But by the Lemma, whatever x_0 may be, an increasing series of positive integers m, ... can be found such that $(x_0 - \tan m\alpha)^2$, and therefore also $(x_0 - \tan m\alpha)^2 + a^{-2m}$, is less than any assignable quantity. Hence, there must be terms in the series for $f(x)$ which can only be expanded in negative powers of $x - x_0$, whatever value x_0 may have; and therefore for no real value of x_0 can $f(x)$ be expanded in a series of positive powers.

ON RICHELOT'S INTEGRAL OF THE DIFFERENTIAL EQUATION $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$.

By Prof. *Cayley*.

In the Memoir "Einige Neue Integralgleichungen des Jacobi'schen Systems Differentialgleichungen" *Crelle* t. 25 (1843) pp. 97-118, Richelot, working with the more general problem of a system of $n-1$ differential equations between n variables, obtains a result which in the particular case $n=2$ (that is for the differential equation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,\ X = a + bx + cx^2 + dx^3 + ex^4,$$

and Y the same function of y), is in effect as follows: an integral is

$$\left\{\frac{\sqrt{X}(\theta - y) - \sqrt{Y}(\theta - x)}{x - y}\right\}^2$$
$$= \square\,(\theta - x)(\theta - y) + \Theta + e(\theta - x)^2(\theta - y)^2,$$

where $\square$, θ are arbitrary constants, and Θ denotes the quartic function $a + b\theta + c\theta^2 + d\theta^3 + e\theta^4$;
viz. this is theorem 3, p. 107, taking therein $n=2$, and writing θ, $\square$ for Richelot's α and const.

The peculiarity is that the integral contains apparently *two* arbitrary constants, and it is very interesting to show how these really reduce themselves to a single arbitrary constant.

Observe that on the right-hand side there are terms in θ^4, θ^3 whereas no such terms present themselves on the left-hand side. But by changing the constant $\square$, we can get rid of these terms, and so bring each side to contain only terms in θ^2, θ, 1; viz. writing $\square=-2e\theta^2-d\theta-c+C$, where C is a new arbitrary constant, the equation becomes

$$\left\{\frac{\sqrt{X}(\theta-y)-\sqrt{Y}(\theta-x)}{x-y}\right\}^2$$
$$\begin{aligned}=\ &\theta^2[\quad e(x+y)^2 \quad +d(x+y)+C \quad]\\ &+\theta\,[-2e\,xy(x+y)-dxy \quad -(C-c)(x+y)+b]\\ &+\ [\quad e\,x^2y^2 \quad +(C-c)\,xy \quad +a],\end{aligned}$$

which still contains the two arbitrary constants θ, C.

But this gives the three equations

$$\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}=e(x+y)^2+d(x+y)+C,$$

$$-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2}$$
$$=-2e\,xy(x+y)-dxy-(C-c)(x+y)+b$$

$$\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}=e\,x^2y^2+(C-c)\,xy+a.$$

The first of these is Lagrange's integral containing the arbitrary constant C; and it is necessary that the three equations shall be one and the same equation; viz. the second and third equations must be each of them a mere transformation of the first equation.

It is easy to verify that this is so. Starting from the first equation, we require first the value of

$$-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2},=\Omega,\text{ for a moment.}$$

We form a rational combination, or combination without any term in $\sqrt{XY}$; this is

$$(x+y)\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2}$$
$$=e(x+y)^3+d(x+y)^2+C(x+y)+\Omega,$$

where the left-hand side is

$$\frac{(x-y)(X-Y)}{(x-y)^2},=\frac{X-Y}{x-y},$$

which is

$$=e(x^3+x^2y+xy^2+y^3)+d(x^2+xy+y^2)+c(x+y)+b,$$

and we thence have for

$$\Omega, =-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2},$$

the value given by the second equation.

Secondly, starting again from the first equation, and proceeding in like manner to find the value of

$$\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}, =\Omega, \text{ for a moment,}$$

we form a rational combination

$$-xy\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}+\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}$$
$$=-e\,xy(x+y)^2-d\,xy(x+y)-Cxy+\Omega,$$

where the left-hand side is

$$\frac{(x-y)(-yX+xY)}{(x-y)^2}, =\frac{-yX+xY}{x-y},$$

which is

$$=-e\,xy(x^2+xy+y^2)-d\,xy(x+y)-c\,xy+a;$$

and we thence have for

$$\Omega, =\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}$$

the value given by the third equation.

In conclusion, I give what is in effect the process by which Richelot obtained his integral. The integral is $v=\square$, where

$$v=\frac{-\Theta}{\theta-x.\theta-y}-e(\theta-x.\theta-y)+(\theta-x.\theta-y)\Omega^2,$$

if, for shortness,

$$\Omega=\frac{\sqrt{X}}{\theta-x.x-y}+\frac{\sqrt{Y}}{\theta-y.y-x},$$

and it is required thence to show that $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, or, what is the same thing, to show that v satisfies the partial differential equation

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=0.$$

We have

$$\frac{dv}{dx}=\frac{-\Theta}{(\theta-x)^2(\theta-y)}+e(\theta-y)-(\theta-y)\Omega^2$$
$$+2(\theta-x)(\theta-y)\Omega\frac{d\Omega}{dx},$$

$$\frac{dv}{dy}=\frac{-\Theta}{(\theta-x)(\theta-y)^2}+e(\theta-x)-(\theta-x)\Omega^2$$
$$+2(\theta-x)(\theta-y)\Omega\frac{d\Omega}{dy},$$

and thence, attending to the value of Ω,

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=\frac{-\Theta}{\theta-x\,.\,\theta-y}(x-y)\Omega$$
$$+(e-\Omega^2)(\theta-x)(\theta-y)(x-y)\Omega$$
$$+2(\theta-x)(\theta-y)\Omega\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right),$$

or say

$$-\frac{\left(\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}\right)}{(\theta-x)(\theta-y)(x-y)\Omega}$$
$$=\frac{\Theta}{(\theta-x)^2(\theta-y)^2}-e+\Omega^2-\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right),$$

and it is consequently to be shown that the function on the right hand side is $=0$. We have

$$\sqrt{X}\frac{d\Omega}{dx}=\frac{\frac{1}{2}X'}{(\theta-x)(x-y)}+\frac{X}{(\theta-x)^2(x-y)}$$
$$-\frac{X}{(\theta-x)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-y)(x-y)^2},$$

$$\sqrt{Y}\frac{d\Omega}{dy}=\frac{\frac{1}{2}Y'}{(\theta-y)(y-x)}+\frac{Y}{(\theta-y)^2(y-x)}$$
$$-\frac{Y}{(\theta-y)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-x)(x-y)^2},$$

and thence

$$\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}=\frac{\tfrac{1}{2}X'}{(\theta-x)(x-y)}-\frac{\tfrac{1}{2}Y'}{(\theta-y)(y-x)}$$
$$+\left\{\frac{X}{(\theta-x)^2}+\frac{Y}{(\theta-y)^2}\right\}\frac{1}{x-y}$$
$$-\left(\frac{X}{\theta-x}-\frac{Y}{\theta-y}\right)\frac{1}{(x-y)^2}$$
$$-\frac{\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)},$$

or multiplying by $\dfrac{2}{x-y}$, we may put the result in the form

$$\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right)=\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}+\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(\theta-y)^2}$$
$$+\frac{2X}{(\theta-x)^2(x-y)^2}+\frac{2Y}{(\theta-x)^2(x-y)^2}-\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2}.$$

and the equation to be verified thus is

$$0=\frac{\Theta}{(\theta-x)^2(\theta-y)^2}-e+\Omega^2$$
$$-\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}-\frac{2X}{(\theta-x)^2(x-y)^2}$$
$$-\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}-\frac{2Y}{(\theta-x)^2(x-y)^2}$$
$$+\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2}.$$

But decomposing the first term into simple fractions, we have

$$\frac{\Theta}{(\theta-x)^2(\theta-y)^2}=+e$$
$$+\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}+\frac{X}{(\theta-x)^2(x-y)^2}$$
$$+\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}+\frac{Y}{(\theta-y)^2(x-y)^2}.$$

Also for the third term, we have

$$\Omega^2 = \frac{X}{(\theta - x)^2 (x-y)^2} + \frac{Y}{(\theta - y)^2 (x-y)^2} - \frac{2\sqrt{(XY)}}{(\theta - x)(\theta - y)(x-y)^2},$$

and substituting these values the several terms destroy each other, so that the right-hand side is $=0$ as it should be.

LIMITS OF THE EXPRESSION $\dfrac{x^p - y^q}{x^q - y^q}$.

By H. W. Segar.

§ 1. In a paper with the above title in *Messenger*, XXII., 165–171, Mr. S. R. Knight gives the theorem:—

'If the quantities x and y are positive, and if the quantities p and q are real, then $\dfrac{x^p - y^p}{x^q - y^q}$ lies between $\dfrac{p}{q} x^{p-q}$ and $\dfrac{p}{q} y^{p-q}$,' which is really not more general than that of which Prof. Chrystal makes such frequent application in the second volume of his 'Algebra'; and he discusses the inequalities that exist between these three expressions when p or q, or both are negative.

The same theorem and all these inequalities in the different cases are practically given in *Messenger*, XXII., 47, and they there appear in the form

$$\frac{1 - \left(\frac{c}{b}\right)^n}{n} > \frac{1 - \left(\frac{c}{b}\right)^m}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where, as is at once evident from the method of proof, b and c are any two unequal positive quantities, m is numerically greater than n, and n may be positive or negative, but the