EVEN MAGIC SQUARES.

By *W. W. Rouse Ball.*

A MAGIC square consists of a number of integers arranged in the form of a square, so the sum of the numbers in every row, in every column, and in each diagonal is the same. If the integers are the consecutive numbers from 1 to n^2, the square is said to be of the n^{th} order, and in this case the sum of the numbers in any row, column, or diagonal is equal to N, where $N = \frac{1}{2}n\,(n^2+1)$.

The construction of a magic square of any odd order presents no difficulty, and rules for the formation of such squares have been well known for more than two centuries.

The construction of a magic square of any even order higher than two is possible, and rules for the formation of such squares have been established, but the application of these rules is not always simple. Other methods which are partly empirical cannot be considered as mathematically satisfactory. I propose here to establish a general method which is easy to apply, and which I believe, at any rate as far as singly-even squares are concerned, to be new. The substance of it was communicated to me by Mr. C. H. Harrison, but for the presentation and some additions on the selection of the cells whose numbers are to be interchanged I am responsible.

Probably the following notation is known to the reader, though for the sake of completeness I add it. (i) The square may be divided by horizontal and vertical lines into n^2 small squares in each of which a number has to be written: the small squares are called *cells.* (ii) Two rows which are equidistant, the one from the top, the other from the bottom, are said to be *complementary.* (iii) Two columns which are equidistant, the one from the left-hand side, the other from the right-hand side, are said to be *complementary.* (iv) Two cells in the same row, but in complementary columns, are said to be *horizontally related.* (v) Two cells in the same column, but in complementary rows, are said to be *vertically related.* (vi) Two cells in complementary rows and columns are said to be *skewly selected*; thus, if the cell b is horizontally related to the cell a, and the cell d is vertically related to the cell a, then the cells b and d are skewly related; in such a case if the cell c is vertically related to the cell b, it will be horizontally related to the cell d, and the cells a and c are skewly related: the cells a, b, c, d constitute an *associated group*, and if the square is divided into four equal quarters, one cell of an associated group is in each quarter.

A *horizontal interchange* consists in the interchange of the numbers in two horizontally related cells. A *vertical interchange* consists in the interchange of the numbers in two vertically related cells. A *skew interchange* consists in the interchange of the numbers in two skewly related cells. A *cross interchange* consists in the change of the numbers in any cell and in its horizontally related cell with the numbers in the cells skewly related to them; hence, it is equivalent to two vertical interchanges and two horizontal interchanges.

I suppose that the cells are initially filled with the numbers $1, 2, \ldots, n^2$ in their natural order commencing (say) with the top left-hand corner, writing the numbers in each row from left to right, and taking the rows in succession from the top. I will begin by proving that a certain number of horizontal and vertical interchanges in such a square must make it magic, and will then give a rule by which the cells whose numbers are to be interchanged can be at once picked out.

First, we may notice that the sum of the numbers in each diagonal is equal to N; hence the diagonals are already magic, and will remain so if the numbers therein are not altered.

Next, consider the rows. The sum of the numbers in the x^{th} row from the top is $N - \frac{1}{2}n^2(n-2x+1)$. The sum of the numbers in the complementary row, that is, the x^{th} row from the bottom, is $N + \frac{1}{2}n^2(n-2x+1)$. Also the number in any cell in the x^{th} row is less than the number in the cell vertically related to it by $n(n-2x+1)$. Hence, if in these two rows we make $\frac{1}{2}n$ interchanges of the numbers which are situated in vertically selected cells, then we increase the sum of the numbers in the x^{th} row by $\frac{1}{2}n \times n(n-2x+1)$, and therefore make that row magic; while we decrease the sum of the numbers in the complementary row by the same number, and therefore make that row magic. Hence, if in every pair of complementary rows we make $\frac{1}{2}n$ interchanges of the numbers situated in vertically related cells, the square will be made magic in rows. But, in order that the diagonals may remain magic, either we must leave both the diagonal numbers in any row unaltered, or we must change both of them with those in the cells vertically related to them.

The square is now magic in diagonals and in rows, and it remains to make it magic in columns. Taking the original arrangement of the numbers (in their natural order) we might have made the square magic in columns in a similar way to that in which we made it magic in rows. The sum of the numbers originally in the y^{th} column from the left-hand side is $N - \frac{1}{2}n(n-2y+1)$. The sum of the numbers originally in the complementary column, that is, the y^{th} column from the right-hand side, is $N + \frac{1}{2}n(n-2y+1)$. Also the number originally in any cell in the y^{th} column was less than the number in the cell horizontally related to it by $n-2y+1$. Hence, if in these two columns we had made $\frac{1}{2}n$ interchanges of the numbers situated in horizontally related cells, we should have made the sum of the numbers in each column equal to N. If we had done this in succession for every pair of complementary colums, we should have made the square magic in columns. But, as before, in order that the diagonals might remain magic, either we must have left both the diagonal numbers in any column unaltered, or we must have changed both of them with those in the cells horizontally related to them.

It remains to shew that the vertical and horizontal interchanges, which have been considered in the last two paragraphs, can be made independently, that is, that we can make these interchanges of the numbers in complementary columns in such a manner as will not affect the numbers already interchanged in complementary rows. This will require that in every column there shall be exactly $\frac{1}{2}n$ interchanges of the numbers in vertically related cells, and that in every row there shall be exactly $\frac{1}{2}n$ interchanges of the numbers in horizontally related cells. I proceed to shew how we can always ensure this, if n is greater than 2. I continue to suppose that the cells are initially filled with the numbers 1, 2, ..., n^2 in their natural order, and that we work from that arrangement.

A doubly-even square is one where n is of the form $4m$. If the square is divided into four equal quarters, the first quarter will contain $2m$ columns and $2m$ rows. In each of these columns take m cells so arranged that there are also m cells in each row, and change the numbers in these $2m^2$ cells and the $6m^2$ cells associated with them by a cross interchange. The result is equivalent to $2m$ interchanges in every row and in every column, and therefore renders the square magic.

One way of selecting the $2m^2$ cells in the first quarter is to divide the whole square into sixteen subsidiary

a	b	b	a
b	a	a	b
b	a	a	b
a	b	b	a

squares each containing m^2 cells, which we may represent by the diagram above, and then we may take either the cells in the a squares or those in the b squares; thus, if every number in the eight a squares is interchanged with the number skewly related to it the resulting square is magic.

Another way of selecting the $2m^2$ cells in the first quarter would be to take the first m cells in the first column, the cells 2 to $m+1$ in the second column, and so on, the cells $m+1$ to $2m$ in the $(m+1)^{\text{th}}$ column, the cells $m+2$ to $2m$ and the first cell in the $(m+2)^{\text{th}}$ column, and so on, and finally the $2m^{\text{th}}$ cell and the cells 1 to $m-1$ in the $2m^{\text{th}}$ column.

A singly-even square is one where n is of the form $2(2m+1)$. If the square is divided into four equal quarters, the first quarter will contain $2m+1$ columns and $2m+1$ rows. In each of these columns take m cells so arranged that there are also m cells in each row: as, for instance, by taking the first m cells in the first column, the cells 2 to $m+1$ in the second column, and so on, the cells $m+2$ to $2m+1$ in the $(m+2)^{\text{th}}$ column, the cells $m+3$ to $2m+1$ and the first cell in the $(m+3)^{\text{th}}$ column, and so on, and finally the $(2m+1)^{\text{th}}$ cell and the cells 1 to $m-1$ in the $(2m+1)^{\text{th}}$ column. Next change the numbers in these $m(2m+1)$ cells and the $3m(2m+1)$ cells associated with them by cross interchanges. The result is equivalent to $2m$ interchanges in every row and in every column. In order to make the square magic we must have $\frac{1}{2}n$, that is, $2m+1$ such interchanges in every row and in every column, that is, we must have one more interchange in every row and in every column. This presents no difficulty, for instance, in the arrangement indicated above the numbers in the $(2m+1)^{\text{th}}$ cell of the first column, in the first cell of the second column, in the second cell of the third column, and so on, to the $2m^{\text{th}}$ cell in the $(2m+1)^{\text{th}}$ column may be interchanged with the numbers in their vertically related cells; this will make all the rows magic. Next, the numbers in the $2m^{\text{th}}$ cell of the first column, in the $(2m+1)^{\text{th}}$ cell of the second column, in the first cell of the third c lumn, in the second cell of the fourth column, and so on, to the $(2m-1)^{\text{th}}$ cell of the $(2m+1)^{\text{th}}$ column may be interchanged with those in the cells horizontally related to them; and this will make the columns magic without affecting the magical properties of the rows.

It will be observed that we have implicitly assumed that m is not zero, *i.e.* that n is greater than 2; also it would seem that, if $m=1$ and therefore $n=6$, then the numbers in the diagonal cells must be included in those to which the cross interchange is applied, but, if $n>6$, this is not necessary, though it may be convenient.

A RECTANGULAR HYPERBOLA CONNECTED WITH A TRIANGLE.

By *W. W. Taylor*.

If P, Q are any two points in the plane of the triangle ABC, there exists a third point R, such that the following pencils are equal in pairs;

$$A[BPRC]=A[CQRB],\ B[CPRA]=B[AQRC],$$
$$C[APRB]=C[BQRA].$$

If coordinates of P, Q, R be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; h, k, l respectively, the anharmonic ratio of the pencil $A[BPRC]$

$$=-\frac{\gamma_1}{\beta_1}\Big/\left(\frac{l}{k}-\frac{\gamma_1}{\beta_1}\right);$$

therefore, on reducing, the above equations become

$$\alpha_1\alpha_2:\beta_1\beta_2:\gamma_1\gamma_2=h^2:k^2:l^2.$$

We shall call Q the *reciprocal of P with respect to R*. It follows from this definition that, if through R the straight line be drawn, whose intercept between AB and AC is bisected at R, the intercept between AP and AQ on the same straight line is also bisected at R, and similarly for the other sides. Now let us consider the rectangular hyperbola whose equation in trilinear coordinates is

$$(b^2-c^2)\,\alpha^2+(c^2-a^2)\,\beta^2+(a^2-b^2)\,\gamma^2=0.$$