ON THE FLEX-LOCUS OF A SYSTEM OF PLANE CURVES WHOSE EQUATION IS A RATIONAL INTEGRAL FUNCTION OF THE COORDINATES AND ONE ARBITRARY PARAMETER.

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Let $$f(x, y, c) = 0 \qquad \text{(I)}$$

be the equation of the system of curves, rational and integral with regard to the coordinates x, y and the parameter c.

There is a point of inflexion on a curve of the system, where $d^2y/dx^2 = 0$.

Using square brackets to enclose the variable with regard to which partial differential coefficients of $f(x, y, c)$ are taken,

$$[x] + [y]\frac{dy}{dx} = 0 \qquad \text{(II)},$$

$$[x, x] + 2[x, y]\frac{dy}{dx} + [y, y]\left(\frac{dy}{dx}\right)^2 + [y]\frac{d^2y}{dx^2} = 0 \qquad \text{(III)},$$

or, substituting for dy/dx from (II) in (III),

$$[x, x][y]^2 - 2[x, y][x][y] + [y, y][x]^2 = -[y]^3\frac{d^2y}{dx^2} \qquad \text{(IV)}.$$

Hence if $d^2y/dx^2 = 0$, in general

$$[x, x][y]^2 - 2[x, y][x][y] + [y, y][x]^2 = 0 \qquad \text{(V)}.$$

The left-hand side of (V) is the Hessian.

Consequently let (V) be written in the form

$$H = 0 \qquad \text{(VI)}.$$

In (VI) H is a function of x, y, c.

Let the roots of (I) considered as an equation for c be $c_1, c_2, \ldots, c_n$.

Let the result of substituting any root c_r for c in H be denoted by H_r.

Let the result of eliminating c between (I) and (VI) be denoted by $E = 0$.

Let the locus of the points of inflexion, or flex-locus, of the curves (I) be $F = 0$. Let the locus of their double points be $N = 0$. Let the locus of their cusps be $C = 0$.

Then the object of this paper is to show that E contains the factors F, N^6, C^8.

1. *The differential coefficients of H as far as the third order.*

Let ∂ denote partial differentiation when x, y are independent variables, c being expressed as a function of x, y by means of (I).

$$\frac{\partial H}{\partial x} = [x, x, x][y]^2 - 2[x, x, y][x][y] + [x, y, y][x]^2$$

$$+ 2[x]\{[x, x][y, y] - [x, y]^2\}$$

$$+ \frac{\partial c}{\partial x}\left(\begin{array}{l} \quad [x, x, c][y]^2 - 2[x, y, c][x][y] + [y, y, c][x]^2 \\ + 2[x]\{[x, c][y, y] - [y, c][x, y]\} \\ - 2[y]\{[x, c][x, y] - [y, c][x, x]\} \end{array}\right),$$

$$\frac{\partial^2 H}{\partial x^2} = [x, x, x, x][y]^2 - 2[x, x, x, y][x][y] + [x, x, y, y][x]^2$$

$$+ 2[x]\{[x, x, x][y, y] - 3[x, x, y][x, y] + 2[x, y, y][x, x]\}$$

$$+ 2[y]\{[x, x, x][x, y] - [x, x, y][x, x]\}$$

$$+ 2[x, x]\{[x, x][y, y] - [x, y]^2\}$$

$$+ 2\frac{\partial c}{\partial x}\left(\begin{array}{l} \quad [x, x, x, c][y]^2 - 2[x, x, y, c][x][y] + [x, y, y, c][x]^2 \\ + 2[x]\{[x, x, c][y, y] - 2[x, y, c][x, y] + [y, y, c][x, x] \\ \qquad\qquad + [x, y, y][x, c] - [x, x, y][y, c]\} \\ + 2[y]\{[x, x, x][y, c] - [x, x, y][x, c]\} \\ + 2[x, c]\{[x, x][y, y] - [x, y]^2\} \end{array}\right)$$

$$+ \left(\frac{\partial c}{\partial x}\right)^2 \left(\begin{array}{l} \quad [x, x, c, c][y]^2 - 2[x, y, c, c][x][y] + [y, y, c, c][x]^2 \\ + 2[x]\{[y, y][x, c, c] - [x, y][y, c, c] \\ \qquad\qquad + 2[x, c][y, y, c] - 2[y, c][x, y, c]\} \\ + 2[y]\{[x, x][y, c, c] - [x, y][x, c, c] \\ \qquad\qquad - 2[x, c][x, y, c] + 2[y, c][x, x, c]\} \\ + 2\{[x, x][y, c]^2 - 2[x, y][x, c][y, c] + [y, y][x, c]^2\} \end{array}\right)$$

$$+ \frac{\partial^2 c}{\partial x^2}\left(\begin{array}{l} \quad [x, x, c][y]^2 - 2[x, y, c][x][y] + [y, y, c][x]^2 \\ + 2[x]\{[x, c][y, y] - [y, c][x, y]\} \\ - 2[y]\{[x, c][x, y] - [y, c][x, x]\} \end{array}\right).$$

In forming $\frac{\partial^3 H}{\partial x^3}$ it is necessary only to calculate the terms which obviously do not vanish through containing a factor $[x]$ or $[y]$.

The terms retained will then be

$$6[x,x]\{[x,x,x][y,y]-2[x,x,y][x,y]+[x,y,y][x,x]\}$$

$$+6\frac{\partial c}{\partial x}\left(\begin{array}{l} \quad [x,x,x]\{[x,c][y,y]+[y,c][x,y]\} \\ - \quad [x,x,y]\{3[x,c][x,y]+[y,c][x,x]\} \\ +2[x,y,y][x,c][x,x] \\ + \quad [x,x,c]\{2[x,x][y,y]-[x,y]^2\} \\ + \quad [x,x]\{[x,x][y,y,c]-2[x,y][x,y,c]\} \end{array}\right)$$

$$+6\left(\frac{\partial c}{\partial x}\right)^2\left(\begin{array}{l} \quad [x,x,x][y,c]^2-2[x,x,y][x,c][y,c]+[x,y,y][x,c]^2 \\ +2[x,c]\{[x,x,c][y,y]-2[x,y,c][x,y]+[y,y,c][x,x]\} \\ +2[x,c,c]\{[x,x][y,y]-[x,y]^2\} \end{array}\right)$$

$$+6\left(\frac{\partial c}{\partial x}\right)^3\left(\begin{array}{l} \quad [x,x,c][y,c]^2-2[x,y,c][x,c][y,c]+[y,y,c][x,c]^2 \\ +[x,c,c]\{[x,c][y,y]-[y,c][x,y]\} \\ -[y,c,c]\{[x,c][x,x]-[y,c][x,x]\} \end{array}\right).$$

$$+6\frac{\partial^2 c}{\partial x^2}[x,c]\{[x,x][y,y]-[x,y]^2\}$$

$$+6\frac{\partial c}{\partial x}\frac{\partial^2 c}{\partial x^2}\{[x,x][y,c]^2-2[x,y][x,c][y,c]+[y,y][x,c]^2\}.$$

2. To prove that at a point on the node-locus

$$H=0, \quad \frac{\partial H}{\partial x}=0, \quad \frac{\partial^2 H}{\partial x}=0.$$

At a point ξ, η on the node-locus, the equations

$$[x]=0, \ [y]=0, \ [c]=0$$

hold; see a paper by the Author on the c- and p- discriminants of Ordinary Integrable Differential Equations of the First Order (*Proceedings of the London Mathematical Society*, Vol. XIX., p. 562).

Let the value of c corresponding to the curve which has the node at ξ, η be γ.

Then $x=\xi$, $y=\eta$, $c=\gamma$ satisfy

$$[x]=0, \ [y]=0, \ [c]=0.$$

Hence, they also make

$$H=0,$$

$$\frac{\partial H}{\partial x}=0,$$

$$\frac{\partial^2 H}{\partial x^2} = 2[x, x]\{[x, x][y, y] - [x, y]^2\}$$

$$+ 4\frac{\partial c}{\partial x}[x, c]\{[x, x][y, y] - [x, y]^2\}$$

$$+ 2\left(\frac{\partial c}{\partial x}\right)^2 \{[x, x][y, c]^2 - 2[x, y][x, c][y, c] + [y, y][x, c]^2\}.$$

But $x = \xi$, $y = \eta$, $c = \gamma$ also make

$$\begin{vmatrix} [x, x] & [x, y] & [x, c] \\ [y, x] & [y, y] & [y, c] \\ [c, x] & [c, y] & [c, c] \end{vmatrix} = 0;$$

see paper cited above, p. 563.

Therefore

$$[x, x][y, c]^2 - 2[x, y][x, c][y, c] + [y, y][x, c]^2$$
$$= [c, c]\{[x, x][y, y] - [x, y]^2\},$$

therefore

$$\frac{\partial^2 H}{\partial x^2} = 2\{[x, x][y, y] - [x, y]^2\}\left\{[x, x] + 2[x, c]\frac{\partial c}{\partial x} + [c, c]\left(\frac{\partial c}{\partial x}\right)^2\right\}.$$

Now to determine $\frac{\partial c}{\partial x}$, there is the equation

$$[x] + [c]\frac{\partial c}{\partial x} = 0,$$

which is indeterminate since $[x] = 0$, $[c] = 0$.

Hence, differentiating

$$[x, x] + 2[x, c]\frac{\partial c}{\partial x} + [c, c]\left(\frac{\partial c}{\partial x}\right)^2 + [c]\frac{\partial^2 c}{\partial x^2} = 0.$$

But $[c] = 0$,

therefore $$[x, x] + 2[x, c]\frac{\partial c}{\partial x} + [c, c]\left(\frac{\partial c}{\partial x}\right)^2 = 0,$$

therefore $$\frac{\partial^2 H}{\partial x^2} = 0.$$

3. To shew that at a point on the cusp-locus

$$H = 0, \ \frac{\partial H}{\partial x} = 0, \ \frac{\partial^2 H}{\partial x^2} = 0, \ \frac{\partial^3 H}{\partial x^3} = 0.$$

At a point on the cusp-locus (see paper cited above, pages 563, 564),

$$[x, x]:[x, y]:[x, c]$$
$$=[y, x]:[y, y]:[y, c]$$
$$=[c, x]:[c, y]:[c, c],$$

wherein $x=\xi,\ y=\eta,\ c=\gamma.$

Put in the above $[c, x]=\sigma\ [c, c],$

$$[c, y]=\rho\ [c, c],$$

therefore $[x, x]=\sigma^2\ [c, c],$

$$[x, y]=\sigma\rho\ [c, c],$$
$$[y, y]=\rho^2\ [c, c].$$

As the cusp is a node, it is only necessary in this case to prove $\frac{\partial^3 H}{\partial x^3}=0.$

On making the above substitutions in the value of $\frac{\partial^3 H}{\partial x^3}$, the coefficients of $\frac{\partial^2 c}{\partial x^2}$ and $\frac{\partial c}{\partial x}\frac{\partial^2 c}{\partial x^2}$ both vanish.

The terms remaining in $\frac{\partial^3 H}{\partial x^3}\Big/[c, c]^2$ are

$$6\sigma^2\{\rho^2[x, x, x]-2\rho\sigma[x, x, y]+\sigma^2[x, y\ y]\}$$
$$+3\frac{\partial c}{\partial x}\left[\begin{array}{l}4\rho^2\sigma[x, x, x]-8\rho\sigma^2[x, x, y]+4\sigma^3[x, y, y]\\+2\rho^2\sigma^2[x, x, c]-4\rho\sigma^3[x, y, c]+2\sigma^4[y, y, c]\end{array}\right]$$
$$+3\left(\frac{\partial c}{\partial x}\right)^2\left[\begin{array}{l}2\rho^2[x, x, x]-4\rho\sigma[x, x, y]+2\sigma^2[x, y, y]\\+4\rho^2\sigma[x, x, c]-8\rho\sigma^2[x, y, c]+4\sigma^3[y, y, c]\end{array}\right]$$
$$+\left(\frac{\partial c}{\partial x}\right)^3\left[6\rho^2[x, x, c]-12\rho\sigma[x, y, c]+6\sigma^2[y, y, c]\right]$$
$$=6\left(\sigma+\frac{\partial c}{\partial x}\right)^2\left(\begin{array}{l}\rho^2[x, x, x]-2\rho\sigma[x, x, y]+\sigma^2[x, y, y]\\+\frac{\partial c}{\partial x}\left\{\rho^2[x, x, c]-2\rho\sigma[x, y, c]+\sigma^2[y, y, c]\right\}\end{array}\right).$$

But the equation to determine $\frac{\partial c}{\partial x}$ is in this case

$$[x, x]+2[x, c]\frac{\partial c}{\partial x}+[c, c]\left(\frac{\partial c}{\partial x}\right)^2=0,$$

and this becomes

$$\sigma^2 + 2\sigma \frac{\partial c}{\partial x} + \left(\frac{\partial c}{\partial x}\right)^2 = 0.$$

Hence $\frac{\partial^3 H}{\partial x^3} = 0$ at points on the cusp-locus.

4. To show that if $F=0$ be the flex-locus, E must contain F as a factor.

$$E = AH_1 H_2 \ldots H_n,$$

where A is the rationalising factor.

If $x=\xi$, $y=\eta$ be a point on the flex-locus, then when $x=\xi$, $y=\eta$ the equations

$$f(x, y, c) = 0,$$
$$H = 0$$

are satisfied by a common value of c.

Hence one of the quantities H_1, H_2, ..., H_n vanishes.

therefore $$E = 0.$$

Hence E contains F as a factor.

5. To show that if $N=0$ be the node-locus, E contains N^6 as a factor.

At a point on the node-locus, $H=0$, $\frac{\partial H}{\partial x} = 0$, $\frac{\partial^2 H}{\partial x^2} = 0$.

At a point ξ, η on the node-locus, the values of x, y, c satisfy

$$f(x, y, c) = 0,$$
$$[x] = 0,$$
$$[y] = 0,$$
$$[c] = 0.$$

Hence (I) treated as an equation for c has equal roots. Suppose that when $x=\xi$, $y=\eta$ the roots c_1, c_2 become equal, then writing E for brevity in the form

$$E = BH_1 H_2,$$

and forming all the partial differential coefficients of E with regard to x up to the 5th order, every term in the result must contain H_1 or H_2 or a first or second differential coefficient of H_1 or H_2. Hence all these differential coefficients vanish. Hence E must contain N^6 as a factor.

6. To show that if $C=0$ be the cusp-locus, E must contain C^8 as a facter.

If $x=\xi$, $y=\eta$ be a point on the cusp-locus the same equations hold as in the case of the node-locus, but in addition $\frac{\partial^3 H}{\partial x^3}$ vanishes.

Consequently if all the differential coefficients of E with regard to x up to the 7th order be formed, every term in the result must contain H_1 or H_2 or a first or second or third differential coefficient of H_1 or H_2. Hence all these differential coefficients of E must vanish. Hence E must contain C^8 as a factor.

7. Putting together the results of the last three articles it follows that the result of eliminating c between

$$f(x, y, c)=0,$$

and

$$[x, x][y]^2-2[x, y][x][y]+[y, y][x]^2=0$$

contains the factors

$$F, N^6, C^8.$$

8. The preceding results agree with Plücker's Formula

$$3n(n-2)=i+6\delta+8\kappa.$$

For every point of intersection of the curve and its Hessian, there is a factor in the eliminant.

As the Hessian cuts the curve once at a point of inflexion, 6 times at a double point, and 8 times at a cusp, the factors of the eliminant might be expected to be the flex-locus once, the node-locus 6 times, and the cusp-locus 8 times.

Example I.

Take the curves

$$y-c-x^3=0,$$

Therefore

$$[x]=-3x^2,$$

$$[y]=1,$$

$$[x, x]=-6x,$$

$$[x, y]=0,$$

$$[y, y]=0.$$

Therefore $[x]^2[y, y]-2[x][y][x, y]+[y]^2[x, x]=0$

becomes

$$x=0.$$

Hence the result of eliminating c between

$$y - c - x^3 = 0,$$

and

$$x = 0,$$

is

$$x = 0.$$

Hence $x = 0$ is the flex-locus, and x occurs to the first power in the eliminant.

Example II.

$$(x - c)^3 - (y - c) = 0.$$

Therefore

$$[x] = 3(x - c)^2,$$

$$[y] = -1,$$

$$[x, x] = 6(x - c),$$

$$[x, y] = 0,$$

$$[y, y] = 0.$$

Hence $[y]^2 [x, x] - 2[x][y][x, y] + [x]^2 [y, y] = 0$

becomes

$$6(x - c) = 0.$$

Eliminating c from

$$(x - c)^3 - (y - c) = 0,$$

and

$$x - c = 0,$$

the result is $x - y = 0$.

This is the flex-locus, and occurs only once.

Example III.

$$(y - c)^2 - x(x - a)(x - b) = 0,$$

i.e.

$$(y - c)^2 - x^3 + x^2(a + b) - xab = 0 \;\ldots\ldots\ldots\ldots(\text{I}),$$

Therefore

$$[x] = -3x^2 + 2x(a + b) - ab,$$

$$[y] = 2(y - c),$$

$$[x, x] = -6x + 2(a + b),$$

$$[x, y] = 0,$$

$$[y, y] = 2;$$

Therefore $[y]^2 [x, x] - 2[x][y][x, y] + [x]^2 [y, y] = 0$

becomes

$$4(y - c)^2 \{-6x + 2(a + b)\} + 2\{-3x^2 + 2x(a + b) - ab\}^2 = 0 \ldots(\text{II}).$$

Now the result of eliminating c from (I) and (II) is

$$[8x(x-a)(x-b)(a+b-3x)+2\{-3x^2+2x(a+b)-ab\}^2]^2=0,$$

$$i.e.\ [4x(x-a)(x-b)(a+b-3x)+\{3x^2-2x(a+b)+ab\}^2]^2=0,$$

$$i.e.\qquad \{3x^4-4(a+b)x^3+6abx^2-a^2b^2\}^2=0.$$

Now this is the flex-locus, for

$$y-c=[x(x-a)(x-b)]^{\frac{1}{2}},$$

$$\frac{dy}{dx}=\frac{1}{2}\,\frac{3x^2-2x(a+b)+ab}{[x(x-a)(x-b)]^{\frac{1}{2}}},$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left\{\frac{[6x-2(a+b)]\{x(x-a)(x-b)\}^{\frac{1}{2}}-\frac{1}{2}\,\dfrac{(3x^2-2x(a+b)+ab)^2}{\{x(x-a)(x-b)\}^{\frac{1}{2}}}}{x(x-a)(x-b)}\right\}$$

$$=\frac{4x(x-a)(x-b)(3x-a-b)-(3x^2-2x(a+b)+ab)^2}{4[x(x-a)(x-b)]^{\frac{3}{2}}}.$$

Hence $\dfrac{d^2y}{dx^2}=0$, when

$$4x(x-a)(x-b)(a+b-3x)+\{3x^2-2x(a+b)+ab\}^2=0.$$

The reason why this factor occurs twice is this:—

The curve being symmetrical with regard to the axis of x, if $x=\xi$, $y=\eta$ is a point of inflexion, so is $x=\xi$, $y=-\eta$.

Now the system of curves is formed by shifting the curve

$$y^2=x(x-a)(x-b)$$

parallel to the axis of y.

Hence, if $x=\xi$, $y=\eta$ be one point of inflexion on the curve, then the straight line $x=\xi$ is a part of the flex-locus. But it is the locus not of one point of inflexion only, but of two, for as the curve $y^2=x(x-a)(x-b)$ is moved parallel to the axis of y, two of its points of inflexion describe the line $x=\xi$.

Example IV.

Take the curves $(y-c)^2-x(x-a)^2=0$.

The results may be deduced from the last example by putting $b=a$.

Hence the locus to be considered is now

$$(3x^4-8ax^3+6a^2x^2-a^4)^2=0,$$

$$i.e.\qquad (x-a)^6(3x+a)^2=0.$$

In this $x = a$ is the double point locus, hence $x - a$ occurs 6 times as a factor.

Again $3x + a = 0$ is the locus of the two points of inflexion; every point on this locus contains two points of inflexion. Consequently $3x + a$ occurs twice as a factor.

That $3x + a = 0$ is the flex-locus is seen at once, since

$$y - c = x^{\frac{3}{2}} - ax^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \tfrac{3}{2}x^{\frac{1}{2}} - \tfrac{1}{2}ax^{-\frac{1}{2}},$$

$$\frac{d^2y}{dx^2} = \tfrac{1}{4}(3x + a)\,x^{-\frac{3}{2}}.$$

Hence $\dfrac{d^2y}{dx^2} = 0$ when $3x + a = 0$.

Example V.

$$(y - c)^2 = x^3.$$

This is obtained by putting $a = 0$ in the last result.

The locus to be considered becomes now

$$x^8 = 0.$$

Now $x = 0$ is the cusp-locus.

Hence it occurs 8 times.

Example VI.

$$(x - c)^3 - y + c^2 = 0,$$
$$[x] = 3(x - c)^2,$$
$$[y] = -1$$
$$[x, x] = 6(x - c),$$
$$[x, y] = 0,$$
$$[y, y] = 0,$$

therefore $[x]^2[y, y] - 2[x, y][x][y] + [y]^2[x, x] = 0$

becomes $$x - c = 0.$$

Eliminating c from

$$x - c = 0,$$

and $$(x - c)^3 - y + c^2 = 0,$$

the result is $$x^2 - y = 0.$$

Now $x^2 - y = 0$ is the flex-locus.

Hence it occurs only once as a factor.
