ON A PROPERTY OF CERTAIN DETERMINANTS.

By *W. Burnside*.

It is well known that, if the 2nd, 3rd,... and nth rows of a determinant are formed from the first by permuting its elements cyclically, the determinant is expressible as the product of n linear factors. For instance

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} = (a_1 + a_2 + a_3)(a_1 + \omega a_2 + \omega^2 a_3)(a_1 + \omega^2 a_2 + \omega a_3)$$

where $\omega^3 = 1$.

The permutations by which all the rows proceed from the first form a group in the ordinary sense of the word : namely, if the permutations by which any two rows are derived from the first are performed successively, the resulting permutation will be that by which some other line of the determinant is derived from the first. In the case considered this group is of the simplest possible nature, all its permutations being obtainable by the repetition of one, which may be called the generating permutation ; such a group is called a cyclical group.

Now the property referred to above is not confined to determinants whose successive rows proceed from the first by a cyclical group of permutations. Thus it may be easily verified that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix} = (a_1 + a_2 + a_3 + a_4)(a_1 + a_2 - a_3 - a_4)$$
$$\times (a_1 - a_2 - a_3 + a_4)(a_1 - a_2 + a_3 - a_4).$$

In this case it may be proved that the permutations by which the different rows proceed from the first again form a group (evidently not a cyclical group) and that the different permutations are commutative : for instance, if the permutations by which the 2nd and 3rd rows are obtained are performed in *either* order, there results the permutation giving the 4th row.

A group with this property, that all its operations are commutative, is called an Abelian group. It is shewn in the text-books dealing with the theory of groups (*e.g.*, Netto : *Substitutionentheorie*, p. 146) that the operations of an

Abelian group can be represented, each once, symbolically by the form

$$P^\alpha Q^\beta R^\gamma...,$$

where the commutative symbols P, Q, R... represent certain individual operations of the group, whose orders are p, q, r... (that is, which lead to identity when repeated p, q, r...times) and the indices take all possible integral values less than p, q, r... respectively. The total number of operations including identity is then pqr....

The property it is proposed here to prove is the following—

If in a determinant of n rows the successive rows proceed from the first by permutations which form an Abelian group of order n (including identity), the determinant is expressible as the product of n linear factors.

Suppose the Abelian group given symbolically in the form

$$P^\alpha Q^\beta R^\gamma, \quad 0 \le \alpha \le p-1, \quad 0 \le \beta \le q-1, \quad 0 \le \gamma \le r-1,$$

so that $n = pqr$ is the order of the group. (It will be seen that the proof applies equally well whatever the number of symbols involved). Form the determinant whose first row consists of the n different symbols

$$P^\alpha Q^\beta R^\gamma$$

written in any order, while the elements of any other row are derived from the corresponding elements of the first row by multiplying them by $P^a Q^b R^c$; a, b, c taking all possible values except simultaneous zeros. If then account is taken of the symbolical equations

$$P^p = 1, \quad Q^q = 1, \quad R^r = 1,$$

the elements in every row will consist of the elements of the first row permuted; and since the symbols are commutative the permutations by which any two rows proceed from the first are also commutative, while that they form a group is self-evident.

Now let ω_1, ω_2, ω_3 be respectively a pth, a qth, and an rth root of unity, and multiply by $\omega_1^a \omega_2^b \omega_3^c$ the row that was formed from the first by multiplying by $P^a Q^b R^c$, doing this for all values of a, b, c. When the determinant is thus prepared form a new first row by adding together all the elements of each of the columns. The term in the first row corresponding to the column headed $P^\alpha Q^\beta R^\gamma$ is now

$$P^\alpha Q^\beta R^\gamma \Sigma \, (P\omega_1)^a \, (Q\omega_2)^b \, (R\omega_3)^c,$$

where the summation extends to all values of a, b, c less than p, q, r respectively and including simultaneous zero values.

The expression may be written

$$\omega_1^{-\alpha}\omega_2^{-\beta}\omega_3^{-\gamma} \Sigma (P\omega_1)^{\alpha+a} (Q\omega_2)^{\beta+b} (R\omega_3)^{\gamma+c},$$

or finally, in consequence of the symbolical equations

$$P^p = 1, \quad Q^q = 1, \quad R^r = 1,$$

and the actual equations

$$\omega_1^p = 1, \quad \omega_2^q = 1, \quad \omega_3^r = 1,$$

in the form

$$\omega_1^{-\alpha}\omega_2^{-\beta}\omega_3^{-\gamma} \Sigma (P\omega_1)^a (Q\omega_2)^b (R\omega_3)^c.$$

Every term of the first row and therefore the determinant itself is thus seen to be divisible by

$$\Sigma (P\omega_1)^a (Q\omega_2)^b (R\omega_3)^c,$$

and since ω_1, ω_2, ω_3 may be any roots of their respective equations, the determinant has n different factors of this form.

The result is clearly independent of the particular symbols used for the elements of the rows, and the factors are linear in the elements with roots of unity for coefficients. Since the determinant is of the nth degree in the elements and the existence of n different linear factors has been demonstrated, it is therefore expressible as the product of n linear factors.

The simplest case in which the group is not cyclical is that given above. Another simple illustration is given by

$$
\begin{vmatrix}
a_1, & a_2, & a_3, & b_1, & b_2, & b_3, & c_1, & c_2, & c_3 \\
a_2, & a_3, & a_1, & b_2, & b_3, & b_1, & c_2, & c_3, & c_1 \\
a_3, & a_1, & a_2, & b_3, & b_1, & b_2, & c_3, & c_1, & c_2 \\
b_1, & b_2, & b_3, & c_1, & c_2, & c_3, & a_1, & a_2, & a_3 \\
b_2, & b_3, & b_1, & c_2, & c_3, & c_1, & a_2, & a_3, & a_1 \\
b_3, & b_1, & b_2, & c_3, & c_1, & c_2, & a_3, & a_1, & a_2 \\
c_1, & c_2, & c_3, & a_1, & a_2, & a_3, & b_1, & b_2, & b_3 \\
c_2, & c_3, & c_1, & a_2, & a_3, & a_1, & b_2, & b_3, & b_1 \\
c_3, & c_1, & c_2, & a_3, & a_1, & a_2, & b_3, & b_1, & b_2
\end{vmatrix}
$$

$$= (A + B + C)(A + B\omega + C\omega^2)(A + B\omega^2 + C\omega),$$
$$(A' + B' + C')(A' + B'\omega + C'\omega^2)(A' + B'\omega^2 + C'\omega),$$
$$(A'' + B'' + C'')(A'' + B''\omega + C''\omega^2)(A'' + B''\omega^2 + C''\omega),$$

where $A = a_1 + a_2 + a_3$, $A' = a_1 + a_2\omega + a_3\omega^2$, $A'' = a_1 + a_2\omega^2 + a_3\omega$,

$$B = b_1 + b_2 + b_3, \quad B' = \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots$$
$$C = c_1 + c_2 + c_3 \quad C' = \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots$$

and

$$\omega^3 = 1.$$