where $m = 0, 1$; $n = 0, 1, 2$; and α, β take all positive or negative values.

Again in this case a relation

$$(PQP^2Q)^m = 1$$

obviously reduces the group to one of order $6m^2$.

The distinct triangles corresponding to the finite groups in these cases can always be chosen so as to fill a square in the one case and a regular hexagon in the other. The figure corresponding to the last case is a particularly interesting one.

TWO THEOREMS ON PRIME NUMBERS.

By *N. M. Ferrers.*

1. If $2p+1$ be any prime number, the sum of the products of the integers 1, 2, ..., $2p$, taken r together, r being any integer less than $2p$ is divisible by $2p+1$.

For, x denoting any integer whatever,

$$x(x+1)\dots(x+2p) \text{ is divisible by } 2p+1,$$

therefore if x be not divisible by $2p+1$,

$$(x+1)(x+2)\dots(x+2p) \text{ is so,}$$

or, denoting the sum of the products of the integers 1, 2, ..., $2p$ taken r together by S_r,

$$x^{2p} + S_1 x^{2p-1} + \dots + S_{2p-1} x + 1.2\dots 2p$$

is divisible by $2p+1$.

But by Fermat's Theorem,

$$x^{2p} - 1 \text{ is so divisible,}$$

and by Wilson's Theorem,

$$1.2\dots 2p + 1 \text{ is so divisible,}$$

therefore $x^{2p} + 1.2\dots 2p$ is so.

Therefore removing these terms, and dividing by x,

$$S_1 x^{2p-2} + S_2 x^{2p-3} + \dots + S_{2p-1}$$

is divisible by $2p+1$ for all values of x, from 1 to $2p$ inclusive.

Hence, putting respectively 1, 2, ..., $2p-1$ for x, we get

$$S_1 + S_2 + \ldots + S_{2p-1} = M_1(2p+1),$$
$$S_1 . 2^{2p-2} + S_2 . 2^{2p-3} + \ldots + S_{2p-1} = M_2(2p+1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots,$$
$$S_1 . (2p-1)^{2p-2} + S_2(2p-1)^{2p-3} + \ldots + S_{2p-1} = M_{2p-1}(2p+1),$$

M_1, M_2, ..., M_{2p-1} denoting positive integers.

Hence, if all but one of the $2p-1$ quantities S be eliminated from the $2p-1$ equations, the coefficient of the remaining one will be the determinant

$$\begin{array}{l} 1,\ 1,\ \ldots,\ 1, \\ 2^{2p-2} . 2^{2p-3} \ldots 1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ (2p-1)^{2p-2} . (2p-1)^{2p-3} \ldots 1, \end{array}$$

which is equal to the product of the several differences of pairs of the integers 1, 2, ..., $(2p-1)$, and therefore can involve no factor greater than $2p-2$. And the right-hand side of the resulting equation is necessarily a multiple of $2p+1$. Hence, each of the quantities S is so.

2. The sum of the products of the squares of the integers 1, 2, ..., p, taken r together, r being any integer less than p, is divisible by $2p+1$.

For the product $(x-p)\{x-(p-1)\} \ldots x(x+1) \ldots (x+p)$ is divisible by $2p+1$.

Therefore, if x be not so divisible,

$$(x^2-1^2)(x^2-2^2) \ldots (x^2-p^2) \text{ is so divisible.}$$

Now by Wilson's Theorem,

$$1.2 \ldots 2p+1 \text{ is divisible by } 2p+1,$$

therefore subtracting $2p+1$ and dividing by $2p$,

$$1^2 . 2 \ldots (2p-1) - 1 \text{ is so.}$$

Therefore multiplying by 2, adding $2p+1$, and dividing by $2p-1$,

$$1^2 . 2^2 \ldots (2p-2) + 1 \text{ is so.}$$

Repeating this process, we at length find that

$$1^2 . 2^2 \ldots p^2 + (-1)^p \text{ is so.}$$

Also $x^{2p} - 1$ is so divisible,

therefore $\quad x^{2p} + (-1)^p \, 1^2 . 2^2 \ldots p^2$ is so divisible.

Hence, since we have seen that $(x^2-1^2)\dots(x^2-p^2)$ is so divisible, we find that

$$\Sigma_1 x^{2p-2} - \Sigma_2 x^{2p-4} + \dots \pm \Sigma_{p-1} x^2 \text{ is so,}$$

where Σ_r denotes the sum of the squares of the quantities 1, 2, ..., p, taken r together,

therefore $\quad \Sigma_1 x^{2p-4} - \Sigma_2 x^{2p-6} + \dots \pm \Sigma_{p-1}$ is so divisible,

therefore, writing successively 1, 2, ..., $(p-1)$ for x, we see, as before, that each of the quantities Σ is so divisible.

[In vol. xxii., p. 51, Mr. Osborn proved that, if p be a prime greater than 3, the numerator of the harmonical progression

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-1}$$

is divisible by p^2, remarking that it seemed very unlikely that this theorem should not have been given before. Dr. Ferrers has pointed out to me that it was given by Wolstenholme in vol. v., p. 35 (1861) of the *Quarterly Journal*. It is there shown (1) that, if p is a prime greater than 3, the numerator of

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-1}$$

is divisible by p^2, (2) that the numerator of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(p-1)^2}$$

is divisible by p, and (3) that the number of combinations of $2p-1$ things, taken $p-1$ together, diminished by 1, is divisible by p^3.

In giving me this reference, Dr Ferrers mentioned that he had had by him for many years some other results of the same class. These theorems which he kindly wrote out with their demonstrations, at my request, form the above paper.—*Ed.*].
