{n^2+2n+\frac{11}{12}}e^{-\frac{1}{2}n^2+\frac{1}{4}\gamma-T},$$

where $$T = \frac{S_2}{2.3}\frac{1}{2^2} - \frac{S_3}{3.4}\frac{1}{2^3} + \frac{S_4}{4.5}\frac{1}{2^4} - \&c.;$$

and the left-hand member

$$= 3^3.5^5\ldots(2n-1)^{2n-1}(2n)^{2n+1}e$$

$$= A^{-1}2^{n^2+2n+1\frac{3}{12}}n^{n^2+2n+1\frac{1}{12}}e^{-\frac{1}{2}n^2+1} \quad (\S\,37).$$

Equating these values, we find

$$3\log A = 1 + \tfrac{7}{12}\log 2 - \tfrac{1}{2}\log\pi - \tfrac{1}{4}\gamma$$
$$+ \frac{S_2}{2.3}\frac{1}{2^2} - \frac{S_3}{3.4}\frac{1}{2^3} + \frac{S_4}{4.5}\frac{1}{2^4} - \&c.$$

§ 52. Now, it was found in § 23 that

$$3\log A = \tfrac{1}{2} + \tfrac{7}{12}\log 2 - \tfrac{1}{4}\gamma - \frac{S_3}{3.4}\frac{1}{2^3} - \frac{S_5}{5.6}\frac{1}{2^5} - \frac{S_7}{7.8}\frac{1}{2^7} - \&c.;$$

whence, by equating the values of $3\log A$,

$$\frac{S_2}{2.3}\frac{1}{2^2} + \frac{S_4}{4.5}\frac{1}{2^4} + \frac{S_6}{6.7}\frac{1}{2^6} + \&c. = \tfrac{1}{2}\log\pi - \tfrac{1}{2}.$$

§ 53. To verify this equation, we notice that

$$\tfrac{1}{2}\log\frac{\pi x}{\sin\pi x} = \frac{S_2}{2}x^2 + \frac{S_4}{4}x^4 + \frac{S_6}{6}x^6 + \&c.;$$

whence, by integration,

$$\frac{S_2}{2.3}\frac{1}{2^2} + \frac{S_4}{4.5}\frac{1}{2^4} + \&c. = \int_0^{\frac{1}{2}}(\log\pi + \log x - \log\sin\pi x)\,dx$$

$$= \tfrac{1}{2}\log\pi + \tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2} - \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\log\sin x\,dx,$$

$$= \tfrac{1}{2}\log\pi + \tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = \tfrac{1}{2}\log\pi - \tfrac{1}{2}.$$

§ 54. Putting $-x$ for x in § 49 and multiplying the results, we have

$$\frac{(1-x)^{1-x}(1+x)^{1+x}\ldots(n-x)^{n-x}(n+x)^{n+x}}{1^2.2^4.3^6\ldots n^{2n}}$$
$$= e^{2\left(\frac{S_1}{1.2}x^2 + \frac{S_2}{3.4}x^4 + \frac{S_3}{5.6}x^6 + \&c.\right)};$$

whence we deduce that

$$(a-1)^{a-1}(a+1)^{a+1}(2a-1)^{2a-1}$$
$$\times(2a+1)^{2a+1}\dots(na-1)^{na-1}(na+1)^{na+1}$$
$$=A^{2a}(na)^{(n^2+n)a}\,n^{\frac{1}{6}a+\frac{1}{a}}\,e^{-\frac{1}{2}an^2+\frac{\gamma}{a}+2W},$$

where $$W=\frac{S_3}{3.4}\,\frac{1}{a^3}+\frac{S_5}{5.6}\,\frac{1}{a^5}+\frac{S_7}{7.8}\,\frac{1}{a^7}+\&c.$$

§ 55. Putting $a=2$, the left-hand member

$$=3^6.5^{10}\dots(2n-1)^{2(2n-1)}(2n+1)^{2n+1}$$
$$=A^{-2}\,2^{2n^2+\frac{1}{6}}\,n^{2n^2-\frac{1}{6}}\,e^{-n^2}(2n)^{2n+1}e$$
$$=A^{-2}\,2^{2n^2+2n+\frac{7}{6}}\,n^{2n^2+2n+\frac{5}{6}}\,e^{-n^2+1},$$

and the right-hand member

$$=A^4\,2^{2n^2+2n}\,n^{2n^2+2n+\frac{5}{6}}\,e^{-n^2+\frac{1}{2}\gamma+2W},$$

where $$W=\frac{S_3}{3.4}\,\frac{1}{2^3}+\frac{S_5}{5.6}\,\frac{1}{2^5}+\frac{S_7}{7.8}\,\frac{1}{2^7}+\&c.$$

Equating these expressions,

$$6\log A=1+\tfrac{7}{6}\log 2-\tfrac{1}{2}\gamma-2W,$$

which is the same relation as that found in § 23, W being the series which was there denoted by Q.

Products in which the exponent is the reciprocal of the number, §§ 56–69.

§ 56. I cannot close this paper without drawing attention to the very elegant expressions that Prof. Rogel has obtained for the products corresponding to those in § 1, but in which the exponent is the reciprocal of the number.*

* *Educational Series Reprint*, Vol. LX. (1894), p. 66. Question 11968.

Starting with the formula

$$\frac{\log 2}{2}\sin 4\mu\pi + \frac{\log 3}{3}\sin 6\mu\pi + \frac{\log 4}{4}\sin 8\mu\pi + \&c.$$

$$= \pi\,\{\log\Gamma(\mu) + \tfrac{1}{2}\log\sin\mu\pi - (1-\mu)\log\pi$$

$$- (\tfrac{1}{2}-\mu)\,\gamma - (\tfrac{1}{2}-\mu)\log 2\},$$

and, taking the exponential of both sides, he obtains the general result,

$$2^{\frac{\sin 4\mu\pi}{2}}\,.\,3^{\frac{\sin 6\mu\pi}{3}}\,.\,4^{\frac{\sin 8\mu\pi}{4}}\ldots = \left\{\frac{(\sin\mu\pi)^{\frac{1}{2}}\,\Gamma(\mu)}{2^{\frac{1}{2}-\mu}\pi^{1-\mu}e^{(\frac{1}{2}-\mu)\gamma}}\right\}^{\pi}.$$

§ 57. Putting $\mu = \frac{1}{4}$, and replacing $\Gamma(\frac{1}{4})$ by its value $2\pi^{\frac{1}{4}}K^{0\frac{1}{2}}$, K^0 being the complete elliptic integral of the first kind corresponding to the modulus $\frac{1}{\sqrt{2}}$, the formula gives

$$\frac{5^{\frac{1}{5}}.9^{\frac{1}{9}}.13^{\frac{1}{13}}\ldots(4n+1)^{\frac{1}{4n+1}}}{3^{\frac{1}{3}}.7^{\frac{1}{7}}.11^{\frac{1}{11}}\ldots(4n-1)^{\frac{1}{4n-1}}} = \left(\frac{2K^0}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{1}{2}\pi}.$$

This result is very curious, involving as it does all the four constants e, π, γ, K^0.

§ 58. Similarly by putting $\mu = \frac{1}{3}$ the corresponding ratio involving the subeven and supereven numbers to modulus 3 is obtained, viz.,

$$\frac{4^{\frac{1}{4}}.7^{\frac{1}{7}}.10^{\frac{1}{10}}\ldots(3n+1)^{\frac{1}{3n+1}}}{2^{\frac{1}{2}}.5^{\frac{1}{5}}.8^{\frac{1}{8}}\ldots(3n-1)^{\frac{1}{3n-1}}} = \left(\frac{3^{\frac{1}{4}}\Gamma(\frac{1}{3})}{2^{\frac{2}{3}}\pi^{\frac{2}{3}}e^{\frac{1}{6}\gamma}}\right)^{\frac{2\pi}{\sqrt{3}}}.$$

This result may, as pointed out by Prof. Rogel, be expressed in terms of the complete elliptic integral to modulus sin 15° by means of the relation

$$\Gamma^3(\tfrac{1}{3}) = 2^{\frac{1}{3}}3^{-\frac{1}{4}}\pi K_1,$$

K_1 being the value of K when $k = \sin 15°$.

The value of the product so expressed is found to be

$$\left(\frac{2^{\frac{1}{3}}3^{\frac{1}{2}}K_1}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{2\pi}{3\sqrt{3}}}.$$

§ 59. For the modulus $k = \sin 15^\circ$ we know that

$$\frac{K'}{K} = \sqrt{3};$$

if therefore we denote by K_2 the value of K for the modulus $k = \sin 75^\circ$, we have $K_2 = 3^{\frac{1}{2}}K_1$.

The value of the product in the last section may therefore be written also

$$\left(\frac{2^{\frac{1}{3}}K_2}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{2\pi}{3\sqrt{3}}}.$$

§ 60. The series from which the products are derived involve the subeven and supereven numbers to the moduli 4 and 3, with contrary signs. They seem therefore deserving of notice on their own account. Taking the results in §§ 57 and 58 the series-formulæ may be written

$$\frac{\log 3}{3} - \frac{\log 5}{5} + \frac{\log 7}{7} - \frac{\log 9}{9} + \&c. = \tfrac{1}{2}\pi\left(\tfrac{1}{2}\gamma - \log\frac{2K^0}{\pi}\right);$$

and

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 5}{5} - \frac{\log 7}{7} + \&c. = \frac{2\pi}{3\sqrt{3}}\left(\tfrac{1}{2}\gamma - \log\frac{2^{\frac{1}{3}}K_2}{\pi}\right).$$

§ 61. By putting $\mu = \frac{1}{6}$ in § 56, we find

$$\frac{\log 2}{2} - \frac{\log 4}{4} - \frac{\log 5}{5} + \frac{\log 7}{7} + \frac{\log 8}{8} - \&c.$$

$$= \frac{2\pi}{\sqrt{3}}\log\left\{\frac{3^{\frac{1}{4}}\,\Gamma(\frac{1}{6})}{2^{\frac{5}{6}}\pi^{\frac{5}{6}}e^{\frac{1}{3}\gamma}}\right\}.$$

The terms in the series have the positive sign when the number $\equiv 1$ and 2, mod. 6, and the negative sign if it $\equiv -1$ and -2, mod. 6.

§ 62. The formula of § 58 may be written

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 8}{8} - \&c.$$

$$= -\frac{2\pi}{\sqrt{3}} \log \left\{ \frac{3^{\frac{1}{4}} \Gamma\left(\frac{1}{3}\right)}{2^{\frac{2}{3}} \pi^{\frac{2}{3}} e^{\frac{1}{6}\gamma}} \right\}.$$

Thus, by adding and subtracting, we find

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 8}{8} - \frac{\log 10}{10} + \&c.$$

$$= -\frac{\pi}{\sqrt{3}} \log \left\{ 2^{\frac{1}{6}} \pi^{\frac{1}{6}} e^{\frac{1}{6}\gamma} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{6}\right)} \right\},$$

$$\frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 11}{11} - \frac{\log 13}{13} + \&c.$$

$$= -\frac{\pi}{\sqrt{3}} \log \left\{ \frac{3^{\frac{1}{2}} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{1}{3}\right)}{2^{\frac{3}{2}} \pi^{\frac{3}{2}} e^{\frac{1}{2}\gamma}} \right\}.$$

In the first series the terms are positive or negative, according as the number $\equiv 2$ or -2, mod. 6; and in the second series they are positive or negative, according as the number $\equiv -1$ or 1, mod. 6. Thus the second series depends upon the subeven and supereven numbers to modulus 6, taken with contrary signs.

§ 63. Since

$$\Gamma\left(\tfrac{1}{6}\right) = 2^{\frac{11}{9}} 3^{\frac{1}{3}} \pi^{\frac{1}{6}} K_1^{\frac{2}{3}} = 2^{\frac{11}{9}} \pi^{\frac{1}{6}} K_2^{\frac{2}{3}},$$

we may express the values of the two series in terms of K_1 or K_2 instead of Gamma Functions. We thus find

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 8}{8} - \frac{\log 10}{10} + \&c. = \frac{\pi}{3\sqrt{3}} \log \left\{ \frac{2^{\frac{5}{6}} 3^{\frac{3}{4}} K_2}{\pi e^{\frac{1}{2}\gamma}} \right\};$$

and

$$\frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 11}{11} - \frac{\log 13}{13} + \&c. = -\frac{\pi}{\sqrt{3}} \log \left\{ \frac{2^{\frac{1}{2}} 3^{\frac{1}{4}} K_2}{\pi e^{\frac{1}{2}\gamma}} \right\}.$$

§ 64. Expressed as a product, the last formula becomes

$$\frac{7^{\frac{1}{7}}.13^{\frac{1}{13}}.19^{\frac{1}{19}}\ldots(6n+1)^{\frac{1}{6n+1}}}{5^{\frac{1}{5}}.11^{\frac{1}{11}}.17^{\frac{1}{17}}\ldots(6n-1)^{\frac{1}{6n-1}}}=\left(\frac{2^{\frac{1}{2}}3^{\frac{1}{4}}K_2}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{\pi}{\sqrt{3}}}.$$

§ 65. We may obtain also the value of the corresponding quotient involving subeven and supereven numbers to modulus 8.

For, by putting $\mu=\frac{1}{8}$ in the general formula of § 56, we have

$$\frac{\log 2}{2}-\frac{\log 6}{6}+\frac{\log 10}{10}-\frac{\log 14}{14}+\frac{\log 18}{18}-\&c.$$

$$+\frac{1}{\sqrt{2}}\left(\frac{\log 3}{3}-\frac{\log 5}{5}-\frac{\log 7}{7}+\frac{\log 9}{9}+\frac{\log 11}{11}-\&c.\right)$$

$$=\pi\log\left\{\frac{(\sin\frac{1}{8}\pi)^{\frac{1}{2}}\,\Gamma(\frac{1}{8})}{2^{\frac{5}{8}}\pi^{\frac{7}{8}}e^{\frac{3}{8}\gamma}}\right\}.$$

The first line

$$=\frac{\log 2}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\&c.\right)$$

$$-\frac{1}{2}\left(\frac{\log 3}{3}-\frac{\log 5}{5}+\frac{\log 7}{7}-\frac{\log 9}{9}+\&c.\right)$$

$$=\frac{\log 2}{2}\cdot\frac{\pi}{4}+\tfrac{1}{2}\pi\log\left\{\frac{(\sin\frac{1}{4}\pi)^{\frac{1}{2}}\,\Gamma(\frac{1}{4})}{2^{\frac{1}{4}}\pi^{\frac{3}{4}}e^{\frac{1}{4}\gamma}}\right\}$$

$$=\frac{\pi}{2}\log\left\{\frac{(\sin\frac{1}{4}\pi)^{\frac{1}{2}}\,\Gamma(\frac{1}{4})}{\pi^{\frac{3}{4}}e^{\frac{1}{4}\gamma}}\right\}.$$

Thus, substituting this value,

$$\frac{1}{\sqrt{2}}\left(\frac{\log 3}{3}-\frac{\log 5}{5}-\frac{\log 7}{7}+\frac{\log 9}{9}+\frac{\log 11}{11}-\&c.\right)$$

$$=\frac{\pi}{2}\log\left\{\frac{\sin\frac{1}{8}\pi\,\Gamma^2(\frac{1}{8})}{2^{\frac{1}{2}}\pi e^{\frac{1}{2}\gamma}\,\Gamma(\frac{1}{4})}\right\},$$

and therefore

$$\frac{\log 3}{3}-\frac{\log 5}{5}-\frac{\log 7}{7}+\frac{\log 9}{9}+\frac{\log 11}{11}-\&c.$$
$$=\frac{\pi}{\sqrt{2}}\log\left\{\frac{\sin\frac{1}{8}\pi\,\Gamma^2(\frac{1}{8})}{2^{\frac{1}{2}}\pi e^{\frac{1}{2}\gamma}\,\Gamma(\frac{1}{4})}\right\}.$$

The terms in the series are positive when the number $\equiv 1$ and 3, mod. 8, and negative when it $\equiv -1$ and -3, mod. 8.

§ 66. Combining this result by addition and subtraction with

$$\frac{\log 3}{3}-\frac{\log 5}{5}+\frac{\log 7}{7}-\frac{\log 9}{9}+\frac{\log 11}{11}-\&c.$$
$$=\pi\log\left\{\frac{\Gamma(\frac{1}{4})}{2^{\frac{1}{2}}\pi^{\frac{3}{4}}e^{\frac{1}{4}\gamma}}\right\},$$

we obtain the values of the series

$$\frac{\log 3}{3}-\frac{\log 5}{5}+\frac{\log 11}{11}-\frac{\log 13}{13}+\&c.,$$

and
$$\frac{\log 7}{7}-\frac{\log 9}{9}+\frac{\log 15}{15}-\frac{\log 17}{17}+\&c.,$$

in the former of which the terms are positive or negative according as the number $\equiv 3$ or -3, mod. 8; and, in the latter, according as it $\equiv -1$ or 1, mod. 8.

§ 67. In conclusion I may remark that the value of the series

$$\frac{\log 3}{3^2}+\frac{\log 5}{5^2}+\frac{\log 7}{7^2}+\frac{\log 9}{9^2}+\&c.$$

may be expressed by means of the constant A.

For, integrating the general formula of § 56 with respect to μ between the limits 0 and μ, we obtain the equation

$$\frac{1}{\pi}\left(\frac{\log 2}{2^2}\sin^2 2\mu\pi+\frac{\log 3}{3^2}\sin^2 3\mu\pi+\frac{\log 4}{4^2}\sin^2 4\mu\pi+\&c.\right)$$
$$=\pi\left\{\int_0^\mu \log\Gamma(\mu)\,d\mu+\tfrac{1}{2}\int_0^\mu \log\sin\mu\pi\,d\mu\right.$$
$$\left.-(\mu-\tfrac{1}{2}\mu^2)\log\pi-\tfrac{1}{2}(\mu-\mu^2)(\gamma+\log 2)\right\}.$$

Putting $\mu = \frac{1}{2}$, we have

$$\frac{\log 3}{3^2} + \frac{\log 5}{5^2} + \frac{\log 7}{7^2} + \frac{\log 9}{9^2} + \&c.$$

$$= \pi^2 \left\{ \int_0^{\frac{1}{2}} \log \Gamma(\mu)\, d\mu + \tfrac{1}{2} \int_0^{\frac{1}{2}} \log \sin \mu\pi \, d\mu - \tfrac{3}{8} \log \pi - \tfrac{1}{8} \log 2 - \tfrac{1}{8}\gamma \right\}$$

$$= \pi^2 \left\{ \int_0^{\frac{1}{2}} \log \Gamma(\mu)\, d\mu - \tfrac{3}{8} \log 2 - \tfrac{3}{8} \log \pi - \tfrac{1}{8}\gamma \right\}.$$

§ 68. Now, from § 23,

$$3 \log A = 1 + \tfrac{7}{12} \log 2 - \tfrac{1}{2} \log \pi + 2 \int_0^{\frac{1}{2}} \log \Gamma(1+x)\, dx,$$

and

$$\int_0^{\frac{1}{2}} \log \Gamma(1+x)\, dx = \int_0^{\frac{1}{2}} \log x \, dx + \int_0^{\frac{1}{2}} \log \Gamma(x)\, dx$$

$$= \tfrac{1}{2} \log \tfrac{1}{2} - \tfrac{1}{2} + \int_0^{\frac{1}{2}} \log \Gamma(x)\, dx.$$

Thus the equation becomes

$$3 \log A = -\tfrac{5}{12} \log 2 - \tfrac{1}{2} \log \pi + 2 \int_0^{\frac{1}{2}} \log \Gamma(x)\, dx,$$

and therefore

$$\int_0^{\frac{1}{2}} \log \Gamma(x)\, dx = \tfrac{5}{24} \log 2 + \tfrac{1}{4} \log \pi + \tfrac{3}{2} \log A.$$

§ 69. Substituting this value of the integral in the formula of § 66, we find

$$\frac{\log 3}{3^2} + \frac{\log 5}{5^2} + \frac{\log 7}{7^2} + \frac{\log 9}{9^2} + \&c.$$

$$= \pi^2 \left(\tfrac{3}{2} \log A - \tfrac{1}{6} \log 2 - \tfrac{1}{8} \log \pi - \tfrac{1}{8}\gamma \right);$$

and therefore, expressing this result as a product,

$$3^{\frac{1}{9}} 5^{\frac{1}{25}} 7^{\frac{1}{49}} 9^{\frac{1}{81}} \ldots = \left(\frac{A^3}{2^{\frac{1}{3}} \pi^{\frac{1}{4}} e^{\frac{1}{4}\gamma}} \right)^{\frac{1}{2}\pi^2}.$$