ON AN APPLICATION OF THE THEORY OF GROUPS TO KIRKMAN'S PROBLEM.

By *W. Burnside.*

In the solution of Kirkman's problem it is convenient from some points of view, first to form a complete set of 35 triplets of the 15 symbols and then to consider the possibility of dividing them into 7 sets, each containing all the symbols and representing a day's walking order according to the popular way of presenting the problem.

A complete set of triplets of a given number n of symbols is a set such that every pair of symbols enters once and no pair enters more than once in a triplet. In order that this may be possible n must be of the form $6m+1$ or $6m+3$: and conversely it has been shewn recently by Mr. E. H. Moore (*Math. Ann.*, XLIII.) that if n has one of these two forms it is always possible to form a complete set of triplets and that in at least two distinct ways.

The problem it is proposed here to deal with is that of determining those solutions of Kirkman's problem which are unchanged by as great a number of permutations of the 15 symbols as possible. It will be seen that when this limitation on the problem is introduced, the solution is no longer of the extremely tentative nature that has marked all attempts at the solution of the problem in its general form; and it appears possible that a corresponding method may perhaps be applicable to the general problem.

The permutations of the 15 symbols which leave a solution of the problem unchanged necessarily form a group; for if the solution is unchanged (the 7 days walks being of course permuted among themselves) by any two permutations, it is unchanged by any combination or repetition of these permutations.

The only primes that can enter into the order of a permutation of 15 symbols are 2, 3, 5, 7, 11, 13. A permutation of order 11 or 13 could not, however, possibly permute the 7 days walks among themselves, and also could not leave each day's walk unchanged, and therefore it is only necessary to consider permutations whose orders contain 2, 3, 5 and 7 as factors.

It may further be shewn that no permutations of order 5 can leave a solution unchanged. To prove this the forms of permutations which can change a complete set of triplets into itself must be considered. Such permutations must, in fact, either permute all the 15 symbols or they must keep either 1 or 3 of them unaltered.

Thus, if such a permutation keeps 2 symbols unaltered it must also keep that third symbol which enters with these two into a triplet unchanged, while if it keeps 4 symbols unaltered it must keep each symbol which enters with any two of these with a triplet unchanged, and this may easily be shewn to lead to all the symbols being unchanged so that the permutation reduces to identity.

A permutation of order 5 which can change a complete set of triplets into itself must therefore be of the form

$$(1.2.3.4.5)\,(6.7.8.9.10)\,(11.12.13.14.15).$$

The corresponding set of triplets must contain 10 triplets, each having a pair of symbols from the first bracket, 10 each having a pair from the second, and 10 each having a pair from the third bracket, and finally 5 triplets each having one symbol from each bracket. Now if from this set of triplets a solution could be obtained which is transformed into itself by the above permutation, it would be necessary that 5 of the 7 days' walks should be interchanged cyclically by the permutation, while the other two were changed into themselves. But the above partial analysis of this set of triplets shews at once that only one day's walk can be formed which is transformed into itself by the permutation, namely that consisting of the 5 triplets each of which has one symbol from each bracket of the permutation.

It follows therefore that there can be no solution which is transformed into itself by a permutation of order 5; and that the order of a group of permutations which can transform a solution into itself must be of the form $2^a 3^b 7^c$.

If now c were greater than unity there would be at least two commutative permutations of order 7, one not being a power of the other, which would transform the solution into itself. But two such permutations of order 7 of 15 symbols can only be commutative when each consists of a single cycle of 7 symbols, the symbols in the two cycles being all distinct. Each of these would by itself leave 8 symbols unchanged, and therefore as has been shewn above could not transform the set of triplets into itself. The index c must therefore be zero or unity.

Next, if b were greater than unity there would necessarily be a permutation of order 9, or two commutative permutations of order 3 which would leave the solution unchanged. Since a group of order 9 cannot be expressed transitively in terms of 7 symbols, this set of permutations would have to transform one day's walk into itself, and the rest among themselves in sets of three. But since a group of order 9 cannot be expressed at all in terms of 3 symbols, at least one permutation of order 3 contained in it would necessarily transform four day's walks each into themselves; and this is impossible for it would involve changing 8 triplets into themselves, namely the two in each of the four days' walks which are not interchanged cyclically. It follows from this reasoning that the index b is either zero or unity.

Lastly, if a were greater than 3 there would be a group of order 16, which permuting the 7 days' walks among themselves would necessarily transform one into itself. One of the 5 triplets would then remain unchanged by all the permutations of the group, and the remaining 4 would be permuted among themselves. Since, however, a group of order 16 cannot be expressed in terms of 4 symbols, at least one permutation of order two would leave all 5 triplets unchanged, and since this involves that the permutation consists of 5 transpositions and therefore leaves 5 symbols unchanged, it is impossible. It follows that a is not greater than 3, and that the greatest possible order of a group of permutations which can transform a solution into itself is $2^3.3.7$ or 168.

A complete set of triplets must now be formed which shall be transformed into itself by a group of 168 permutations. This group will contain a permutation of order 7 and, as seen above, such a permutation must be of the form

$$S \equiv (abcdefg)\,(1234567)\,8.^*$$

Without loss of generality one of the triplets containing 8 may be taken as $8a1$, and the remainder will then proceed from this by the permutation S.

Suppose now, if possible, that no triplet contains 3 symbols from the second bracket in S. Then from the triplets containing 12, 13, 14 there proceed by the application of S 21 triplets containing every pair of the 7 symbols 1, 2, 3, 4, 5, 6, 7, The remaining 7 triplets must therefore each contain 3 symbols

* The notation is here changed partly in view of the form of the result, and partly to avoid double figure symbols.

from the first bracket of S. A set of triplets which is transformed into itself by S must therefore contain a set of 7 triplets consisting entirely of symbols taken from one of the two brackets of S, and again without loss of generality this may be assumed to be the first bracket. This set of 7 triplets must proceed from either abd or abf by the application of S, for if ab occurs with either c, e or g, S will produce in each case triplets containing a common pair of symbols: moreover this set of triplets is evidently a complete set for the symbols a, b, c, d, e, f, g.

There are now 3 remaining triplets containing a, namely those in which a enters with the 6 symbols 2, 3, 4, 5, 6, 7. These 6 symbols may be arranged in 3 pairs in 6 different ways, but it may be at once verified that 24, 37, 56; 26, 34, 57 and 27, 36, 55 are the only ways which do not, on the application of S, lead to triplets containing a common pair.

Hence, all types of complete sets of triplets which are transformed into themselves by S arise from the application of S to one of the sets of 5 contained in the table

$$a81 \begin{cases} abd \\ abf \end{cases} \begin{cases} a24, a37, a56 \\ a26, a34, a57 \\ a27, a36, a45 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{I}).$$

In any case the complete set of triplets contains within it a complete set of triplets of the 7 symbols a, b, c, d, e, f, g, and of no other set of 7 symbols; and the permutations which transform it into itself must therefore form an intransitive group permuting $a, b, \ldots, g$ and $1, 2, \ldots, 8$ respectively among themselves.

The question therefore presents itself: by what permutations is a complete set of triplets of seven symbols such as

$$abd,\ bce,\ cdf,\ deg,\ efa,\ fgb,\ gac$$

transformed into itself? To answer this question it may be noticed, first that in all 30 such complete sets can be formed, for when one of the 5 remaining symbols has been chosen to go with ab, it is found that the other 6 triplets may be put together in 6 different ways; secondly that any one of these 30 complete sets can be transformed into any other by a suitable permutation. Hence the order of the permutation-group by which any one is transformed into itself is $7! \div 30$ or 168.

Now there is only one group of permutations of 7 symbols of order 168, which is the well-known simple group of this order, first recognised in analysis as the group of the modular equation for transformation of the seventh order. Hence, if a solution of Kirkman's problem can be found which is transformed into itself by a group of the greatest order which has been shewn to be possible, namely 168, this group must be isomorphous with the above mentioned group.

Now it is well known that this simple group of order 168 can also be expressed as a transitive group in 8 symbols, and that it can be generated from any one of its operations of order 7 combined with any one of its operations of order 2. Considering it first in connection with the complete set of triplets of 7 symbols written above, these from their mode of formation are changed into themselves by the cyclical permutation of order 7,

$$S' \equiv (abcdefg),$$

and a permutation of order 2 which changes the set into itself can be determined at once by supposing the symbols of one triplet to remain if possible unchanged. Thus, if a, b and d remain unchanged, aef and age, bce and bfg, dcf and deg, must, each pair of them, be interchanged or remain unchanged, and any one of the three pairs of transpositions

$$(ef)(gc),\ (ec)(fg),\ (eg)(cf)$$

will produce this result.

Hence, in particular, the group of order 168 which transform the set of triplets into itself can be generated from

$$S' \equiv (abcdefg), \quad T' \equiv abd\,(ef)\,(gc),$$

Taking now the first of the complete set of triplets given by the above table, the 28 triplets, each of which contains only one of the symbols a, b, ..., g, may be written in the form

$$\begin{array}{c|cccc} a & 81 & 24 & 37 & 56 \\ b & 82 & 35 & 41 & 67 \\ c & 83 & 46 & 52 & 71 \\ d & 84 & 57 & 63 & 12 \\ e & 85 & 61 & 74 & 23 \\ f & 86 & 72 & 15 & 34 \\ g & 87 & 13 & 26 & 45 \end{array} \qquad \text{......................(II).}$$

The 7 sets of 4 pairs in this table are permuted among themselves cyclically by the permutation (1234567), say S'', exactly as the symbols a, b, ..., g, prefixed to each set of pairs are permuted by S'; and it remains to see whether a permutation T''', of order two, of the symbols 1, 2, ..., 8, can be found which will permute the sets of pairs in the same way that T' permutes the symbols prefixed to them.

Such a permutation T'', if it exists, will necessarily with S'' generate a permutation group of the 8 symbols of order 168.

Since T'' is to change into themselves the sets of pairs to which a, b and d are prefixed, and since the symbols 8, 1, 2, 4 enter into each of these sets in two pairs, T'' must contain either the transpositions (81) (24), (82) (14) or (84) (12) taken with (36) (57), (37) (56) or (35) (67). The conditions that the lines prefixed e and f are interchanged by T'', and also those prefixed c and g, suffice to determine it completely as

$$(84)(12)(35)(67),$$

and the 7 sets of 4 pairs are therefore permuted among themselves by the permutations

$$S' \equiv (1234567), \quad T'' \equiv (84)(12)(35)(67),$$

exactly as the symbols prefixed to them are permuted by S' and T'. The complete set of 35 triplets is therefore changed into itself by the intransitive group of permutations, order 168, generated by

$$S \equiv (abcdefg)(1234567)8, \quad T \equiv abd(ef)(cg)(84)(12)(35)(67).$$

Of the other 5 types of complete sets of triplets given by the first table, it will be found by a similar investigation that the set arising from

$$abf,\ a81,\ a26,\ a34,\ a57$$

by operation of S is also changed into itself by a group of 168 permutations, and that the remaining four are not. The group except as regards the symbols in which it is expressed is necessarily the same in this second case as in the first, so that it is not a distinct solution.

Finally, the arrangement of the separate day's walks from this set of triplets has to be considered. No day's walk or set of 5 triplets can contain two triplets one of which proceeds from the other by S, as the solution could not then be transformed into itself by S. Hence, that day's walk which contains abd must contain one triplet from each vertical line in table II. At the same time since c, e, f, g must all be represented, it must contain one triplet from each horizontal line of the table except the first, second and fourth.

It is easily verified that this is only possible in two ways, corresponding to

abd		*abd*
*c*46		*c*52
*e*23	and	*e*61
*f*15		*f*34
*g*87		*g*87.

The two sets of 7 days' walks proceeding from these by the operation of S are then the only two solutions from the complete set of triplets considered which are transformed into themselves by the permutation S. If now these are written out at length it is found that the second is transformed by T into a new solution, while the first, or

I	II	III	IV	V	VI	VII
abd,	*bce*,	*cdf*,	*deg*,	*efa*,	*fgb*,	*gac*,
*c*46,	*d*57,	*e*61,	*f*72,	*g*13,	*a*24,	*b*35,
*e*23,	*f*34,	*g*45,	*a*56,	*b*67,	*c*71,	*d*12,
*f*15,	*g*26,	*a*37,	*b*41,	*c*52,	*d*63,	*e*74,
*g*87,	*a*81,	*b*82,	*c*83,	*d*84,	*e*85,	*f*86

is changed into itself, the permutation of the day's walks corresponding to T being

II IV V (I VII) (III VI).

The solution thus obtained is therefore the only distinct solution which is transformed into itself by the maximum group of order 168. Moreover, if the group is given by its generating permutations as an intransitive group of 15 symbols, interchanging them in two sets of 7 and 8 respectively, the solution is unique. To the generating permutations S and T of the 15 symbols, there correspond for the permutations of the day's walks

(I II III IV V VI VII) and II IV V (I VII) (III VI).

Note.—The solution thus arrived at is one that is given by almost every one who has offered solutions of the problem. The object of this paper is to call attention to an interesting property of the solution, and to the method by which it is here obtained.