so that finally

$$x : y : z : w,$$

$$:: \sqrt{(\beta-\gamma)}\,\{(pu-e_1)^2+e_2e_3\} : \sqrt{(\gamma-\alpha)}\,\{(pu-e_2)^2+e_3e_1\}$$

$$: \sqrt{(\alpha-\beta)}\,\{(pu-e_3)^2+e_1e_2\} : \tfrac{1}{2}p'u\sqrt{\{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\}}.$$

The invariants g_2, g_3 of the elliptic functions used are the two invariants of the quartic (i), so that the absolute invariant of the elliptic functions is the absolute invariant of the quartic, and the elliptic differential involved is

$$\frac{d\lambda}{\sqrt{(\lambda^4\Delta-\lambda^3\Theta+\lambda^2\Phi+\lambda\Theta'+\Delta')}}.$$

ON THE FACT THAT SEMINVARIANTS OF A BINARY QUANTIC ARE INVARIANTS OF THAT QUANTIC AND ITS DERIVATIVES.

By Professor *E. B. Elliott.*

1. The object of the following is to exhibit without graphical or symbolical reasoning a fact which Mr. Kempe has brought into notice in a paper on the application of graphs to ordinary binary quantics (*Proc. Lond. Math. Soc.*, 1893, pp. 105, &c.).

He has shewn that $S(a_0, a_1, a_2, \ldots, a_n)$, any seminvariant of the binary quantic

$$\alpha_n \equiv (a_0, a_1, a_2, \ldots, a_n)(x, y)^n,$$

is an invariant of that quantic and its successive derivatives with regard to x,

$$\alpha_{n-1} \equiv (a_0, a_1, a_2, \ldots, a_{n-1})(x, y)^{n-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\alpha_2 \equiv (a_0, a_1, a_2)(x, y)^2,$$

$$\alpha_1 \equiv a_0x + a_1y,$$

$$\alpha_0 \equiv a_0,$$

or some of them. He obtains, in fact, that

$$a_0^{\,W}\, S(a_0, a_1, a_2, \ldots, a_n),$$

where W is the weight of S, can be expressed rationally and integrally in terms of

$$\mathfrak{A}_0 \equiv a_0,$$
$$\mathfrak{A}_2 \equiv a_0 a_2 - a_1^2,$$
$$\mathfrak{A}_3 \equiv a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$\mathfrak{A}_n \equiv a_0^{-1}(a_0, a_1, a_2, \dots, a_n)(-a_1, a_0)^n,$$

and that these are invariants of the system. They are indeed, after the first which is one of the system, the resultants, each divided by a_0, of α_1 and the remaining quantics of the system. Also, a_0 or $\mathfrak{A}_0$ being an invariant of the system, we may remove the factor a_0^W, and conclude that $S(a_0, a_1, a_2, \dots, a_n)$ is itself an invariant of $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$.

We shall see, however, later that $a_0^W S$ has the specially important quality, which S itself does not as a rule possess, of replacing an *integral* invariant of n quantics of degrees 0, 1, 2, ..., n with different coefficients.

It is of course easy to see that $a_0^{W-i} S$, where i is the order of S, is a rational integral function of the protomorphic invariants $\mathfrak{A}_0, \mathfrak{A}_2, \mathfrak{A}_3, \dots, \mathfrak{A}_n$. For the result of depriving α_n of its second term by linear transformation of x, is

$$(a_0, a_1, a_2, \dots, a_n)(x, y)^n$$
$$\equiv \left(a_0, 0, \frac{\mathfrak{A}_2}{a_0}, \frac{\mathfrak{A}_3}{a_0^2}, \dots, \frac{\mathfrak{A}_n}{a_0^{n-1}}\right)\left(x + \frac{a_1}{a_0} y, y\right)^n,$$

so that, by the fundamental property of seminvariants,

$$S(a_0, a_1, a_2, \dots, a_n) = S\left(a_0, 0, \frac{\mathfrak{A}_2}{a_0}, \dots, \frac{\mathfrak{A}_n}{a_0^{n-1}}\right),$$

the right-hand member of which equality has a_0^{W-i} for the denominator of terms, if there be any, in which the first two arguments do not occur, and lower powers of a_0 for denominators of other terms.

2. It is instructive to have in mind the close connexion which exists between the theory of seminvariants of α_n, and that of elimination of x between α_n and its derivatives.

We know that, for every suffix r from 0 to n inclusive,

$$\alpha_r = e^{\frac{x}{y}\Omega} a_r y^r,$$

where Ω denotes

$$a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \ldots + na_{n-1} \frac{d}{da_n},$$

and consequently that, $F(a_0, a_1, a_2, \ldots, a_n)$ being a rational integral function of weight W,

$$F(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) = e^{\frac{x}{y}\Omega} F(a_0, a_1, a_2, \ldots, a_n)\, y^W.$$

If now $F(a_0, a_1, a_2, \ldots, a_n)$ be a seminvariant $S(a_0, a_1, a_2, \ldots, a_n)$ of α_n it is annihilated by Ω, and therefore

$$S(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) = S(a_0, a_1, a_2, \ldots, a_n)\, y^W.$$

Thus, a seminvariant of α_n is such a function of the coefficients $a_0, a_1, a_2, \ldots, a_n$ that, if these are replaced by the quantics $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$, the result is free from x.*

Now $S(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ being a rational integral function of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$, is an absolute covariant of those quantics. Thus

$$S(a_0, a_1, a_2, \ldots, a_n)\, y^W$$

is an absolute covariant of $\alpha_0, \alpha_1, \alpha_2 \ldots, \alpha_n$.

In other words it is the only term which does not vanish in virtue of $\Omega S = 0$, in an absolute covariant of degree W in x and y, and of weight W,

$$e^{\frac{x}{y}\Omega} S(a_0, a_1, a_2, \ldots, a_n)\, y^W.$$

3. To make this clearer let $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$ denotes the quantics with different coefficients which $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ become when we accent their coefficients no times, once, twice, ..., n times respectively. Any integral function of these, and in particular

$$S(\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}),$$

where S is a seminvariant in its arguments, is an absolute covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, a_n^{(n)}$. Also, if $\Sigma\Omega$ denote

$$\sum_{r=1}^{r=n} \left\{ a_0^{(r)} \frac{d}{da_1^{(r)}} + 2a_1^{(r)} \frac{d}{da_2^{(r)}} + \ldots + ra_{r-1}^{(r)} \frac{d}{da_r^{(r)}} \right\},$$

$$S(\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}) \equiv e^{\frac{x}{y}\Sigma\Omega} S(a_0, a_1', a_2'', \ldots, a_n^{(n)})\, y^W.$$

* Perhaps it is new, as an explicit statement, that, as follows from the above, if $z\,(=\alpha_n)$ be a rational integral function of x of degree n, and the products of degree W in x of i of the functions $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \ldots, \frac{d^nz}{dx^n}$ be formed, the number of linear functions of these products which are free from x is exactly that of seminvariants of type $W; i, n$, *i.e.* is $(W;\ i, n) - (W-1;\ i, n)$ or zero according as $in - 2W$ is not $<$ or < 0.

Thus $e^{\frac{x}{y}\Sigma\Omega} S(a_0, a_1', a_2'', \ldots, a_n^{(n)})\, y^W$ is an absolute covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$. And $S(a_0, a_1, a_2, \ldots, a_n)\, y^W$ is what this becomes when we remove all accents.

The resultant of this covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$ and α_1', *i.e.* $a_0'x + a_1'y$, is an invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$. Now this resultant is

$$R \equiv \left\{a_0'^W - a_0'^{W-1} a_1' \Sigma\Omega + \frac{1}{1.2} a_0'^{W-2} a_1'^2 (\Sigma\Omega)^2 - \ldots \right.$$

$$\left. + (-1)^W \frac{1}{W!} a_1'^W (\Sigma\Omega)^W \right\} S(a_0, a_1', a_2'', \ldots, a_n^{(n)}).$$

And what this becomes when we remove all accents is an invariant

$$a_0^W S(a_0, a_1, a_2, \ldots, a_n)$$

of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$.

We see then that $a_0^W S$ is an integral invariant of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ given by an integral invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$, but that S itself, though an integral invariant of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ is integral only in consequence of the cancelling against one another of its fractional parts in virtue of the special equalities among the coefficients, being in fact the representative of the, as a rule, fractional invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$,

$$a_0^{-W} R.$$

ON TWISTED CUBICS AND THE CUBIC TRANSFORMATION OF ELLIPTIC FUNCTIONS.

By *A. C. Dixon, M.A.*

IN the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIII, p. 352, it is found that the modular equation for the cubic transformation of elliptic functions expresses the condition that four straight lines should touch the same twisted cubic curve. The question naturally arises, have the elliptic functions themselves any connexion with the matter? An answer to that question may be given as follows:—

Let the parameters of two points on the cubic be θ and ϕ. Then the six coordinates of the chord joining them are of the second degree in θ and ϕ separately. The condition that this