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Compliance for uncertain inventories: Yet another look?

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Abstract The paper deals with the problem of compliance when the inventories are uncertain. This problem is treated here from the view of comparison of uncertain alternatives. We emphasize inadequacy of using reported emission estimates, as those obtained from emission inventories, when they are subject to high uncertainty. Meanwhile, there exists a number of techniques to rank such uncertain estimates. Several of them are presented in the paper. Probabilistic and fuzzy approaches are considered and compared. Many of them can be adapted to check fulfillment of obligations on the basis of knowledge of uncertain emission estimates.

Keywords: greenhouse gases inventories, compliance, uncertain alternatives, ranking

1 Introduction

A handful of solutions have been proposed to cope with the problem of commitment verification for emission obligations in case of uncertain inventories, see [16]. Many of them pointed to methodological incompetence in using reported (crisp) values in clearing pollutant emission targets. For most environmental problems, only highly inexact knowledge on emission values is available, as is the case of greenhouse gases, see e.g. [15, 17, 20].

To give an example of a paradox arising from dealing with uncertain inventories, let us consider verification of a single emission inventory x against a given limit K, i.e. $x \leq K$. A distribution of an inventory uncertainty $\mu(x)$ may

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be asymmetric. See an example in Figure 1 depicting a histogram of emission inventory uncertainty for Austria, obtained by the Monte Carlo method, [29]. Let us suppose that an emission target for Austria is like that marked in Figure 1. The question of interest is whether this party fulfills its commitment, or not. Ignoring uncertainty, the answer is *yes*, as the reported value of inventory lies below the target. However, although the reported value is just below the target, it is more likely that the actual emission is above the target, because most of the possible emissions (probability mass) is placed to the right of the target value. Can we then responsibly accept the answer *yes*?

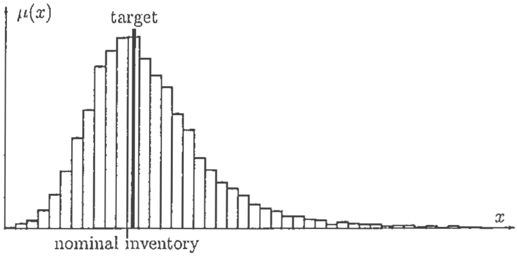

Fig. 1 An example of an asymmetric uncertainty distribution of a national inventory: compliant or noncompliant?

Another example with simplified uncertainty distributions is depicted in Figure 2, with the x axis placed *ad hoc*. Let us consider two parties with known triangular distributions of emission uncertainties. The reported inventories (the dominant values of the distributions densities) of both parties are very close to each other. Ignoring uncertainty, the party A will be considered compliant (fulfilling the commitment), while the party B will be considered noncompliant. However, confidence in the inventory value of the party B is high, while the confidence in the inventory value of the party A is much lower. Therefore, which party is more credible? Should the party A be considered compliant, while the party B should not?

According to IPCC Good Practice Guidelines [12], the reports should be "consistent, comparable and transparent". It is, thus, reasonable to require that decision on fulfillment of obligations should be fair among parties, in the way, that ordering of inventories should make it possible to decide which inventory outperforms others. From the above examples we can see that, when dealing with uncertain values of possibly asymmetric distributions, taking decisions on fulfillment of obligations or comparison of inventories only on the basis of reported inventories may contradict simple conclusions inferred from the uncertainty distributions interpreted as a probabilty distribution.

For the greenhouse gases, reduction of inventory is often defined in percents, i.e. $x_c \leq \rho x_b$, where x_c is an emission inventory in the compliance period, x_b

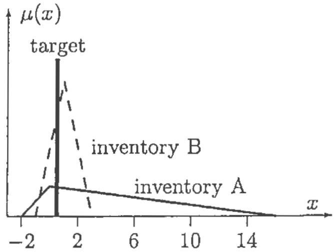

Fig. 2 Which inventory is smaller, A or B?

is an emission inventory in the basic year (at the beginning of the reduction period), and ρ is a required fraction of emission reduction. Here, again, the task is to compare uncertain inventories in the compliance year, x_c, with inventories reduced from the basic year, ρx_b, and to decide whether the former is lower than the latter. Thus, the method presented here an be also useful for making decisions in his case.

In the paper it is assumed that the distribution of uncertainty of an inventory is given. This not always can be true. Some countries undergo an effort of carrying out Monte Carlo calculations from which one can get quite a good insight of how the country's uncertainty distribution looks like. Some other countries report only either uncertainty interval or even standard deviation. Many of the methods presented in the paper will work even with the interval information. It can be there interpreted as a uniform distribution of uncertainty.

In Section 2 we present different probabilistic, and in Section 3 fuzzy-rooted techniques, which can deal with the uncertain inventories. Section 4 concludes.

2 Probabilistic approaches

2.1 Introductory remarks

Treating an inventory as a random value with probabilistic distribution seems to be self-imposing, although it perhaps does not completely comply with the randomness asumptions.

Comparison of uncertain random values has been already considered in various fields. The problem of selection from risky projects has a long history in such areas as finance, R&D projects, IT projects, [9]. Several methods have been proposed there to compare such projects. The methods can be divided into groups. All the methods presented below are adopted to the considered problem of emission inventories.

In the sequel two uncertain inventories, A and B of Figure 2, will help us to illustrate the described techniques. The question to be answered is as follows:

inventory of which party can be considered "'smaller"'. The answer can be either used for ordering inventories of two parties, or checking if x_c is smaller than ρx_b, i.e. verifying fulfillment of reduction expressed in percents.

2.2 Statistical moments

Mean value and variance. The most elementary technique is based on *the mean value and the variance* (MV). The smaller is the mean value and the variance, the better the inventory is. This method is explained on the case presented in Figure 3. Although the reported value of the inventory A is smaller than that of B, the mean value of A is greater than the mean value of B. The same is true for the standard deviations. Even this simple criterion shows, that an inventory of the party B should be considered smaller than that of the party A. This is contrary to the result for reported values, which ignores uncertainty. Let us mention that in this approach fulfilling of the limit would be related to comparison of the mean value rather and not the reported value.

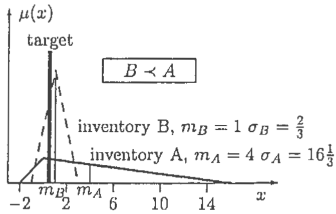

Fig. 3 Comparison of means and variances

Semivariance. However, many inventories would be not possible to compare in pairs, as using two indices, mean value and standard deviations, may lead to contradictory results. Taking this into account, a notion of *the semivariance* can be applied (MSV), which is defined as

$$s_S^2 = \int_K^\infty (x-K)^2 \mu(x) dx \tag{1}$$

where K is a chosen value and $\mu(x)$ is the distribution density function of an inventory. The smaller the value of s_S^2 is, the better is the inventory. In our case K can be conveniently chosen as a given target, and this value is used in the example of Figure 2, surveyed in Table 1. In the example considered $s_{SA}^2 > s_{SB}^2$. Thus, according to this criterion, an inventory B is smaller than A. Using this criterion, an inventory satisfies the target, if the semivariance is smaller than a preselected value.

2.3 Critical probability

Critical probability. A large group of techniques uses the notion of *critical probability* (CP), the notion introduced already in 1952 [25]. Most of the methods in this group require knowledge of the related probability distributions. The measure used to compare inventories is the probability of surpassing the target K

$$crp = \int_K^\infty \mu(x)dx \tag{2}$$

A smaller value of *crp* indicates better inventory. As seen in Figure 4, again, an inventory of the party B is evaluated as the smaller one.

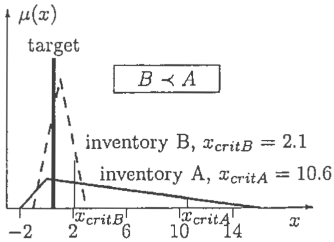

Fig. 4 Calculation of critical values

Risk. In other related methods, as *the Baumol's risk measure* and *the value at risk* (VaR), the probability α of inventory x to be above a critical value x_{crit} is fixed, and then the value x_{crit} is calculated. Without going into details, an inventory is smaller when x_{crit} is smaller. In our example, presented in Figure 4, fixing probability to 0.1, the inventory B is chosen as the smaller one.

A technique similar in spirit has been proposed to ensure a reliable compliance. It is called *undershooting*, [6,7,21,22,23], and is illustrated in Figure 5. In this approach, it is required that only a small enough α-th part of an inventory distribution may lie above a target. This idea, when used to order inventories, becomes equivalent to the CP technique.

In all these techniques satisfaction of a given limit would be connected with specifying the critical probability, which should be not greater than a prescribed value, or requiring that the related value x_{crit} is not greater than the limit.

2.4 Stochastic dominance

Stochastic dominance. In *the stochastic dominance technique* an inventory B is smaller than A, if their cumulative probability functions (cpfs) satisfy

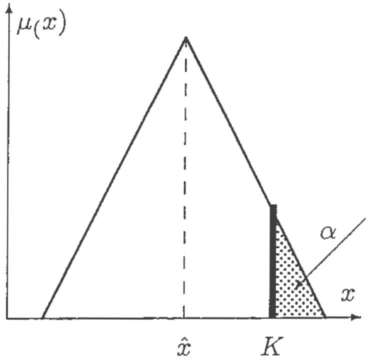

Fig. 5 Illustration of compliance in the undershooting approach.

$F_A(x) \leq F_B(x)$ for all x, and the condition is strict for at least one x. It is obvious that not all inventories can be decisively compared this way, see cpfs of our examplary inventories A and B depicted in Figure 6. Although cpf of the party B is greater for most values of x, it is lower than cpf of the party A for a small range of low value arguments. This possible lack of an answer *yes* or *not* is not convenient for comparison of inventories. However, some modifications have been proposed to extend the set of inventories which can be compared.

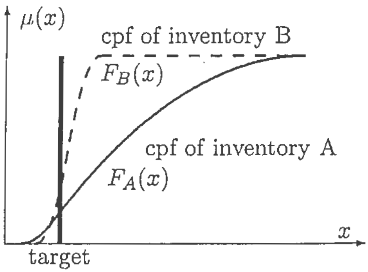

Fig. 6 Stochastic dominance criterion for comparison of inventories A and B.

Almost stochastic dominance. In *the almost stochastic dominance* (ASD)[1] the inventory B is smaller than A, if the area between both cpfs for $F_B(x) < F_A(x)$ is small enough (ε times smaller, usually with $0 < \varepsilon < 0.5$) part of the

[1] This is the first order ASD. For the second order ASD see [9].

whole area between pdfs, $\int_x |F_B(x) - F_A(x)|dx$. It can be seen by inspection in Figure 6 that this condition is satisfied in our example of Figure 2. Thus, also this technique indicates inventory B as the smaller one in this case.

A simplified comparison of inventories could confine to checking the values at $x = K$. This would be equivalent to a variant of critical probability approach. But this way analysis of fulfillment of the limit in the stochastic dominance techniques could be reduced to checking if the value of the inventory cpf at the limit is big enough.

2.5 Examples and resumé of probabilistic approaches

The results obtained so far for the inventories from Figure 2 are summerized in Table 1. As can be seen there, all methods point to the inventory B as the smaller one that is contrary to the conclusion taken when only the reported values are considered.

Table 1 Criteria values for comparison of inventories A and B for the inventories from Figure 2.

Method	Criterion value for A	Criterion value for B	Inventory chosen
MV	$m_A = 4$ $\sigma_A = 16\frac{1}{3}$	$m_B = 1$ $\sigma_B = \frac{2}{3}$	B
MSV	$s^2_{SA} = 13.45$	$s^2_{SB} = 0.35$	B
CP	$crp_A = \frac{8}{9}$	$crp_B = \frac{7}{8}$	B
risk	$c_{critA} = 10.6$	$c_{critB} = 2.1$	B

We perform another test, using rather difficult example presented in Figure 7. The distributions are shifted to the origin, so 0 in the figure corresponds to the value $\hat{x}$. Intuitively, B would be considered "better", as this inventory is more credible. But using the reported inventories only, they are considered equivalent.

The mean of both distributions is 0 and the variance is $\sigma^2 = a^2/6$, where $[-a, a]$ is the interval, on which the distributions are nonzero: in the figure a equals 2 for the inventory A and 1 for the inventory B. Thus, the mean and variance method obviously prefers B. Denoting $k = K/a$, the semivariance is equal to $s_S^2 = a^2(1-k)^4/12$ for $k \geq 0$ and $s_S^2 = a^2[1 - k^3(4-k)]$ for $k < 0$. It can be simply checked that for $k \geq 0$ it holds $s_S^2(a) > s_S^2(b)$ if $a > b$. The same is true for $k < 0$, if k is sufficiently close to zero, see an example in Figure 7.

As to the critical probability, $cpr = (1-k)^2/2$ for $k \geq 0$, and $cpr = (1 - 2k - k^2)/2$ for $k < 0$. Thus, for $k > 0$ it holds $cpr(b) > cpr(a)$ if $b > a$, and opposite, for $k < 0$ there is $cpr(b) < cpr(a)$ if $b > a$.

Similar for risk values, we get $x_{crit} = a(1 - \sqrt{2\alpha})$ for $K > 0$. As $\alpha < 0.5$ there, then $x_{crit}(b) > x_{crit}(a)$ for $b > a$. For $K < 0$ there is $x_{crit} = a\sqrt{2(1-\alpha)} - 1) < 0$. As $\alpha > 0.5$ there, then $x_{crit}(b) < x_{crit}(a)$ for $b > a$.

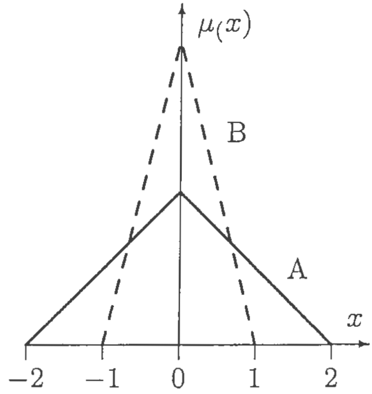

Fig. 7 Examplary distributions of inventories A and B.

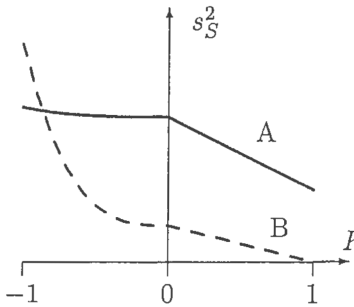

Fig. 8 Dependence of semivariances s_S^2 in function of K for inventories A and B from Figure 7.

Comparison of cumulative probability functions is not decisive, see Figure 11. But almost stochastic dominance can be used with $\varepsilon = 0.5$. This criterion points to the inventory B as the smaller one.

Concluding, decision about fulfillment of obligation, which is based on deterministic (reported value) comparison of an inventory with a target, does not agree with a common sense understanding of comparison and ordering of uncertain values within the probabilistic approach. The deterministic comparison also contradicts the already existing scientific knowledge on ordering projects with stochastic uncertainty.

Alternatively, almost all of the techniques presented in this section could be used for comparing uncertain inventories, except those, which may fail to provide an answer, like the mean value and variance or the stochastic dom-

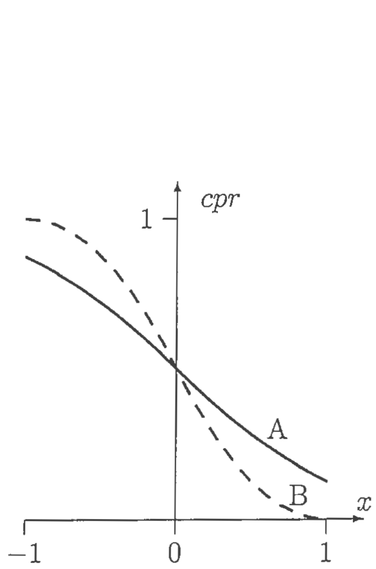

Fig. 9 Dependence of critical probability *cpr* in function of K for inventories A and B from Figure 7.

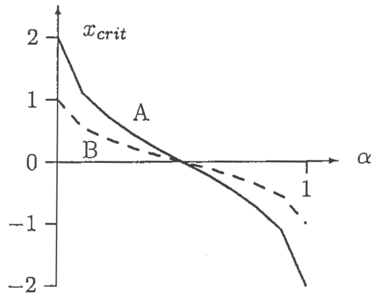

Fig. 10 Dependence of critical values x_{crit} in function of α for inventories A and B from Figure 7.

inance ones. The critical probability group seems to be particularly suitable for this.

But practically all the methods guarantee proper ordering of inventories only when the reported value is smaller or at most equal to the limit. In the other case the ordering may be opposite to the expected one, see the analysis above for two inventories with the same reported inventory values and different variations. These problem is irrelevant in the methods like undershooting, where only those inventories, which are smaller enough than the limit, are considered to fulfill it, and the other are considered noncompliant.

We could, however, like to consider and order all inventories. When the reported inventory value is greater than the limit, the proper ordering can be

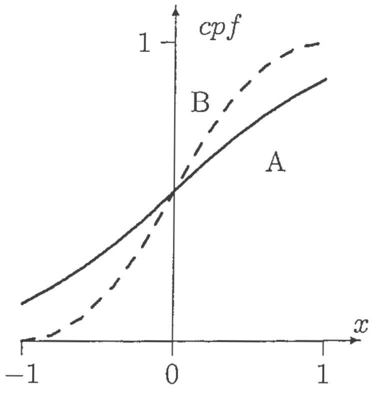

Fig. 11 Cumulative probability functions cpf for inventories A and B from Figure 7.

achieved using a probability β that the inventory is smaller than the limit, see Figure 12. Then we are convinced with the probability $1-\beta$ that the inventory is noncompliant. In this case there will be an indecision interval in the inventory, see Figure 12. When $K \in (x^l_{crit}, x^u_{crit})$, then we are not convinced enough if the inventory fulfills the limit or not.

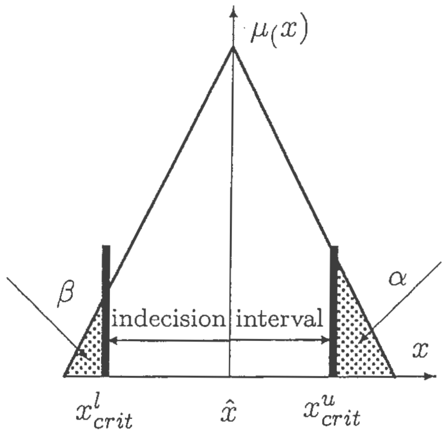

Fig. 12 Illustration of compliance, noncompliance and the indecision interval.

The question arises what can be done when the limit falls in the indecision interval of an inventory. It is actually quite fair to say that for these inventories no decision can be taken with high enough confidence.

One of the answers proposed for such cases in [14,8] was to wait until one or another exceedance occur in the inventories assessed in the consequent years. A rough method to estimate when this may take place was also designed, called *the verification time*. It is based on a linear or quadratic prognosis of future emission trajectory. This approach requires for compliance an additional obligatory undershooting so that the countries emission reductions and limitations become detectable.

Another solution for the inventories, which can not be classified as compliant or noncompliant, and which has been actually already proposed to be used, might be to add a penalty in the next clearing time for not fulfilling the compliance condition in the previous period. This penalty could depend on risks of fulfilling (α) and/or not fulfilling (β) the limit.

3 Fuzzy set approaches

3.1 Conceptual difference

A fuzzy set is a generalization of a common, crisp set. A crisp set can be defined by its characteristic function χ_A, taking either value 0 or 1, see Figure 13. A fuzzy set is characterized by a similar membership function μ_A, which takes values from the interval $[0, 1]$. The fuzzy sets, whose membership functions have values equal to 1, are called *normalized*. We consider here only normalized fuzzy sets. This membership grade can have one of three interpretations [5]: veristic, possibilistic and as truthvalues. The *veristic* interpretation means, that an element fully belongs to the set, if it has a membership grade equal to 1, it does not belong for the value equal to 0, and only "partly belongs" for the intermediate values. In a *possibilistic* interpretation, the fuzzy set represents a number of possible elements, the membership grades of each element indicates how possible this event is: ranging from a value of 0 if it is impossible, over values between 0 and 1 if it is somewhat possible, to a value of 1 if it is perfectly possible. As such, the fuzzy set describes imprecision and can be compared to the probability density function. However, it's normalization condition is less strict that in probability theory: the constraint now is that the highest value must be 1 (this implies that there must be at least one element that is perfectly possible). The difference in normalization also implies different algebra rules. The use of membership grades as *truthvalues* is an extension of boolean logic: 1 is considered to represent true, 0 represents false and intermediate values express a partial truth. This can be used when evaluating statements: saying that 90 is a big number can be considered only partly true.

A fuzzy number is a fuzzy set in the numerical domain $\mathbb{R}$ that satisfies a number of criteria (there is some discussion as to which criteria are absolutely necessary), for more details we refer to [19,31].

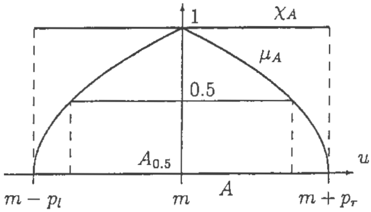

Fig. 13 The characteristic function χ_A and a membership functions μ_A of a set A.

A fuzzy set can be fully characterized by a family of so called α-cuts denoted by A_α, i. e. points u, for which the value $\mu_A(u)$ assumes at least the value α, see Figure 13, where an example of a α-cut for $\alpha = 0.5$ is depicted and denoted $A_{0.5}$. Let us notice here that now α has another meaning than in the probabilistic part. The problem is that both the notion of α as the probability of not fulfilling the target and the notion of α-cut in the fuzzy set research area are commonly used. To solve this overlap, in this section of the paper we switch to using η (instead of α) for the measure of that part of distribution where the target is not satisfied (that is the probability of not fulfilling the target in the probabilistic approach).

Two additional notions connected with a fuzzy set are worth to mention. One is *the support*, called supp A, which is the set of points u, for which the membership function is strictly positive. The second notion is *the core* of the fuzzy set, called core A, which is the set of points, for which the membership function is equal to 1.

Fuzzy set and posibilistic models of uncertainty can be considered as a competitive approach to the probabilistic one, described above. A few arguments can be given in favour of this approach. There are a number of interpretations of probability theory, we will only consider the ones related to the current context; for an overview we refer to [10]. First, the probabilistic approach is intrinsically related to the frequency of variable appearance (*frequentist approach*, [27]), while it is hardly possible to have frequent inventories at the same year. The use of the probability as degree of belief has been proposed in [24], but [18] showed that many people do not adhere to the probability calculus in this interpretation. Second, in the fuzzy set approach determination of the distribution is much more flexible. The distributions can be freely shaped and do not need to follow any known probabilistic distributions to be practically useful. For example, they can be estimates given by experts. Uncertainty of emission inventories often have an expert-quantified character, even if the Monte Carlo simulation is used to estimate its distribution. Third, the algebra in the fuzzy set approach is simpler, in the sense that for complicated problems more often it is possible to get a final analytic solution using the fuzzy approach than using the probabilistic one, see e. g. [22,21]. As in the case of inventories the data is often obtained through a mixture of statistical methods and corrections, interpretations and estimates by experts who express some belief and label the data accordingly, fuzzy set theory may be well suited for the analysis of compliance.

The fuzzy sets have been used in the undershooting technique [21] to calculate the difference $x_c - \rho x_b$. But their role was only instrumental there, as the rest of the technique was close to the idea used in probabilistic CP technique.

Regardless on how the data is processed, fuzzy set theory can inspire us for other aspects. In fuzzy set theory, the ranking of fuzzy (or inaccurate) values is a common problem to which different solutions have been proposed. And just as there are issues with the ranking of the inventories (Fig. 2), similar issues occur with the ranking of fuzzy numbers as illustrated on Fig. 21. Ignoring conceptual differences, there are sufficient similarities to warrant investigating how the possibilistic ranking methods hold up against the other methods. In the following sections we will briefly present some of the textbook cases of ranking methods in possibility theory and see how different cases appear in the different ranking methods.

In the following subsections, we will list four conceptually different groups of methods that are used to rank fuzzy numbers. Some of these methods resemble those from the probabilistic approaches, other use quite different paradigms. These methods were chosen to illustrate that various approaches can be used to tackle the ranking problem.

3.2 On the underlying assumptions

Most of the fuzzy ranking methods have been developed for fuzzy sets over the domain $[0, 1]$. The main reason for this is that there are some specific advantages in developing ranking methods (e.g. integrals over the domain cannot yield a result greater than 1. For the application of the methods in ranking different inventories, the methods could be modified to suit a different domain. This is possible for all the methods, but may complicate the formulas somewhat. To keep the formulas simple and to remain true to the original definitions, it was chosen not to do this. An alternative option would be to rescale the domain of the inventories to the interval $[0, 1]$ to allow for a direct application of the methods. If the supports of the fuzzy number is finite, as we assume here, and in the original support $x \in [l, r]$, the new variable is defined as $z = (x - l)/(r - l)$.

The ranking methods below are ranking methods in the sense that they put an ordering on at least two fuzzy numbers. Some authors have chosen to rank from lowest to highest, others rank from highest to lowest. The concept of this article is to present different methods and how difficult cases are distinguished differently. As such, these are minor details that can easily be overcome and should not deter from the message.

There were quite a number of different techniques proposed for ranking fuzzy sets. Not all are mentioned below. Some of those not mentioned can be found in a review paper [3].

Yager F1. In [30], the author presents three different ranking methods. They are pure ranking methods in the sense that a number is derived for every element. The number is independent of the other elements in the set. A weight function g is introduced to add weights to the fuzzy set A. This basically allows us to specify which values are more important, based on their possibility. Common weight functions are either $g(z) = 1$ (reflecting that all possible values are equally important) and $g(z) = z$ (indicating that the higher the possibility of a value, the more important it is and the more it will contribute to determine the rank).

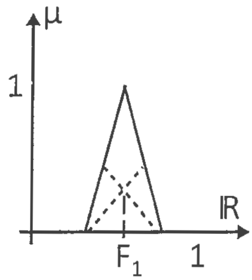

Fig. 14 F1 ranking function proposed by Yager.

The first ranking function is defined as follows:

$$F_1(A) = \frac{\int_0^1 g(z)\mu_A(z)dz}{\int_0^1 \mu_A(z)dz} \tag{3}$$

If the weight function $g(z) = z$ is used, then F_1 represents the mean value of the membership function, called usually the center of gravity of the fuzzy set. This is illustrated in Figure 14. Note that if the weight function $g(z) = 1$ is used; no ranking conclusions could be drawn: F_1 would result in 1 for every fuzzy set.

When $g(z) = z$, this technique can be compared with the mean value technique in the probabilistic approach. The ranking function may be defined in a more general way, and one option could be to take $g(z) = \left[z - F_1(A)|_{g(z)=1}\right]^2$, analogous to the variance. Also an analog of semivariance could be defined here, which shows similarity of this fuzzy approach technique with the probabilistic one.

3.4 Analogues to critical values

Nahorski et al. A strict analogue to a critical value technique in probabilistic approach has been proposed in Nahorski et al. [23, 22, 21]. To get an analogue to probability, which defines the critical value, the corresponding area is normalized there by dividing it by the area under the membership function, as in Figure 5. This approach assumes a rather precise knowledge of the membership function.

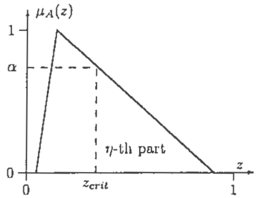

Fig. 15 Dermination of the critical value z_{crit} in the Nahorski et al. (calculation of the η-th part of the distribution area) and Adamo (calculation of the α-cut) techniques.

Adamo. On the other hand, Adamo [1] proposed to consider points satisfying $\mu_A(z) = \alpha$, $0 \leq \alpha \leq 1$, and choose the highest value of z as a ranking criterion. In another wording, the criterion value is the most right value of the α-cut of the fuzzy number A. The critical value depends now on the choice of α, but in this case it has clear fuzzy set interpretation connected with the α-cut. This idea may be compared with the one by Nahorski et al. [22, 21], where the critical area has a more probabilistic origin, while that of Adamo has a more fuzzy set flavour, see Figure 15. No doubt, for a given membership function both techniques can be related by mathematical expressions.

These techniques can be simply used for derivation of criterions for checking fulfillment od the limit, analogously to the ones which stem from the similar probabilistic approaches.

Yager F2. The second ranking function introduced by Yager [30] compares the given fuzzy set A to the linear fuzzy set B, defined by $\mu_B(z) = z$. The second ranking function is then defined as follows:

$$F_2(A) = \max_{z \in S} \min(z, \mu_A(z)) \tag{4}$$

Here, S represents the support of the fuzzy set A; in our case assumed to be the interval $[0, 1]$. Graphically, this yields the intersection point between the linear fuzzy set ($\mu_B(z) = z$) and the given fuzzy set where the possibility is the highest. This is illustrated in Figure 16.

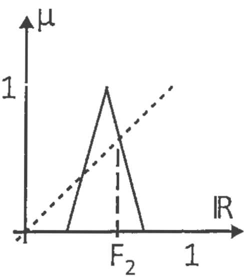

Fig. 16 F2 ranking function proposed by Yager.

This rating function has simple interpretation. The fuzzy set with the membership function $\mu_B(z) = z$ may be interpreted as representing a variable "high". The membership function $\min(z, \mu_A(z))$ represents a variable, which is a conjunction of A and B, i.e. the points, which belong both to the variable "high" and A. In other wording, it represents distribution of the possibility that A is "high". Its maximal point satisfies these two requirements in the "best" way.

The membership function of the variable "high" may be shaped in a different way. Jain [13] proposed more general set of functions $\mu_B(z) = (z/z_{max})^k$, $k > 0$[2]. In this case the result of comparison of fuzzy numbers may, however, strongly depend on the choice of k, and no clear criteria exist, which value of k should be chosen.

Apart of ranking the fuzzy numbers, the critical values could be used to check fulfillment of obligations, analogously to the stochastic approach. The simplest would be strict comparison of F_2 with K. However, the constructions proposed here are of a rather subjective character, difficult to interpret physically, and therefore their use may be limited.

Yager F3. The third ranking function defined by Yager [30] is more complex than the Yager's first and second ranking functions to explain using formulae, although it is simple to interpret geometrically. It is defined as

$$F_3(A) = \int_0^{\alpha_{max}} m(A_\alpha) d\alpha \qquad (5)$$

with A_α the α-cuts of A, α_{max} is the highest occurring possibility in the fuzzy set A[3], and m is the middle point of the α-cut. The formula is relatively easy to grasp graphically: the index is the surface area to the left of the line that runs exactly along the middle of the fuzzy number. For triangular fuzzy numbers, this connects the top of the fuzzy number (i.e. where the possibility is one)

[2] With assumptions taken in this section $z_{max} = 1$

[3] For the normalized sets, as are assumed in this paper, $\alpha_{max} = 1$

with the middle of the support. This is represented by the shaded area in Figure 17.

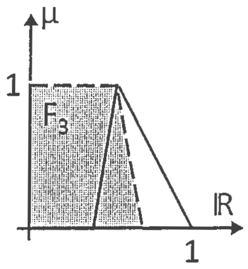

Fig. 17 F3 ranking function proposed by Yager.

This ranking index can be directly used to checking satisfaction of the limit. For this let us notice that $F_3(A)$ is the mean value of the function $m(A_\alpha)$, in which α is the argument. This is because $0 \leq \alpha \leq 1$, so for the triangular membership functions $F_3(A) = \int_0^1 m(A_\alpha)d\alpha = m(A_{0.5})$. Thus, in this case $F_3(A)$ is equal to the middle value of the 0.5-cut of the fuzzy number A. For other membership functions the integral will be equal to the middle value of the same or another α-cut. In any case, this index is closely related with an α-cut, where an appropriate α is determined by the shape of the membership function. This makes this approach a little similar to the Adamo method, with differently determined critical value at the middle of the α-cut instead at the right end. This interpretation encouraged us to classify this technique within the critical values group.

3.5 Fuzzy dominance

Dubois and Prade. In spite of a similar name, the fuzzy dominance techniques proposed up to now in the literature, differ completely in spirit from the stochastic dominance ones, presented in subsection 2.4. It is important to remember here that we use the normalized fuzzy numbers on the domain rescaled to the interval $[0, 1]$. The results of this subsection may be not true, if the normalization or rescaling is not done beforehand.

To compare fuzzy numbers using the fuzzy dominance approach, *possibility and necessity measures* can be used, as introduced by Dubois and Prade [4], see also [11]. A normalized fuzzy set with a membership function $\mu(z)$ induces on the interval $[0, 1]$ a possibility distribution $\pi(z) = \mu(z)$. For simplicity, we refer to defined this way possibility distribution as $\mu(z)$. Given a possibility distribution, the possibility measure of a subset $Z \in U = [0,1]$ is defined as

$$\mathrm{Poss}(Z) = \sup_{z \in Z} \mu(z)$$

It can be interpreted as a degree of possibility that an element is located in the set Z, see an interpretation in Figure 18. Let us notice that using a characteristic function $\chi_Z(z)$ of the set Z, the possibility measure can be equivalently defined as

$$\mathrm{Poss}(Z) = \sup_{z \in [0,1]} \min\{\mu(z), \chi_Z(z)\}$$

Let us notice that when $Z = [r, 1]$, then the above index can be interpreted as a measure that an element x is not smaller than r, i.e. $r \leq x$.

Comparing these notions to the probabilistic ones, the possibility distribution corresponds to the probabilistic distribution, and the possibility measure $\mathrm{Poss}(Z)$ corresponds to the probability of the subset Z.

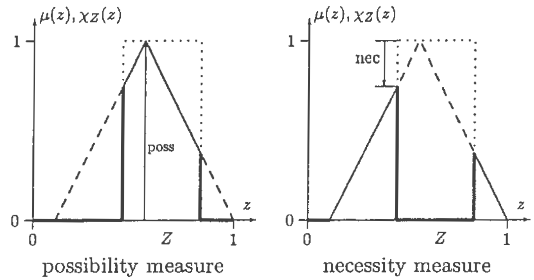

Fig. 18 Illustration of possibility and necessity measures for a crisp set Z.

However, in the possibility theory an additional measure is introduced, called the necessity measure. It is defined as

$$\mathrm{Nec}(Z) = 1 - \mathrm{Poss}(\bar{Z})$$

where $\bar{Z}$ is the complementary set of Z in $[0,1]$, see Figure 18. It can be interpreted as a degree that an element is located necessarily in the set Z. Similarly as in the possibility case, an equivalent definition may be

$$\mathrm{Nec}(Z) = 1 - \sup_{z \in [0,1]} \min\{\mu(z), \chi_{\bar{Z}}\} = \inf_{z \in [0,1]} \max\{1 - \mu(z), \chi_Z(z)\}$$

A simple property, which can easily observed in the Figure 18, holds

$$\mathrm{Nec}(Z) \leq \mathrm{Poss}(Z)$$

which may be interpreted that the measures give lower and upper bounds on uncertainty connected with localization of an element in the set Z. The lower one, necessity, is the degree, in the range $[0,1]$, of our conviction that the point is in the set Z. The higher one, possibility, is the degree of our supposition.

Now, taking a fuzzy sets Z instead of a crisp one, the characteristic function $\chi_Z(z)$ is replaced by the membership function $\mu_Z(z)$, providing the following definitions

$$\mathrm{Poss}(Z) = \sup_{z\in[0,1]} \min\{\mu(z), \mu_Z(z)\}$$

$$\mathrm{Nec}(Z) = 1 - \sup_{z\in[0,1]} \min\{\mu(z), \mu_{\bar{Z}}\} = \inf_{z\in[0,1]} \max\{1-\mu(z), \mu_Z(z)\}$$

where the membership function of the complementary set of Z is given by $\mu_{\bar{Z}}(z) = 1 - \mu_Z(z)$.

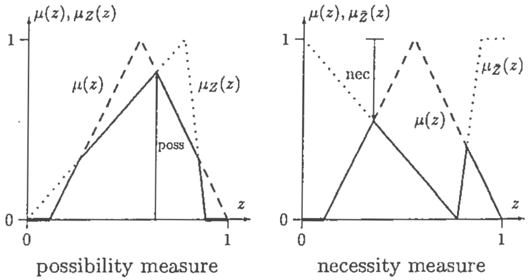

Fig. 19 Illustration of possibility and necessity measures for a fuzzy set Z.

Having introduced the above notions we can pass to defining fuzzy dominance indices. To calculate the possibility and necessity indices, the membership functions are analyzed on the two-dimensional plane (z, y), and more specifically either on the upper right or the bottom left half of the square $[0,1] \times [0,1]$, compare Figure 20. This is analogous to consideration of two-dimensional probability density function for independent variables. To compare two fuzzy numbers, one of them, say B, is taken as a reference one. Its membership function plays a role of a reference possibility distribution. The dominance of a fuzzy set A over B is denoted below as $A \succeq B$, and strict dominance as $A \succ B$.

The *possibility of dominance* (PD) index of a fuzzy set A over a fuzzy set B is defined as

$$PD = \mathrm{Poss}\,(A \succeq B) = \sup_{z,y;z\geq y} \min\left\{\mu_A(z), \mu_B(y)\right\} \tag{6}$$

The index PD is a measure of possibility that the fuzzy numbers A is greater than B, or that the set A dominates the set B. This index has been first proposed by Baas and Kwakernaak [2]. A probabilistic analogue of this index would be probability that $A \geq B$. This index has to be analyzed on the plane (z, y) in the upper right half of the square $[0,1] \times [0,1]$, see Figure 20, where the projection on the function $\min\{\mu_A(z), \mu_B(z)$ on the square is drawn, with the membership functions $\mu_A(z)$ and $\mu_B(y)$ drawn on the axis. The highest value of this function (equal to 1) is located in the area $y > z$ (at the point marked with $\bullet$), while the value $PD < 1$ is located on the boundary of the upper half of the square, at the point marked with $\circ$. It is now easy to notice that the value PD can be calculated as presented in Figure 21.

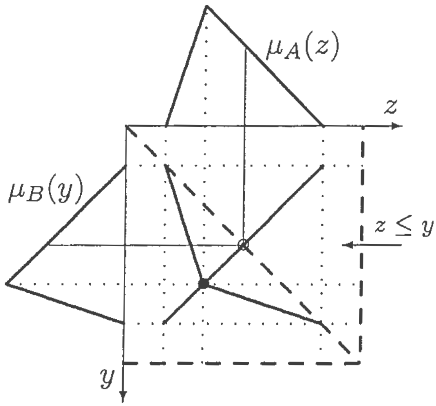

Fig. 20 Calculation of the PD index on the (z, y) plane.

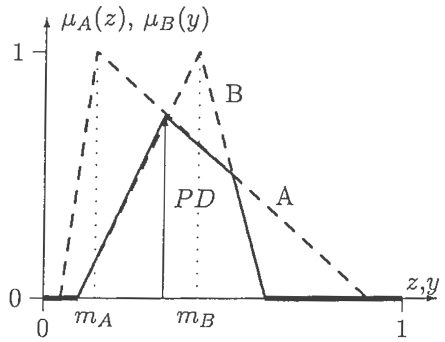

Fig. 21 Calculation of the PD index on a line.

Analyzing the way the value PD is calculated, and using notation from Figure 13, it is seen that

$$\text{Poss}(A \succeq B) = 1 \text{ if } m_A \geq m_B$$

$$\text{Poss}(A \succeq B) = 0 \text{ if } m_A + p_{rA} \leq m_B + p_{lB}$$

Just, the possibility of dominance index PD is equal to 0, if any point of the support of A is smaller than any point of the support of B. If the supports overlap, $PD > 0$. If the core of A is greater or equal to the core of B, then $PD = 1$.

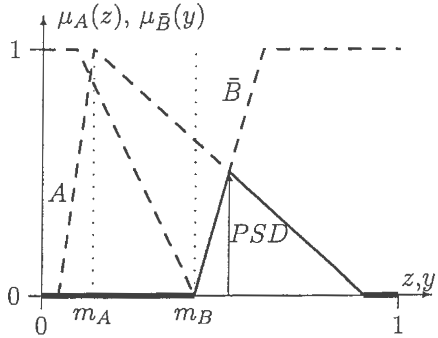

Fig. 22 Calculation of the PSD index.

The *possibility of strict dominance* (PSD) index for a fuzzy set A over a fuzzy set B is defined as

$$PSD = \text{Poss}\,(A \succ B) = \sup_z \inf_{y,y \geq z} \min\{\mu_A(z), 1 - \mu_B(y)\} \qquad (7)$$

where $\mu_A(z)$ and $\mu_B(y)$ are the membership functions of A and B, respectively. Analysis of the function on the two dimensional square brings us on the situation depicted in Figure 22. Now we have

$$\text{Poss}(A \succ B) = 1 \text{ if } m_A \geq m_B + p_{rB}$$

$$\text{Poss}(A \succ B) = 0 \text{ if } m_A + p_{rB} \geq m_B$$

The possibility of strict dominance index is therefore equal to 0, when the support of A is to the left of the core of B. It is positive in the opposite case. It equals 1, if the support of B is to the left of the core of A. Just, the membership function of A has to be more shifted to the right to get the same value of the index require as in the possibility of dominance case.

The *necessity of dominance* (ND) index of a fuzzy set A over a fuzzy set B is defined as

$$ND = \text{Nec}\,(A \succeq B) = \inf_z \sup_{y,y \leq z} \max\{1 - \mu_A(z), \mu_B(y)\} \qquad (8)$$

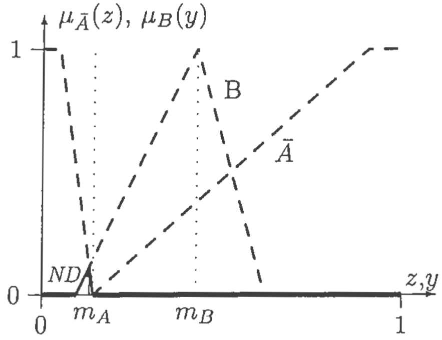

Fig. 23 Calculation of the ND index.

Similarly to the previous analyses, calculation of this index reduces to analysis of the situation presented in Figure 23. It yields

$$\mathrm{Nec}(A \succeq B) = 1 \text{ if } m_A - p_{lA} \geq m_B$$

$$\mathrm{Nec}(A \succeq B) = 0 \text{ if } m_A \leq m_B - p_{lB}$$

Thus, the necessity of dominance index equals 0 when the core of A is to the left of the support of B. It is positive in the opposite case. It is equal to 1, if the support of A is situated to the right of the core of B.

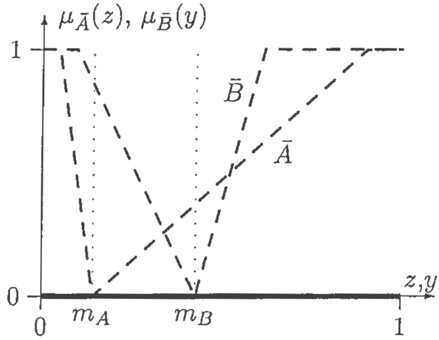

Fig. 24 Calculation of the NSD index.

The *necessity of strict dominance* (NSD) index of a fuzzy set A over a fuzzy set B is defined as

$$\begin{aligned} NSD = \mathrm{Nec}\,(A \succ B) = \inf_{z,y;y\leq z} \max(1 - \mu_A(z), 1 - \mu_B(y)) \\ = 1 - \sup_{z,y;z\leq y} \min\{\mu_A(z), \mu_B(y)\} = 1 - Poss\,(B \succeq A) \end{aligned} \tag{9}$$

This index is the opposite to the measure of possibility that the set B dominates the set A. This index has been first proposed by Watson et al. [28]. The analysis of the index reduces to analysis of the situation presented in Figure 24. There is

$$\text{Nec}(A \succ B) = 1 \text{ if } m_A - p_{lA} \geq m_B + p_{rB}$$

$$\text{Nec}(A \succ B) = 0 \text{ if } m_A \leq m_B$$

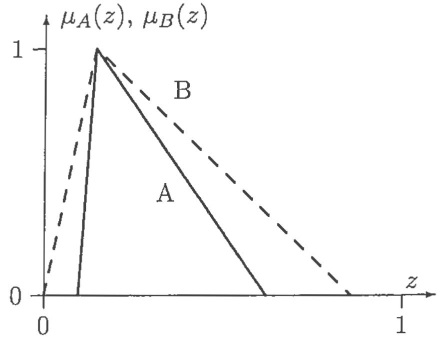

Fig. 25 A test case for ordering of two difficult distributions.

In order to further examine the method let us now consider a difficult ordering of two fuzzy numbers with distributions depicted in Figure 25. Simple inspection provides the following results

$$\text{Poss}(A \succeq B) = \text{Poss}(B \succeq A) = 1$$

and therefore

$$\text{Nec}(A \succ B) = \text{Nec}(B \succ A) = 0$$

Thus, both the possibility and necessity of strict dominance indices do not distinguish these fuzzy numbers. With the other indices, we get

$$\text{Poss}(A \succ B) < \text{Poss}(B \succ A)$$

and

$$\text{Nec}(A \succeq B) > \text{Nec}(B \succeq A)$$

Thus, the possibility of strict dominance index suggests rather B, while the necessity of dominance index rather A, as the "greater" number. This is connected with considering in calculation the right or left slopes of the distributions, respectively.

The Duboit and Prade approach does not prioritize the fuzzy sets itself, like earlier techniques. It answers the question of the degree of possibility or necessity of dominance of a chosen set by another one; it allows for week (soft) comparing of a set against one or more other sets rather than assigning a rank to each set. Thus, comparison of these inventories might give a rather indecisive answer. Similarly as in the undershooting technique, some critical values should be set for making decision on dominance. Moreover, it should be remembered that the indices will not necessary provide consistent results.

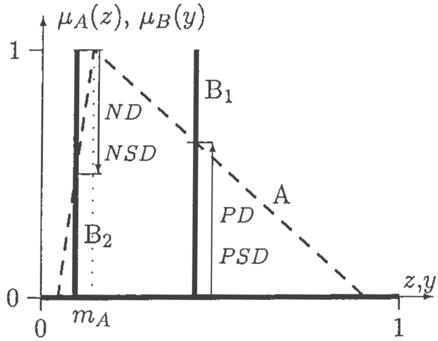

Fig. 26 Calculation of the indices for the crisp limits.

An interesting question is, if these techniques can be used for assessing satisfaction of the limit. For this, the limit can be interpreted as a crisp value, that is a fuzzy variable with the membership function

$$\mu_B(z) = \begin{cases} 1 \; if \; z = \tilde{L} \\ 0 \; if \; z \neq \tilde{L} \end{cases}$$

where $\tilde{L}$ is the rescaled value of the limit L. In these cases $PD = PSD$ and $ND = NSD$, so we can say here only on necessity N and possibility P indices. In Figure 26 two cases are depicted: the limit higher than m_A, and the limit smaller than m_A. In the former case $P > 0$ and $N = 0$. In the latter $P = 1$ and $N > 0$. We see that using the necessity indices is equivalent to the Adamo method with $N = 1 - \alpha$. The possibility indices give an information on a degree of not achieving the limit (recall that the limit is achieved here when A is greater than B), which could be used in inferring noncompliant inventories.

Actually, the situation here is fully analogous to the one presented in Figure 12. Fixing necessity N and possibility P indices we arrive again to the notion of indecision interval, where neither necessity nor possibility of the limit satisfaction is high enough.

Application of the Duboit and Prade method gives useful information on fulfilling the limit. However, analysis of the memberships functions on the plane is rather cumbersome. Simple interpretations on the lines could help in this. Necessity indices give practically the same information as in Adamo and Nahorski at al. method. Possibility indices can possibly be applied in quantifying noncompliance.

3.6 Other approaches

Tran and Duckstein. The authors in [26] first defined a distance measurement between fuzzy sets. From this distance measurement, a ranking method was derived. The concept is that for a given set of elements that need to be ranked, the smallest and largest possible (crisp) values, respectively, denoted *Min* and *Max* are used as predetermined targets. The distance to each of them ($D_{\min}$ and $D_{\max}$) is calculated for each element to be ranked. Note that depending on the shape of the fuzzy set, a greater distance to $D_{\min}$ does not imply a smaller distance to $D_{\max}$ and vice versa. Both distances are then used to rank the fuzzy sets: an element is ranked higher if its distance to $D_{\max}$ is smallest but its distance to $D_{\min}$ is greatest.

When ranking a number of fuzzy sets, the minimum reference target is defined to be the smallest (crisp) value that is possible in any of the fuzzy sets.

$$Min(I) \leq \inf(\bigcup_{i=1}^{I} S_i) \tag{10}$$

Here, I is the set of fuzzy sets A_i that are ranked and S_i is the support of the fuzzy set A_i, i.e. the set of values where $\mu_{A_i} > 0$. In practise, $Min(I)$ will be defined by as being equal to the smallest value of all the supports of the fuzzy sets considered.

The maximum reference target is similarly defined to be the largest possible crisp value of all elements to be ranked:

$$Max(I) \geq \sup(\bigcup_{i=1}^{I} S_i) \tag{11}$$

The notations are similar to before. In practise, $Max(I)$ will be defined by as being equal to the greatest value of all the supports of the fuzzy sets considered.

Similar to the approaches of Yager, a weight function g is used. This method is mainly developed for triangular fuzzy numbers (one single core point and a linear function to the left and to the right of it to model the decrease of possibility - the graph resembles a triangle) and trapezoidal fuzzy numbers (an interval where the core is 1 and a linear function to the left and to the right to model the decrease of possibility - the graph resembles a trapezoid). The latter are not considered here. For triangular fuzzy numbers, the ranking functions (squares of distances) are defined as [26][4]:

$$D^2(A, M, z) = (a_2 - M)^2 + \frac{1}{3}(a_2 - M)((a_3 + a_1) - 2a_2) +$$

[4] The formulas here differ from the formulas in Table 2 in [26]; the author confirmed there was a typographic error in that particular reference and sent us corrected versions - these have not been published elsewhere.

$$+\frac{1}{18}((a_3-a_2)^2+(a_2-a_1)^2)-\frac{1}{18}((a_2-a_1)(a_3-a_2)) \quad (12)$$

for weighting function $g(z) = z$ and

$$D^2(A, M, 1) = (a_2 - M)^2 + \frac{1}{2}(a_2 - M)((a_3 + a_1) - 2a_2)+$$

$$+\frac{1}{9}((a_3-a_2)^2+(a_2-a_1)^2)-\frac{1}{9}((a_2-a_1)(a_3-a_2)) \quad (13)$$

for weighting function $g(z) = 1$.

In these formulas, a_1, a_2 and a_3 define the triangular fuzzy set: $a_1 < a_2 < a_3$, $\mu_{u_i}(a_1) = \mu_{u_i}(a_3) = 0$ and $\mu_{u_i}(a_2)$=1. M can either be Min or Max. Thus, $D_{max} = D(A, Max, z)$ or $D_{max} = D(A, Max, 1)$. Similar for D_{Min}.

This technique has been designed directly for comparison of fuzzy numbers and it is rather difficult to adapt it to checking satisfaction of a limit, which will be typically lying within the support range.

3.7 Examples and resumé of fuzzy approaches

In this section, the different ranking methods will be compared to verify how the ranking of different special cases differ. For the example, triangular fuzzy numbers in the domain [0, 1] will be used.

Same support, different core First, we consider two fuzzy numbers that have the same support, but a different core as shown in Figure 27.

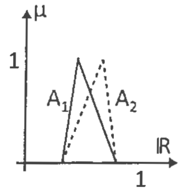

Fig. 27 Two fuzzy sets with the same support and a different core. The numerical ranking values are shown in Table 2.

Table 2 Same support, different core; with Yager ranking functions.

	a_1	a_2	a_3	F1	F2	F3
A_1	0.3	0.4	0.7	0.47	0.54	0.45
A_2	0.3	0.6	0.7	0.53	0.63	0.55

Table 3 Same support, different core; with Tran-Duckstein distance functions ($Min = 0.3$, $Max = 0.7$).

	a_1	a_2	a_3	$D(A, Max, z)$	$D(A, Min, z)$	$D(A, Max, 1)$	$D(A, Min, 1)$
A_1	0.3	0.4	0.7	0.27	0.14	0.26	0.17
A_2	0.3	0.6	0.7	0.14	0.27	0.17	0.26

Intuitively, people would state that $A_2 > A_1$. This ranking is also observed by the Yager's ranking methods (F1, F2 and F3) as shown on table 2. Using the D_{min} and D_{max} values as described in subsection 3.6 – highest ranked has smallest D_{max} and largest D_{min} – also yields the same conclusion as Table 3 indicates (the reference values for Min and Max were: Min=0.3, Max=0.7).

Same core, different support When the core of the different fuzzy numbers is the same, but the supports are different, the numbers become quite a lot more difficult to classify. The examples are illustrated in Figure 28.

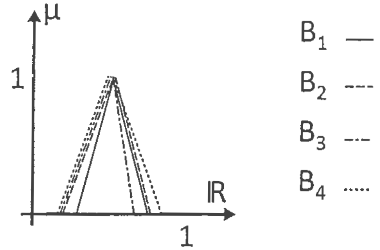

Fig. 28 Four fuzzy sets with the same core but different supports. The numerical ranking values are shown in Table 4.

Consider the example from Table 4. The table lists both the numeric data for the fuzzy sets used, as well as the rankings provided by the three Yager ranking methods. Intuitively, people can agree that $B_2 < B_1$ and that $B_3 < B_1$; we also see that $B_3 < B_2$. The problems start when comparing B_1 with B_4. The latter has nonzero possibility distribution for smaller values than B_1, so following the same reasoning as for B_2, it should be smaller. Yet it

Table 4 Same core, different support; with Yager ranking functions

	a_1	a_2	a_3	F1	F2	F3
B_1	0.3	0.5	0.7	0.5	0.59	0.5
B_2	0.2	0.5	0.7	0.47	0.58	0.48
B_3	0.2	0.5	0.6	0.43	0.54	0.45
B_4	0.2	0.5	0.8	0.5	0.61	0.5

Table 5 Same core, different support; Tran-Duckstein distance functions D^2 ($Min = 0.2$, $Max = 0.8$).

	a_1	a_2	a_3	$D(B, Max, z)$	$D(B, Min, z)$	$D(B, Max, 1)$	$D(B, Min, 1)$
B_1	0.3	0.5	0.7	0.304	0.304	0.307	0.307
B_2	0.2	0.5	0.7	0.322	0.290	0.336	0.288
B_3	0.2	0.5	0.6	0.337	0.272	0.357	0.260
B_4	0.2	0.5	0.8	0.308	0.308	0.316	0.316

also has nonzero possibility distribution for bigger values, so it should also be bigger. Many people would say that both are more or less equal; depending on the use and application, B_1 could be preferred as its shows less uncertainty (smaller range in which the values can occur), and thus greater credibility. The results for Yager's ranking functions are shown in Table 4. For B_1, B_2 and B_3, it clearly yields the same results as the intuitive ranking. While B_1 is considered equal to B_4 in the ranking methods F1 and F3, F2 shows a minor difference, indicating that $B_4 > B_1$. This is rather counter-intuitive, as there is more uncertainty about B_4, but it is obvious from the ranking index that the difference is very small.

When using the ranking methods proposed by Tran and Duckstein (3.6), we see that $B_2 < B_1$, that $B_3 < B_1$ and that $B_3 < B_2$. Comparing B_1 with B_4 is a problem: B_4 has larger D_{max} value, but does not have smaller D_{min} value. The authors propose to leave this issue to the decision maker, indicating the numeric values. A risk prone person may prefer values that can be closer to D_{min} whereas a risk averse person may desire values closer to D_{max}. While this method – unlike the previous one – does not yield a value that allows to make a decision between the two values, it allows for a decision maker to have additional information which might help him decide. In the situation of the inventories, this allows us to prefer B_1 as it has less uncertainty.

In the fuzzy approach it is possible to formulate the problem with only a rather vague information on the inventories uncertainty. The price payed for it is only a week statements on ranking, either much less precise than in the stochastic approach, or indecisive, providing only some indices on possibility or necessity. They would require to set some critical values for making decisions. This is, however, more difficult than in the stochastic case, as intuition on the meaning of these indices is much smaller. Moreover, the indices, like in the fuzzy dominance techniques, may be difficult to be calculated by a computer. For some techniques it is not clear how to check compliance using the idea of ranking used in them.

In some fuzzy sets methods positions of the reported (core) values decide on the primary comparison and only additional indices can indicate, how strong is this conjecture. This is particularly evident in the fuzzy dominance techniques.

Most of the fuzzy techniques require more investigations to elaborate the methods useful for applications.

4 Conclusions

This paper focuses on a presentation of the methods for ranking uncertain values, with application to comparison of uncertain emission inventories and posibility to use the techniques proposed in checking satisfaction of the given limit emission. The review shows a variety of approaches and techniques. Not all of them can be immediately used in analysis of inventories; some other are rather complicated or give no decisive answer. However, they clearly show that the comparison of the reported inventories, without taking into account its uncertainty distribution, leads to paradoxes and is not well grounded scientifically. There are many possibilities to choose a method for deciding, which inventory satisfies the limit, and which not, consistent with ordering or ranking of the inventories. Some of them, like e.g. the undershooting method, has been proposed earlier for this purpose [7,23], see also [16], and adapted to be used in trading of emissions, see additionally [22,21]. But any use of techniques outlined in this paper or other, which take uncertainty into account, inevitably necessitates changing the presently used rules of checking compliance, which depend only on comparison of the reported inventories. Ignoring uncertainty is more hazardous to the final result for asymmetric uncertainty distribution, which may happen in many national inventories, as well as when inventories with quite different uncertainty distributions are compared, as in the case of emissions from different activities.

In spite of basic conceptual differences between the probabilistic and fuzzy approaches, many techniques of comparison of uncertain values are quite alike. Among them the risk methods in probabilistic approaches and fuzzy dominance provide similar techniques of checking compliance, with actually small technical differences in terminology and decision parameters. Although this paper has not been intended on a thorough comparison of usefulness of the techniques presented in checking compliance, these techniques look to be preferable for closer examination.

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