ON THE SURFACE OF THE ORDER n WHICH PASSES THROUGH A GIVEN CUBIC CURVE.

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It is natural to assume that taking A, B, C to denote the general functions $(x, y, z, w)^{n-2}$ of the order $n-2$, the general surface of the order n which passes through the curve

$$\left\{\begin{matrix} x, & y, & z \\ y, & z, & w \end{matrix}\right\} = 0$$

(or, what is the same thing, the curve $x:y:z:w = 1:\theta:\theta^2:\theta^3$) has for its equation

$$\begin{vmatrix} A, & B, & C \\ x, & y, & z \\ y, & z, & w \end{vmatrix} = 0;$$

but the formal proof is not immediate. Writing the equation in the form $U = S\,ax^\alpha y^\beta z^\gamma w^\delta, = 0$, $\alpha+\beta+\gamma+\delta = n$, then U must vanish on writing therein $x:y:z:w = 1:\theta:\theta^2:\theta^3$; a term $ax^\alpha y^\beta z^\gamma w^\delta$ becomes $= a\theta^p$, where $p = \beta + 2\gamma + 3\delta$ is the weight of the term reckoning the weights of x, y, z, w as 0, 1, 2, 3 respectively; and hence the condition is that for each given weight p the sum Sa of the coefficients of the several terms of this weight shall be $=0$. Using any such equation to determine one of the coefficients thereof in terms of the others, the function U is reduced to a sum of duads $a\,(x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'})$, where in each duad the two terms are of the same degree and of the same weight, and where a is an arbitrary coefficient; it ought therefore to be true that each such duad $x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'}$ has the property in question—or writing $P, Q, R = yw - z^2, zy - xw, xz - y^2$, say that each such duad is of the form $AP + BQ + CR$.

Suppose for a moment that α' is greater than α, but that β', γ', δ' are each less than β, γ, δ respectively: the duad is $x^{\alpha'} y^\beta z^\gamma w^\delta (x^\lambda - y^\mu z^\nu w^\rho)$, where λ, μ, ν, ρ are each positive, and hence $x^\lambda - y^\mu z^\nu w^\rho$ is a duad having the property in question, or changing the notation say $x^\alpha - y^\beta z^\gamma w^\delta$ has the property in question; and in like manner by considering the several cases that may happen we have to show that each of the duads

$$x^\alpha - y^\beta z^\gamma w^\delta,\ y^\beta - x^\alpha z^\gamma w^\delta,\ z^\gamma - x^\alpha y^\beta w^\delta,\ w^\delta - x^\alpha y^\beta z^\gamma,$$
$$x^\alpha y^\beta - z^\gamma w^\delta,\ x^\alpha z^\gamma - y^\beta w^\delta,\ x^\alpha w^\delta - y^\beta z^\gamma,$$

has the property in question; it being of course understood that in each of these duads the two terms have the same degree and the same weight. The first form cannot exist; for we must have therein $\alpha=\beta+\gamma+\delta$ and $0=\beta+2\gamma+3\delta$, which is inconsistent with α, β, γ, δ each of them positive. For the second form $\beta=\alpha+\gamma+\delta$, $\beta=2\gamma+3\delta$, this is $\alpha=\gamma+2\delta$ or the duad is $y^{2\gamma+3\delta}-x^{\gamma+2\delta}z^{\gamma}w^{\delta}, =(y^2)^{\gamma}y^{3\delta}-(xz)^{\gamma}(x^2w)^{\delta}$. Writing $y^2=xz-R$, we have terms containing the factor R, and a residual term $(xz)^{\gamma}\{y^{3\delta}-(x^2w)^{\delta}\}$, and writing herein

$$xw=yz-Q \text{ or } x^2w=xyz-Q,$$

we have terms containing Q as a factor and a residual term $(xz)^{\gamma}\{y^{3\delta}-(xyz)^{\delta}\}, =(xz)^{\gamma}y^{\delta}\{(y^2)^{\delta}-(xz)^{\delta}\}$, and again writing herein $y^2=xz-R$, we see that this term contains the factor R: hence the duad in question consists of terms having the factor R or the factor Q. Similarly for the other cases, either α, β, γ, δ can be expressed as positive numbers, and then the duad consists of terms each divisible by P, Q, or R; or else α, β, γ, δ cannot be expressed as positive numbers, and then the duad does not exist: thus for the third form $z^{\gamma}-x^{\alpha}y^{\beta}w^{\delta}$, here $\gamma=\alpha+\beta+\delta$, $2\gamma=\beta+3\delta$, or say $\gamma=3\alpha+2\beta$, $\delta=2\alpha+\beta$, and the duad is $z^{3\alpha+2\beta}-x^{\alpha}y^{\beta}w^{2\alpha+\beta}, =z^{3\alpha}(z^2)^{2\beta}-(xw^2)^{\alpha}(yw)^{\beta}$, which can be reduced to the required form. But for the duad $x^{\alpha}y^{\beta}-z^{\gamma}w^{\delta}$, we have $\alpha+\beta=\gamma+\delta$, $\beta=2\gamma+3\delta$, which cannot be satisfied by positive values of α, β, γ, δ, and thus the duad does not exist.

A surface of the order n which passes through $3n+1$ points of a cubic curve contains the curve: hence the number of constants or say the capacity of a surface of the order n, through the curve $P=0$, $Q=0$, $R=0$, is

$$\tfrac{1}{6}(n+1)(n+2)(n+3)-1-(3n+1), =\tfrac{1}{6}(n^3+6n^2-7n-6).$$

Primâ facie the capacity of the surface $AP+BQ+CR=0$, A, B, C the general functions of the order $n-2$, is

$$3\,.\tfrac{1}{6}(n-1)n(n+1)-1, =\tfrac{1}{2}(n^3-n-2),$$

but there is a reduction on account of the identical equations $zP+yQ+zR=0$, $yP+zQ+wR=0$ which connect the functions P, Q, R: for $n=2$, the formulæ give each of them as it should do, Capacity $=2$; viz. the quadric surface through the curve is $aP+bQ+cR=0$.