and the final form of the cubic in θ is

$$-\frac{4J\Delta}{I^3}\theta^3 + 27J\theta - 27J = 0,$$

or $$\theta^3 - M(\theta - 1) = 0,$$

where $$\frac{1}{M} = \tfrac{4}{27}\left(1 - \frac{27J^2}{I^3}\right).$$

This is Cayley's cubic. The homographic relation between ζ and θ therefore is

$$\zeta = \frac{3J}{I} \cdot \frac{\theta}{3 - 2\theta}.$$

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A MAP OF THE COMPLEX Z-FUNCTION: A CONDENSER PROBLEM.

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THE object of this note is to consider the correspondence of two planes z, w, whose points are connected by the relation

$$z = Z(w + iK'),$$

where Z is the function of Jacobi so denoted. It appears that the relation gives the solution for an electrical condenser formed of two equal plane strips of infinite length and finite breadth, placed with parallel planes and edges so that a normal section consists of two opposite sides of a rectangle. Writing $z \equiv x + iy$ and $w \equiv \phi + i\psi$, x, y are the rectangular coordinates of the point at which the potential is ψ and the flow ϕ, the plates being at potential $-K'$, K'.

Consider the rectangle in the w plane bounded by $\phi = \pm K$, $\psi = \pm K'$.

The Z-function is single valued, and within this rectangle dz/dw only becomes infinite once, viz. at $w = 0$.

Let us find the path of z when w goes around the rectangle.

Along
$$\psi = -K',$$
$$x = Z(\phi),$$
$$y = 0,$$

so that z lies in the straight line $y=0$.

As ϕ goes from $-K$ to $+K$, x first decreases from 0 to $-Z(\phi_0)$, then increases through 0 to $+Z(\phi_0)$, and finally diminishes to 0. Here ϕ_0 is determined from the equation

$$\frac{dx}{d\phi} = 0,$$

or
$$\mathrm{dn}^2\phi = \frac{E}{K}.$$

Along
$$\phi = K,$$
$$x + iy = Z(i\psi + K + iK');$$

therefore
$$x = 0,$$
$$y = -Z(\psi, k') - \frac{\pi}{2KK'}\psi - \frac{\pi}{2K},$$

so that between $\psi = -K'$ and $\psi = K'$, y goes from 0 to $-\pi/K$ along the line $x=0$.

Along
$$\psi = K',$$
$$x + iy = Z(\phi + 2iK')$$
$$= Z(\phi) - \frac{i\pi}{K},$$

and
$$x = Z(\phi),$$
$$y = -\frac{\pi}{K},$$

Thus between $\phi = K$ and $\phi = -K$, z traverses twice the line $y = -\frac{\pi}{K}$ between $Z(\phi_0)$ and $-Z(\phi_0)$.

Finally, along $\phi = -K$,
$$x + iy = Z(i\psi - K + iK')$$
$$= Z(i\psi + K + iK'),$$

and therefore, between $\psi = K'$ and $\psi = -K'$, z goes along the line $x = 0$ from $-\frac{\pi}{K}$ to 0.

When w is small we have

$$z = \frac{1}{w} \text{ approximately,}$$

consequently, corresponding to an indefinitely small circle around $w = 0$, we have an indefinitely large circle round $z = 0$.

Hence, corresponding to the w area inside the rectangle, we have the whole of the z plane with the parts of the two lines $y = 0$, $y = -\pi/K$ lying between $x = \pm Z(\phi_0)$ as internal boundaries.

The general values of x, y in terms of ϕ, ψ are easily obtained.

We have, generally,

$$Z(u+iv) = Z(u) + Z(iv) - k^2 \operatorname{sn} u \operatorname{sn} iv \operatorname{sn}(u+iv)$$

$$= Z(u) - iZ(v, k') + i\frac{\operatorname{sn}(v, k') \operatorname{dn}(v, k')}{\operatorname{cn}(v, k')} - \frac{i\pi}{2KK'} v$$

$$- ik^2 \operatorname{sn} u \frac{\operatorname{sn}(v, k')}{\operatorname{cn}(v, k')} \frac{\operatorname{sn} u \operatorname{dn}(v, k') + i \operatorname{cn} u \operatorname{dn} u \operatorname{sn}(v, k') \operatorname{cn}(v, k')}{\operatorname{cn}^2(v, k') + k^2 \operatorname{sn}^2 u \operatorname{sn}^2(v, k')}.$$

Writing ψ' for $\psi + K'$, $\operatorname{snc}(\psi')$ for $\operatorname{sn}(\psi', k')$, and so on, we get, after a simple reduction

$$x = Z(\phi) + k^2 \operatorname{sn}\phi \operatorname{cn}\phi \operatorname{dn}\phi \frac{\operatorname{snc}^2 \psi'}{\operatorname{cnc}^2\psi' + k^2 \operatorname{sn}^2\phi \operatorname{snc}^2\psi'},$$

$$y = -Z(\psi', k') - \frac{\pi\psi'}{2KK'}$$

$$+ \operatorname{snc}\psi' \operatorname{cnc}\psi' \operatorname{dnc}\psi' \frac{\operatorname{dn}^2\phi}{\operatorname{cnc}^2\psi' + k^2 \operatorname{sn}^2\phi \operatorname{snc}^2\psi'}.$$

We may now proceed to find an expression for the capacity of such a condenser.

The potentials of the plates are $\pm K'$, the distance between them ϕ/K and their breadth $2Z(\phi_0)$ where ϕ_0 is given by $\operatorname{dn}^2\phi_0 = E/K$. The charge of each plate is $K/2\pi$ per unit length, and therefore the capacity per unit length is $K/4\pi K'$.

When the plates are close together the capacity is large, *i.e.* K/K' large, and therefore k nearly unity.

Putting
$$\chi_0 = \operatorname{am}\phi_0,$$
$$\sin^2\chi_0 = \frac{1}{k^2}\left(1 - \frac{E}{K}\right),$$

and therefore when k' is small χ_0 is nearly $\frac{1}{2}\pi$.

Write
$$\chi_0 = \tfrac{1}{2}\pi - \lambda,$$
giving
$$\sin^2\lambda = \frac{1}{k^2}\left(\frac{E}{K} - k'^2\right);$$

and therefore λ^2 is of the order $\dfrac{1}{K}$ or $\dfrac{1}{\log 4/k'}$, so that λ^2/k'^2 is of the order $\dfrac{1}{k'^2 \log 4/k'}$ which becomes indefinitely great as k' decreases.

To find $Z(\phi_0)$ we therefore require expansions of $E(\frac{1}{2}\pi - \lambda)$, $F(\frac{1}{2}\pi - \lambda)$, when λ, k' are both small and λ/k' is large.

Now
$$E(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \sqrt{(\cos^2\phi + k'^2\sin^2\phi)}\, d\phi,$$
$$F(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \frac{1}{\sqrt{(\cos^2\phi + k'^2\sin^2\phi)}}\, d\phi.$$

Since $\dfrac{k'^2}{\cot^2\phi} < \dfrac{k'^2}{\tan^2\lambda}$ throughout the integration, we can expand the integrals in the convergent series
$$E(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \left[\cos\phi + \tfrac{1}{2}k'^2\frac{\sin^2\phi}{\cos\phi} - \dots\right] d\phi,$$
$$F(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \left[\frac{1}{\cos\phi} - \tfrac{1}{2}k'^2\frac{\sin^2\phi}{\cos^3\phi} + \dots\right] d\phi,$$
and*
$$E(\tfrac{1}{2}\pi - \lambda) = \cos\lambda + \tfrac{1}{2}k'^2(\log\cot\tfrac{1}{2}\lambda - \cos\lambda) - \dots,$$
$$F(\tfrac{1}{2}\pi - \lambda) = \log\cot\tfrac{1}{2}\lambda - \tfrac{1}{2}k'^2\left(\tfrac{1}{2}\frac{\cos\lambda}{\sin^2\lambda} - \tfrac{1}{2}\log\cot\tfrac{1}{2}\lambda\right) + \dots .$$

* Equivalent expansions seem to have been first given by Verhulst in his "Fonctions Elliptiques."

We require further the expansions

$$K = \mu + \tfrac{1}{4}k'^2(\mu - 1) + \ldots,$$

$$E = 1 + \tfrac{1}{2}k'^2(\mu - \tfrac{1}{2}) + \ldots,$$

$$K' = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}k'^2) + \ldots,$$

where $\mu = \log 4/k'$.

Let l be the ratio of the breadth of the condenser plates to the distance between them, c the capacity per unit length.

Then $$l\pi = 2E(\chi_0)K - 2F(\chi_0)E,$$

$$c = K/4\pi K',$$

where $$\sin^2\chi_0 = \frac{1}{k^2}\left(1 - \frac{E}{K}\right).$$

By means of the expansions first given, we can carry the expression of c in terms of l to any order of approximation.

In the cases of greatest interest a very accurate expression is got by retaining k'^2 only.

Here

$$\sin^2\lambda = (1 + k'^2)\left\{\frac{1}{\mu}\,\frac{1 + \tfrac{1}{2}k'^2(\mu - \tfrac{1}{2})}{1 + \tfrac{1}{4}k'^2\left(1 - \dfrac{1}{\mu}\right)} - k'^2\right\}$$

$$= (1 + k'^2)\left[\frac{1}{\mu}\left\{1 + \tfrac{1}{2}k'^2\left(\mu - 1 + \frac{1}{2\mu}\right) - k'^2\right\}\right]$$

$$= \frac{1}{\mu}\left\{1 - \tfrac{1}{2}k'^2\left(\mu - 1 - \frac{1}{2\mu}\right)\right\},$$

$$\sin\lambda = \frac{1}{\sqrt{\mu}}\left\{1 - \tfrac{1}{4}k'^2\left(\mu - 1 - \frac{1}{2\mu}\right)\right\},$$

$$\cos^2\lambda = 1 - \frac{1}{\mu} + \frac{1}{2}\frac{k'^2}{\mu}\left(\mu - 1 - \frac{1}{2\mu}\right)$$

$$= \nu^2\left\{1 + \tfrac{1}{2}k'^2\left(1 - \frac{1}{2\mu^2\nu^2}\right)\right\},$$

where $$\nu^2 = 1 - \frac{1}{\mu},$$

$$\cos\lambda = \nu\left\{1 + \tfrac{1}{4}k'^2\left(1 - \frac{1}{2\mu^2\nu^2}\right)\right\}.$$

Hence

$$\log\cot\tfrac{1}{2}\lambda = \tfrac{1}{2}\log\frac{1+\cos\lambda}{1-\cos\lambda}$$

$$= \tfrac{1}{2}\log\frac{1+\nu+\frac{1}{4}k'^2\nu\left(1-\frac{1}{2\mu^2\nu^2}\right)}{1-\nu-\frac{1}{4}k'^2\nu\left(1-\frac{1}{2\mu^2\nu^2}\right)}$$

$$= \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{4}k'^2\mu\nu\left(1-\frac{1}{2\mu^2\nu^2}\right).$$

So that

$$E(\tfrac{1}{2}\pi-\lambda) = \nu\left\{1+\tfrac{1}{4}k'^2\left(1-\frac{1}{2\mu^2\nu^2}\right)\right\} + \tfrac{1}{2}k'^2\left(\tfrac{1}{2}\log\frac{1+\nu}{1-\nu} - \nu\right)$$

$$= \nu + \tfrac{1}{4}k'^2\left\{\log\frac{1+\nu}{1-\nu} - \nu\left(1+\frac{1}{2\mu^2\nu^2}\right)\right\},$$

$$F(\tfrac{1}{2}\pi-\lambda) = \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{4}k'^2\mu\nu\left(1-\frac{1}{2\mu^2\nu^2}\right)$$

$$- \tfrac{1}{2}k'^2\left(\tfrac{1}{2}\mu\nu - \tfrac{1}{4}\log\frac{1+\nu}{1-\nu}\right)$$

$$= \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{8}k'^2\left(\log\frac{1+\nu}{1-\nu} - \frac{1}{\mu\nu}\right).$$

Therefore

$$\tfrac{1}{2}l\pi = \left[\nu + \tfrac{1}{4}k'^2\left\{\log\frac{1+\nu}{1-\nu} - \nu\left(1+\frac{1}{2\mu^2\nu^2}\right)\right\}\right]\{1+\tfrac{1}{4}k'^2\nu^2\}\,\mu$$

$$- \left[\tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{8}k'^2\left(\log\frac{1+\nu}{1-\nu} - \frac{1}{\mu\nu}\right)\right]\{1+\tfrac{1}{2}k'^2(\mu-\tfrac{1}{2})\}$$

$$= \mu\nu - \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} - \tfrac{1}{4}k'^2\nu,$$

and $$2\pi^2 c = \mu\left[1 + \tfrac{1}{4}k'^2\left(1 - \frac{1}{\mu}\right)\right]\left[1 - \tfrac{1}{4}k'^2\right]$$

$$= \mu - \tfrac{1}{4}k'^2.$$

Hence as a first approximation

$$\mu = 2\pi^2 c,$$

$$k' = 4e^{-2\pi^2 c},$$

and, as a second approximation,

$$\mu = 2\pi^2 c + 4e^{-4\pi^2 c},$$

$$\nu^2 = 1 - \frac{1}{2\pi^2 c} + \frac{1}{\pi^4 c^2} e^{-4\pi^2 c}.$$

Substituting these values in the expression for l, we have l in terms of c, neglecting terms of order k'^4.

If l is as large as 10, the terms in k'^2 would also in general be negligible, and we should then get

$$\tfrac{1}{2}\pi l = \mu\nu - \tfrac{1}{2}\log_e \frac{1 + \sqrt{\left(1 - \frac{1}{\mu}\right)}}{1 - \sqrt{\left(1 - \frac{1}{\mu}\right)}}.$$

and $$\mu = 2\pi^2 c,$$

so that

$$\tfrac{1}{2}\pi l = 2\pi^2 c\sqrt{\left(1 - \frac{1}{2\pi^2 c}\right)} - \tfrac{1}{2}\log_e \frac{1 + \sqrt{\left(1 - \frac{1}{2\pi^2 c}\right)}}{1 - \sqrt{\left(1 - \frac{1}{2\pi^2 c}\right)}}.$$

For the general theory of the conformal representation of polygonal areas on which the present solution is based, it will be sufficient to refer to Chap. XX. of Forsyth's *Theory of Functions*.