ON A THEOREM IN THE DIFFERENTIAL CALCULUS.

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SUPPOSE it is required to express the result of the operation

$$f_n\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \ldots,\ \frac{\partial}{\partial x_p}\right) F\{\phi\,(x_1, x_2, x_3, \ldots, x_p)\},$$

where F, ϕ are any functions, and f_n is a rational integral homogeneous function of degree n in the differential operators; it is clear that the expression can be exhibited in the form

$$\chi_0 \frac{d^n F}{d\phi^n} + \chi_1 \frac{d^{n-1}F}{d\phi^{n-1}} + \ldots + \chi_r \frac{d^{n-r}F}{d\phi^{n-r}} + \ldots + \chi_{n-1} \frac{dF}{d\phi},$$

where $\chi_0, \chi_1, \ldots, \chi_{n-1}$ denote functions of the p variables, the form of these functions being independent of the form of F, and depending only on f_n and ϕ. To determine the functions χ, we may take F to be of any form which is convenient; let $F\{\phi\} = \phi^n$ the n^{th} power of ϕ, we have then

$$f_n\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \ldots,\ \frac{\partial}{\partial x_p}\right) \{\phi\,(x_1, x_2, \ldots, x_p)\}^n$$

$$= n!\,\{\chi_0 + \chi_1\phi + \ldots + \frac{1}{r!}\chi_r\phi^r + \ldots + \frac{1}{(n-1)!}\chi_{n-1}\phi\} \ldots (1);$$

now

$$f_n\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \ldots,\ \frac{\partial}{\partial x_p}\right) \{\phi\,(x_1, x_2, \ldots, x_p)\}^n$$

$$= f_n\left(\frac{\partial}{\partial h_1},\ \frac{\partial}{\partial h_2},\ \ldots,\ \frac{\partial}{\partial h_p}\right) \{\phi\,(x_1 + h_1, x_2 + h_2, \ldots, x_p + h_p)\}^n,$$

where on the right-hand side $h_1, h_2, \ldots, h_p$ are all put equal to zero after the operation is performed.

Using the Binomial Theorem, we have

$$\phi\,(x_1 + h_1,\ x_2 + h_2,\ \ldots,\ x_p + h_p)$$

$$= \sum_{r=0}^{r=n} \frac{n!}{r!\,(n-r)!} \{\phi\,(x_1, x_2, \ldots, x_p)\}^r \{\phi\,(x_1+h_1, x_2+h_2, \ldots, x_p+h_p) - \phi\,(x_1, x_2, \ldots, x_p)\}^{n-r},$$

operating on both sides of this equation with

$$f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right),$$

we obtain an equation which must be equivalent to (1); comparing the coefficients of ϕ^r, we have

$$\chi_r = \frac{1}{(n-r)!} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right)$$
$$\{\phi(x_1+h_1, x_2+h_2, \ldots, x_p+h_p) - \phi(x_1, x_2, \ldots, x_p)\}^{n-r},$$

where $h_1, h_2, \ldots, h_p$ are all put equal to zero after the operation is performed.

We have thus obtained the following theorem—

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right) F\{\phi(x_1, x_2, \ldots, x_p)\}$$
$$= \frac{1}{n!}\frac{d^nF}{d\phi^n} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) P^n$$
$$+ \frac{1}{(n-1)!}\frac{d^{n-1}F}{d\phi^{n-1}} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) P^{n-1}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots,$$
$$+ \frac{1}{(n-r)!}\frac{d^{n-r}F}{d\phi^{n-r}} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) P^{n-r}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

where $P = \phi(x_1+h_1, x_2+h_2, \ldots, x_p+h_p) - \phi(x_1, x_2, \ldots, x_p)$, and $h_1, h_2, \ldots, h_p$ are all put equal to zero in the result.

It is clear that the coefficient of $\dfrac{d^{n-r}F}{d\phi^{n-r}}$ in (2) may be expressed in the form

$$\frac{1}{(n-r)!}\left\{f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)[\phi(x_1, x_2, \ldots, x_p)]^{n-r}\right.$$
$$-(n-r)\,\phi(x_1, x_2, \ldots, x_p)\, f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)$$
$$[\phi(x_1, x_2, \ldots, x_p)]^{n-r-1}$$
$$+\frac{(n-r)(n-r-1)}{2!}[\phi(x_1, x_2, \ldots, x_p)]^2 f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)$$
$$[\phi(x_1, x_2, \ldots, x_p)]^{n-r-2}$$
$$\left. - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right\}.$$

The particular case of the theorem (2) in which there is only one variable x, so that $f_n\left(\frac{d}{dx}\right)=\frac{d^n}{dx^n}$ is given in Schlömilch's *Compendium der höheren Analysis*, Vol. II.

I shall now consider a case in which the theorem (2) takes a simple form; let $\phi(x_1, x_2, \ldots, x_p)=x_1^2+x_2^2+\ldots+x_p^2=\rho^2$; in this case the coefficient of $\frac{d^{n-r}F}{d\phi^{n-r}}$ or $\frac{d^{n-r}F}{d(\rho^2)^{n-r}}$ is

$$\frac{1}{(n-r)!}f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right)\{h_1^2+h_2^2+\ldots+h_p^2 \\ +2(h_1x_1+h_2x_2+\ldots+h_px_p)\}^{n-r},$$

where $h_1=0$, $h_2=0$, ..., $h_p=0$; the only term in this expression which does not vanish is

$$\frac{1}{(n-r)!}f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right)\frac{(n-r)!}{r!\,(n-2r)!}2^{n-2r} \\ (h_1x_1+h_2x_2+\ldots)^{n-2r}(h_1^2+h_2^2+\ldots)^r,$$

for this is the only term in which the degree of the operand in $h_1, h_2, \ldots, h_p$, is the same as that of the operator in

$$\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}.$$

It is easily seen that if f_n, ψ_n are two functions of the same degree n,

$$f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right)\psi_n(h_1, h_2, \ldots, h_p) \\ =\psi_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right)f_n(h_1, h_2, \ldots, h_p);$$

it follows that the coefficient of $\frac{d^{n-r}F}{d(\rho^2)^{n-r}}$ is equal to

$$\frac{1}{r!\,(n-2r)!}2^{n-2r}\left(x_1\frac{\partial}{\partial h_1}+x_2\frac{\partial}{\partial h_2}+\ldots+x_p\frac{\partial}{\partial h_p}\right)^{n-2r} \\ \left(\frac{\partial^2}{\partial h_1^2}+\frac{\partial^2}{\partial h_2^2}+\ldots+\frac{\partial^2}{\partial h_p^2}\right)^r f_n(h_1, h_2, \ldots, h_p);$$

let

$$\left(\frac{\partial^2}{\partial h_1^2}+\frac{\partial^2}{\partial h_2^2}+\ldots+\frac{\partial^2}{\partial h_p^2}\right)^r f_n(h_1, h_2, \ldots, h_p)=\lambda_{n-2r}(h_1, h_2, \ldots, h_p),$$

then the above expression is equal to

$$\frac{1}{r!(n-2r)!}2^{n-2r}\left(x_1\frac{\partial}{\partial h_1}+x_2\frac{\partial}{\partial h_2}+\ldots+x_p\frac{\partial}{\partial h_p}\right)^{n-2r}\lambda_{n-2r}(h_1, h_2, \ldots, h_p);$$

if $\lambda_{n-2r}(h_1, h_2, \ldots, h_p)=\Sigma A h_1^{\alpha_1}h_2^{\alpha_2}\ldots h_p^{\alpha_p}$, the only terms which do not vanish are

$$\frac{1}{r!(n-2r)}2^{n-2r}\Sigma A\frac{(n-2r)!}{\alpha_1!\,\alpha_2!\ldots\alpha_p!}x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_p^{\alpha_p}$$
$$\times\frac{\partial^{\alpha_1}}{\partial h_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial h_2^{\alpha_2}}\ldots, h_1^{\alpha_1}h_2^{\alpha_2}, \ldots,$$

or

$$\frac{1}{r!}2^{n-2r}\Sigma A x_1^{\alpha_1}\,x_2^{\alpha_2}\ldots x_p^{\alpha_p}\text{ which is }\frac{1}{r!}2^{n-2r}\lambda_{n-2r}(x_1, x_2, \ldots, x_p);$$

since

$$\lambda_{n-2r}(x_1, x_2, \ldots, x_p)=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_p^2}\right)^r f_n(x_1, x_2, \ldots, x_p),$$

we see that the coefficient of $\frac{d^{n-r}F}{d(\rho^2)^{n-r}}$ si

$$\frac{2^{n-2r}}{r!}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_p^2}\right)^r f_n(x_1, x_2, \ldots, x_p),$$

we have thus obtained the following theorem:—

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)F(x_1^2+x_2^2+\ldots+x_p^2)$$
$$=\left\{2^n\frac{d^nF}{d(\rho^2)^n}+\frac{2^{n-2}}{1!}\frac{d^{n-1}F}{d(\rho^2)^{n-1}}\nabla^2+\frac{2^{n-4}}{2!}\frac{d^{n-2}F}{d(\rho^2)^{n-2}}\nabla^4\right.$$
$$\left.+\ldots+\frac{2^{n-2r}}{r!}\frac{d^{n-r}F}{d(\rho^2)^{n-2r}}\nabla^{2r}+\ldots\right\}f_n(x_1, x_2, \ldots, x_p)\ldots(3),$$

where $$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_p^2},$$

and $$\rho^2=x_1^2+x_2^2+\ldots+x_p^2.$$

The theorem (3) I have given in a paper* "On a theorem in Differentiation, &c.," where it is deduced from the theory of Spherical Harmonics.

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* See *Proc. Lond. Math. Soc.*, Vol. XXIV., p. 67

In the particular case $F(\rho^2) = \rho^{p-2}$ the theorem (3) becomes

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)\frac{1}{\rho^{p-2}} = \frac{(-1)^n (p-2)\, p(p+2)\ldots(p+2n-4)}{\rho^{p+2n-2}}$$
$$\times\left\{1 - \frac{\rho^2\nabla^2}{2.2n+p-4} + \frac{\rho^4\nabla^4}{2.4\,(2n+p-4)\,(2n+p-6)} - \ldots\right\}$$
$$f_n(x_1, x_2, \ldots, x_p) \ldots\ldots\ldots\ldots\ldots(4).$$

It is well known that $\dfrac{1}{\rho^{p-2}}$ is a solution of the equation $\nabla^2 V = 0$, and it follows at once that

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)\frac{1}{\rho^{p-2}}$$

satisfies the same equation, we see therefore that the expression on the right-hand side of (4) satisfies the equation $\nabla^2 V = 0$; now it can be verified at once that if $V_n(x_1, x_2, \ldots, x_p)$ is a solution of the differential equation, so also is

$$\frac{V_n(x_1, x_2, \ldots, x_p)}{\rho^{2n+p-2}};$$

we see therefore that the expression

$$V = f_n - \frac{\rho^2}{2.2n+p-4}\nabla^2 f_n + \frac{\rho^4}{2.4.2n+p-4.2n+p-6}\nabla^4 f_n \ldots(5)$$

satisfies the differential equation $\nabla^2 V = 0$, when f_n denotes any homogeneous integral function of degree n in the variables $x_1, x_2, \ldots, x_p$. All the solutions of $\nabla^2 V = 0$ which are rational algebraical functions of the variables may be obtained by giving f_n various values; for example, the zonal harmonic is obtained by putting $f_n = x_p^{\,n}$.

A case of (3) which is of considerable importance is obtained by taking $f_n(x_1, x_2, \ldots, x_p)$ to be a solution of $\nabla^2 V = 0$; denoting the solution by $Y_n(x_1, x_2, \ldots, x_p)$, the theorem (3) becomes

$$Y_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)F(\rho^2)$$
$$= \frac{d^n F(\rho^2)}{(\rho d\rho)^n}\, Y_n(x_1, x_2, \ldots, x_p) \ldots\ldots\ldots(6).$$

In particular, we have

$$Y_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)\frac{1}{\rho^{p-2}}$$
$$= (-1)^n (p-2)p(p+2)\ldots(p+2n-4)\frac{Y_n(x_1, x_2, \ldots, x_p)}{\rho^{2n+p-2}} \ldots(7),$$

where, as before, ρ^2 denotes $x_1^{\,2} + x_2^{\,2} + \ldots + x_p^{\,2}$.

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