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A fuzzy approach to determination of compliance and emission trading rules for uncertain pollution emission estimates

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Abstract

National inventories of greenhouse gases emissions are computed with rather low precision. Their uncertainty estimates are, however, calculated in a similar way and, therefore, have similar low precision. This should be accounted for in the compliance and trading rules. In this paper we model the uncertain inventories using fuzzy numbers, which allows us to shape both their uncertainties and ignorance of precise uncertainty parameters. Derived this way compliance and emission trading rules generalize those for the interval uncertainty approach, which were considered in the earlier papers. The final conclusion is, however, that the interval uncertainty rules can be still applied, but the used in them the noncompliance risk should take much higher values. The derivation is then generalized for the nonsymmetric membership functions and the compliance checking condition is derived for this case.

Keywords: national inventories of greenhouse gases emision, uncertainty, compliance, emission permit trading, fuzzy sets.

Contents

Chapter 1

Introduction

Greenhouse gases inventories estimates are far to be exact. Estimation of its uncertainty done for several countries showed that they usually outstript the reductions agreed upon in the Kyoto Protocol. Presented up-to-now ideas to change the compliance checking and emission trading rules to include the uncertainty of inventories assume that the uncertainty estimates are known exactly, see [7] for a review of techniques and specifically [4, 12] for solutions in spirit of the present paper. However, this is far from being true. The uncertainty estimates are calculated in a similar way as the inventories and it may be expected that uncertainty of them is of the similar order as that of inventory itself.

It was shown in [12] that although the stochastic case may be useful for the determination of the compliance rule, it provides a too complicated and practically useless formula for the emission trading rule. Thus, in this paper a fuzzy approach is used, which can be considered as a generalization of the interval one. The fuzzy set calculus basically inherits the rules from the interval calculus, and this way provides linear dependencies in the resulting formulas. But at the same time the fuzzy variables may be shaped to have more concentrated distributions than the interval ones, and this way can better approximate the real distributions.

The fuzzy approach solves also the problem of imprecise knowledge of the uncertainty interval length by considering the whole family of intervals of different length and this way modeling uncertainty of their knowledge. Coming out from this point of view in this paper the uncertainty of the inventory uncertainty estimate is taken into account and new rules for checking compliance and emission trading rules are proposed. They are generalizations of the rules presented in [10] and reduce to them when the uncertainty interval estimate is exact. The results of application of these rules are compared to those obtained earlier, for the assumed exact knowledge of uncertainty estimate. The result is that a convenient interval uncertainty approach may be used, but with much higher noncompliance risk.

In section 2 we formulate the problem and introduce some basic notation. Then, in section 3, we recall conditions for checking compliance and formulas for so called efficient emissions, which can be directly traded, without taking account for the emission uncertainty, for the interval type of uncertainty. In section 4 a family of fuzzy numbers is introduced. They are used to model the full inventory uncertainty and form the basis for derivations of generalized compliance and emission trading rules. These rules are compared with the interval approach rules. Section 5 concludes.

Chapter 2

Notation and problem formulation

Basically, the total emission by a party is calculated by summing up emissions from every type of contributing activity and subtracting the gases absorbed by sinks. Yearly emissions $\hat{x}_i(t)$ of every type of activity are computed as the product

$$\hat{x}_i(t) = \hat{c}_i(t)\hat{a}_i(t)$$

where $\hat{a}_i(t)$ is the activity measure (e.g. in tons of material used) and $\hat{c}_i(t)$ is its emission factor, both in the year t. On the national scale both values on the right hand side are unsure, giving rise to uncertainty. The nature of the uncertainty is a complicated one. It originates from a lack of exact knowledge of some variables and a need for an imperfect modeling of often poorly known processes. Table 2.1 gives a few examples of the uncertainty estimates, in percentages of the emissions. Full details can be found in [5, 6].

In the sequel by $x(t)$ we denote the real, unknown emission of a party in the year t and by $\hat{x}(t)$ its best available estimate. To simplify notation the time argument t will be dropped in the sequel.

The Kyoto Protocol declaration requires that each participating country should reduce a prespecified percent of its basic year emission within the given period (around 20 years), although some countries are granted a possibility of stabilizing the emission at the basic year level or even of a limited increase of its emission.

Let us denote by δ the fraction of the party emission that is to be reduced in the commitment period according to its obligation. The value of δ may be

Country	Kyoto reduction	Uncertainty
AT	8	12
NL	8	5
NO	-1	21
PL	6	6
RU	0	17
UK	8	19

Table 2.1: Examples of Kyoto reduction commitments and published uncertainty estimates of national emissions, in per cents.

negative for parties, which were alloted limitation of the emission increase. Denoting by t_b the basic year and by t_c the commitment year, and by x_b and x_c, respectively, the emissions, the following inequality should be satisfied

$$x_c - (1 - \delta)x_b \leq 0 \tag{2.1}$$

As neither x_c nor x_b are known precisely enough. Instead, only the difference of estimates can be calculated

$$\hat{x}_c - (1 - \delta)\hat{x}_b \tag{2.2}$$

where both $\hat{x}_c$ and $\hat{x}_b$ are known with an intolerable low accuracy.

Chapter 3

Interval type uncertainty

Compliance. Assuming that the uncertainty intervals at the basic and the commitment years are $2d_b$ and $2d_c$, respectively, we have

$$x_b \in [\hat{x}_b - d_b, \hat{x}_b + d_b], \qquad x_c \in [\hat{x}_c - d_c, \hat{x}_c + d_c]$$

from which, using the interval calculus rules, we get

$$x_c - (1-\delta)x_b \in [D\hat{x} - d_{bc}, D\hat{x} + d_{bc}]$$

where

$$D\hat{x} = \hat{x}_c - (1-\delta)\hat{x}_b \tag{3.1}$$

and

$$d_{bc} = d_c + (1-\delta)d_b \tag{3.2}$$

However, the inventories $\hat{x}_b$ and $\hat{x}_c$ are dependent and the values of d_{bc} are usually much smaller than those resulting from the above expression. In [12] it was proposed to modify it to

$$d_{bc} = (1-\zeta)(d_c + (1-\delta)d_b) \tag{3.3}$$

where $0 \leq \zeta \leq 1$ is an appropriate chosen variable. This case will be also considered in this paper.

To be fully credible, that is to be sure that (2.1) is satisfied, the party should prove $D\hat{x} + d_{bc} \leq 0$. We say that the party is *compliant with risk* α, if $D\hat{x} + d_{bc} \leq 2\alpha d_{bc}$, that is, not bigger part of its distribution than α lies above

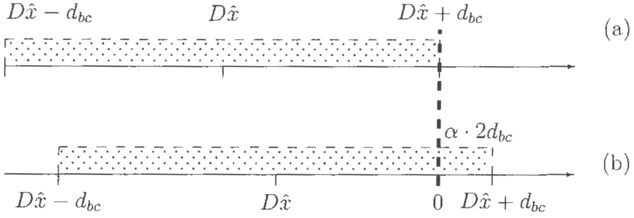

Figure 3.1: Full compliance (a) and the compliance with risk α (b) in the interval uncertainty approach.

zero, see Fig. 3.1 for the geometrical interpretation. After simple algebraic manipulations this gives the condition

$$\hat{x}_c + (1 - 2\alpha)d_{bc} \leq (1 - \delta)\hat{x}_b \tag{3.4}$$

Thus, to prove the compliance with risk α the party has to satisfy its obligation with the inventory emission estimate increased by the value $(1 - 2\alpha)d_{bc}$, dependent on its uncertainty measure expressed by d_{bc}. The condition (3.4) can be also rewritten as

$$\hat{r} = \hat{x}_c/\hat{x}_b \leq 1 - \delta - (1 - 2\alpha)R_{bc}$$

where $\hat{r}$ is the estimated reduction factor and $R_{bc} = d_{bc}/\hat{x}_b$ is the half relative uncertainty interval. Thus, the compliance with risk α can be formally reduced to the form (2.2) by redefinition of the reduction factor

$$\delta \quad \longrightarrow \quad \delta_U = \delta + (1 - 2\alpha)R_{bc} \tag{3.5}$$

Emission trading. Admitting the above compliance proving policy it is possible to consider uncertainty in the emission trading. The main idea of this proposition consists in transferring the uncertainty to the buyer together with the traded quota of emission and then including it in the buyer's emission balance.

Let us denote by $R_c^S = d_c^S/\hat{x}_c^S$ the relative uncertainty of the seller and by $\hat{E}^S$ the traded amount of estimated emission. This emission amount is associated with uncertainty $\hat{E}^S R_c^S$. Before the trade the buying Party checks the following condition

$$\hat{x}_c^B + (1-2\alpha)d_{bc}^B \leq (1-\delta)x_b^B$$

After the transaction the condition changes into

$$\hat{x}_c^B - \hat{E}^S + (1-2\alpha)[d_{bc}^B + \hat{E}^S R_c^S] \leq (1-\delta)x_b^B$$

Due to the partial cancellation of the subtracted estimated emission and its uncertainty in the buyer's emission balance *the effective traded emission* is

$$E_{eff} = \hat{E}^S[1-(1-2\alpha)R_c^S] \qquad (3.6)$$

Thus, the bigger seller's uncertainty is, the less purchased unit is accounted for the buyer. Expression (3.6) reduces emissions estimated with an arbitrary precision to globally comparable values, which can be directly subtracted from country's estimated emission. This way it is possible to construct a market for the effective emissions, see [12].

Chapter 4

A fuzzy type uncertainty

Although the interval approach provides a very simple and convenient solution, its criticism is sometimes aimed at low precision of defining the uncertainty intervals. Similarly to inventory calculation, also calculation of the uncertainty intervals is inexact and its accuracy is of the same order as that of the inventory calculation.

The uncertainty of the interval ends can be modeled using fuzzy set approach, see Appendix A for a short introduction of some basic notions. A common way for this is to use so called fuzzy interval with the trapezoidal membership function, like that presented on Fig. 4.1. The uncertainty of the interval ends is modeled by linear change of the membership function from 0 to 1 at the interval ends.

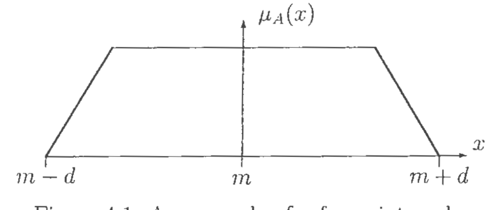

Figure 4.1: An example of a fuzzy interval.

In this paper the fuzzy numbers are used to model imperfect knowledge of the uncertainty. A fuzzy number is a particular case of a fuzzy interval and may be also considered as a straight generalization of an ordinary number, whose value is unsure. This is the situation, which we spot in the greenhouse gas inventories.

An usual problem with the fuzzy set approach is to determine the membership function. Here, we introduce membership functions dependent on parameters. Fixing the parameters, the function best fitting the experimenter expectation can be obtained. To estimate the parameters, the function can also be fitted to the distribution obtained from Monte Carlo simulations, as shown in the sequel.

4.1 Symmetric membership functions

Monte Carlo simulations give either distributions close to symmetric or clearly nonsymmetric [13]. We start the discusion from the symmetric distributions, as this case is much simpler. The nonsymmetric distributions are considered in the next section.

Let us consider a family $\mathcal{F}$ of fuzzy numbers $A^\gamma = \{(x, \mu_A^\gamma(x))|x \in \text{supp } A^\gamma\}$ indexed by a variable $\gamma \in C^+ = \{\gamma \in C|\gamma \geq 0\}$, with the support $\text{supp } A^\gamma = [d_A^l, d_A^r]$. Fig. 4.2 depicts examples of μ_A^γ representing a fuzzy number 0, for few values of γ. The membership function is chosen there as

$$\mu_A^\gamma(x) = \left(1 - \frac{|x|}{d_A}\right)^\gamma$$

with $d_A^l = -d_A$ and $d_A^r = d_A$. This is a special LR type fuzzy number introduced in Appendix A, with $L(u) = R(u) = (1-u)^\gamma$ and $d_A^l = d_A^r$. As can be seen, the introduced family can model a wide arrays of fuzzy uncertainties. It can be even more generalized, if two branches, left and right, with different values of γ and d_A, are used.

It was suggested from the inspection of the results of Monte Carlo simulations [17] that distribution of the inventory error is close to the Gaussian one. It is also depicted in Fig. 4.3. Yet, stochastic approach introduces nonlinearities in derivation of the effective traded emission formulas. As seen in Fig. 4.4, a membership function from the proposed family can also give good fit to Monte Carlo simulation data, presented originally in [16].

Compliance. Let us assume now that the uncertainty of $\hat{x}_b$ and $\hat{x}_c$ are of the fuzzy type with the membership functions from the family $\mathcal{F}$, that is they are fuzzy numbers $\hat{x}_b^\gamma$ and $\hat{x}_c^\gamma$ where

$$\hat{x}_b^\gamma = \{(x, \mu_{\hat{x}_b}^\gamma(x))|x \in \text{supp } \hat{x}_b^\gamma\}$$

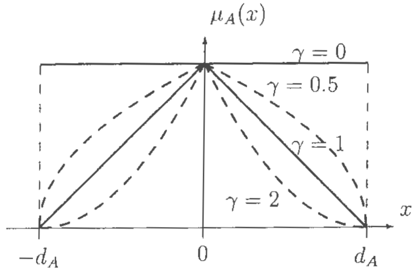

Figure 4.2: Membership functions for $\gamma = 0,\ 0.5,\ 1$ and 2.

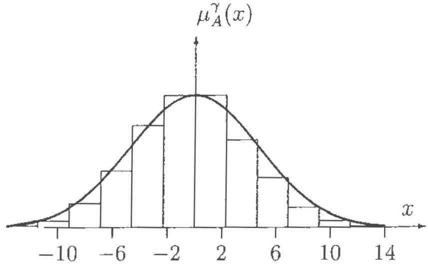

Figure 4.3: Fit of the Gaussian function to the histogram from [16], centered and normalized.

$$\text{supp } \hat{x}_b^\gamma = [\hat{x}_b - d_b, \hat{x}_b + d_b]$$

$$\mu_{\hat{x}_b}^\gamma(x) = \Big(1 - \frac{|x - \hat{x}_b|}{d_b}\Big)^\gamma$$

and similarly

$$\hat{x}_c^\gamma = \{(x, \mu_{\hat{x}_c}^\gamma(x)) | x \in \text{supp } \hat{x}_c^\gamma\}$$

$$\text{supp } \hat{x}_c^\gamma = [\hat{x}_c - d_c, \hat{x}_c + d_c]$$

$$\mu_{\hat{x}_c}^\gamma(x) = \Big(1 - \frac{|x - \hat{x}_c|}{d_c}\Big)^\gamma$$

Then, calculating the difference in analogy to (2.2) a fuzzy number $D\hat{x}^\gamma$ is obtained

$$D\hat{x}^\gamma = \hat{x}_c^\gamma - (1-\delta)\hat{x}_b^\gamma = \{(x, \mu_{D\hat{x}^\gamma}^\gamma(x)) | x \in \text{supp } D\hat{x}^\gamma\} \tag{4.1}$$

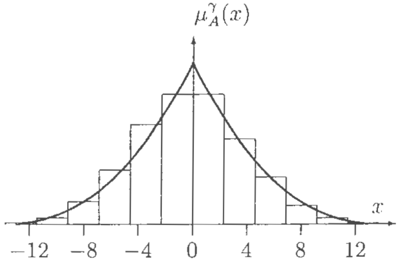

Figure 4.4: Fit of a membership function $\mu_A^\gamma(x)$ for $\gamma = 2.43$ and $d_A = 14.46$ to the histogram from [16], centered and normalized.

whith the support

$$\text{supp } D\hat{x}^\gamma = [D\hat{x} - d_{bc}, D\hat{x} + d_{bc}] \tag{4.2}$$

and the membership function

$$\mu^\gamma_{D\hat{x}^\gamma}(x) = \Big(1 - \frac{|x - D\hat{x}_b|}{d_{bc}}\Big)^\gamma \tag{4.3}$$

where $D\hat{x}$ and d_{bc} are given by (3.1) and (3.3), respectively. The proof of expressions (4.1) to (4.3) may be done easily using slight generalization of the addition and multiplication rules on fuzzy numbers given in the Appendix A. The derivations are given in Appendix C.

For this case we say that a party is *compliant with risk* α when not bigger than the αth part of the area under the membership function (4.3) lies above zero.

The area under the membership function is

$$A = \int_{-d_bc}^{d_bc} \Big(1 - \frac{|x|}{d_{bc}}\Big)^\gamma dx = \frac{2d_{bc}}{\gamma + 1} \tag{4.4}$$

and the area A_α corresponding to the αth part of area A is

$$A_\alpha = \int_{d_{bc}-y}^{d_{bc}} \Big(1 - \frac{x}{d_{bc}}\Big)^\gamma dx = \frac{d_{bc}}{\gamma + 1}\Big(\frac{y}{d_{bc}}\Big)^{\gamma+1} \tag{4.5}$$

Now

$$\alpha = \frac{A_\alpha}{A} = \frac{1}{2}\Big(\frac{y}{d_{bc}}\Big)^{\gamma+1}$$

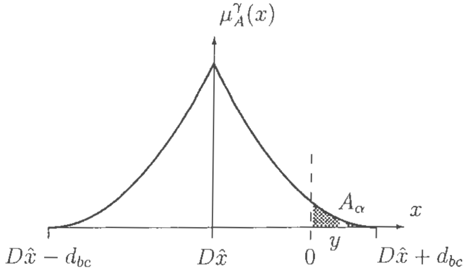

Figure 4.5: Graphical interpretation of the αth part A_α of the area under the membership function.

then

$$y = (2\alpha)^{\frac{1}{\gamma+1}} d_{bc}$$

Thus, the area A_α is placed within the distance $y = (2\alpha)^{\frac{1}{\gamma+1}} d_{bc}$[1] from the right end of the interval $[D\hat{x} - d_{bc}, D\hat{x} + d_{bc}]$, see Fig. 4.5. Thus we get the following condition

$$D\hat{x} + d_{bc} \leq (2\alpha)^{\frac{1}{\gamma+1}} d_{bc}$$

or in a more explicit form

$$\hat{x}_c + [1 - (2\alpha)^{\frac{1}{\gamma+1}}] d_{bc} \leq (1 - \delta)\hat{x}_b \tag{4.6}$$

As before, it can be also transformed to the form

$$\hat{r} = \hat{x}_c / \hat{x}_b \leq 1 - \delta - [1 - (2\alpha)^{\frac{1}{\gamma+1}}] R_{bc}$$

where $R_{bc} = d_{bc}/R_{bc}$, giving rise to redefinition of the reduction factor

$$\delta \quad \longrightarrow \quad \delta_U = \delta + [1 - (2\alpha)^{\frac{1}{\gamma+1}}] R_{bc} \tag{4.7}$$

This formula can be interpreted as an extension of the formula (3.5), as it reduces to (3.5) when $\gamma = 0$.

With a given α the formula (4.7) shows the dependence of the reduction factor on γ. This dependence is illustrated in Fig. 4.6.

[1] For further use let us notice that the value of the membership function corresponding to this point of the x-axis is equal to $\eta = (2\alpha)^{\frac{\gamma}{\gamma+1}}$.

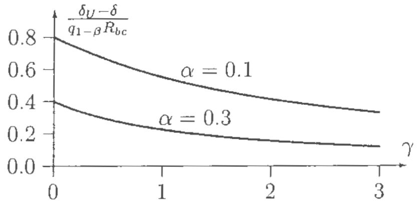

Figure 4.6: Dependence of the reduction factor on γ.

Emission trading. After derivations analogous to the interval case we end with the effective reduction for the fuzzy type uncertainty

$$E_{eff} = \hat{E}^S\{1 - [1 - (2\alpha)^{\frac{1}{\gamma+1}}]R_c^S\} \tag{4.8}$$

It is again an extension of the formula (3.6) for the interval case. In comparison with the interval case it provides smaller differences between E_{eff} and $\hat{E}^S$, see Table 4.1.

Count.	Unc.	Interv.	Fuzzy				
			$\gamma = 0.5$	$\gamma = 1$	$\gamma = 1.5$	$\gamma = 2$	$\gamma = 2.5$
AT	12	0.904	0.965	0.973	0.978	0.981	0.984
NL	5	0.960	0.986	0.989	0.991	0.992	0.993
NO	21	0.832	0.939	0.953	0.961	0.967	0.971
PL	6	0.952	0.983	0.986	0.989	0.991	0.992
RU	17	0.864	0.951	0.962	0.969	0.973	0.977
UK	19	0.848	0.945	0.957	0.965	0.970	0.974

Table 4.1: Comparison of the ratio $E_{eff}/\hat{E}^S$ for the interval and fuzzy approaches for $\alpha = 0.3$ and data from Table 2.1.

Equivalence of approaches. Let us notice that actually the fuzzy approach formulas (4.6) and (4.7) can be considered equivalent to the interval approach ones (3.5) and (3.6) provided appropriate values of α is chosen for both cases. Denoting by the subscript $_I$ the interval and by $_F$ the fuzzy case the equalities of the reduction factors or the effective reductions

$$\delta_I = \delta_F \quad \text{or} \quad E_{eff,I} = E_{eff,F}$$

after simple algebra provide the same condition

$$(2\alpha_I)^{1+\gamma} = 2\alpha_F$$

For the adopted assumptions $0 \leq \alpha_I, \alpha_F \leq 0.5$ and $\gamma \geq 0$ we have

$$\alpha_I \geq \alpha_F$$

with strong inequality for internal points of the assumption set. Dependence of α_I on α_F and γ is shown in Table 4.2. The results show that α_I rises quickly when rises γ. In two cases considered in our calculations estimates of γ close to 2 and 2.5 were obtained. Then, practically it seems that $0.2 \leq \alpha_I \leq 0.3$ should be taken even for small values of α_F.

$\alpha_F \downarrow \gamma \rightarrow$	0.1	0.5	1	1.5	2	2.5
0	0	0	0	0	0	0
0.05	0.06	0.11	0.16	0.20	0.23	0.26
0.10	0.12	0.17	0.22	0.26	0.29	0.32
0.15	0.17	0.22	0.27	0.31	0.33	0.35
0.20	0.22	0.27	0.32	0.35	0.37	0.38
0.25	0.27	0.32	0.35	0.38	0.40	0.41
0.30	0.31	0.36	0.39	0.41	0.42	0.43
0.35	0.36	0.39	0.42	0.43	0.44	0.45
0.40	0.41	0.43	0.45	0.46	0.46	0.47
0.45	0.45	0.47	0.47	0.48	0.48	0.49
0.50	0.50	0.50	0.50	0.50	0.50	0.50

Table 4.2: Dependence of α_I on α_F and γ.

The interpretation of these results is quite straightforward. Ignorance of the exact interval ends knowledge introduces additional uncertainty, which sums up to the uncertainty of the inventory itself. Thus, to obtain the same reduction factor or the same effective reductions a bigger risk should be adopted in the interval approach. An important practical observation is that bigger values of α_I, like 0.2 to 0.3, should be taken to compensate for ignorance of the exact knowledge of the uncertainty interval length, even if a smaller noncompliance risk is actually meant.

4.2 Nonsymmetric membership functions

We consider now a family od fuzzy numbers $A^\gamma = \{(x, \mu_A^\gamma(x)) | x \in \text{supp } A^\gamma\}$ indexed by a vector parameter $\gamma = [\gamma_1\gamma_2] \in C^+ \times C^+$, with the support $\text{supp } A^\gamma = [d_A^l, d_A^u]$. The membership function has the form

$$\mu^\gamma(x) = \begin{cases} a(1 - \frac{x}{d_A^u})^{\gamma^u} & \text{for} \quad 0 \leq x \leq d_A^u \\ a(1 + \frac{x}{d_A^l})^{\gamma^l} & \text{for} \quad d_A^l \leq x < 0 \end{cases}$$

where a is a normalizing factor used for fitting the membership function to empirical distributions. In the theoretical considerations it can be assumed that the membership function has been normalized earlier and then $a = 1$ is taken in the sequel.

Fig. 4.7 presents an estimate of a nonsymmetric membership function obtained using the Monte Carlo method and presented in [18].

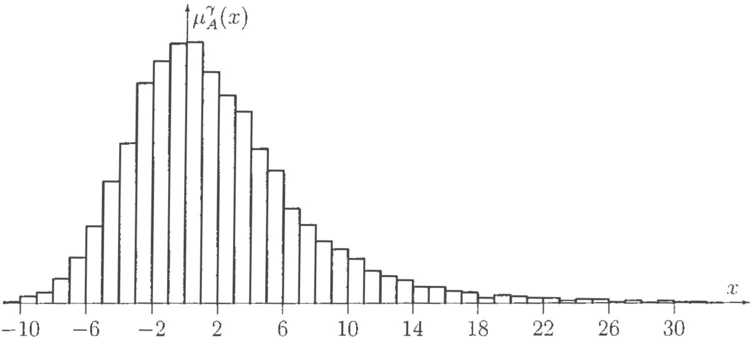

Figure 4.7: An estimate of a membership function $\mu_A^\gamma(x)$ calculated using the Mante Carlo method, [18].

Compliance. It is assumed that the uncertainty of the estimate $\hat{x}_b$ is decribed by the membership function

$$\mu_{\hat{x}_b}^\gamma(x) = \begin{cases} (1 - \frac{x-\hat{x}_b}{d_b^u})^{\gamma_b^u} & \text{for} \quad \hat{x}_b \leq x \leq \hat{x}_b + d_b^u \\ (1 + \frac{x-\hat{x}_b}{d_b^l})^{\gamma_b^l} & \text{for} \quad \hat{x}_b - d_b^l \leq x < \hat{x}_b \end{cases}$$

and for the estimate $\hat{x}_c$

$$\mu^{\gamma}_{\hat{x}_c}(x) = \begin{cases} (1 - \frac{x-\hat{x}_c}{d^u_c})^{\gamma^u_c} & \text{for} \quad \hat{x}_c \le x \le \hat{x}_c + d^u_c \\ (1 + \frac{x-\hat{x}_c}{d^l_c})^{\gamma^l_c} & \text{for} \quad \hat{x}_c - d^l_c \le x < \hat{x}_c \end{cases}$$

To find the membership function of the fuzzy number $D\hat{x} = \hat{x}_c - (1-\delta)\hat{x}_b$ as a linear combination of the fuzzy numbers $\hat{x}_b$ and $\hat{x}_c$, the η-cuts will be used, see Appendix A for explanation of this notion. For the number $\hat{x}_c$ the upper $\hat{x}^u_c$ and the lower $\hat{x}^l_c$ ends of the η-cut are as follows, see Fig. 4.8. For $\hat{x}^u_c$ we have

$$\Big(1 - \frac{\hat{x}^u_c - \hat{x}_c}{d^u_c}\Big)^{\gamma^u_c} = \eta$$

Then

$$\hat{x}^u_c = \hat{x}_c + d^u_c(1 - \eta^{\frac{1}{\gamma^u_c}})$$

In the same way, for $\hat{x}^l_c$

$$\Big(1 + \frac{\hat{x}^l_c - \hat{x}_c}{d^l_c}\Big)^{\gamma^l_c} = \eta$$

and

$$\hat{x}^l_c = \hat{x}_c - d^l_c(1 - \eta^{\frac{1}{\gamma^l_c}})$$

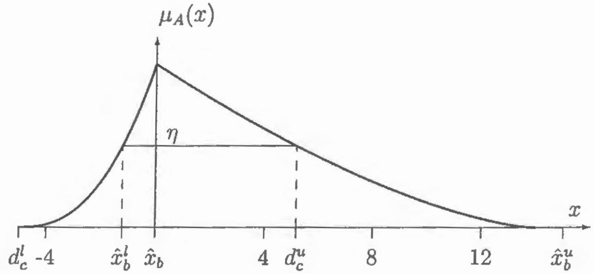

Figure 4.8: Fit of a membership function $\mu^{\gamma}_A(x)$ for $\gamma = 1.91$ and $d_A = 13.7$ to the histogram from [17], centered and normed.

Analogously the fuzzy number $\hat{x}_b$ can be treated. However, we consider the number $-(1-\delta)\hat{x}_b$. There is

$$\Big(1 - \frac{\hat{x}^u_b + (1-\delta)\hat{x}_b}{(1-\delta)d^u_b}\Big)^{\gamma^u_b} = \eta$$

from where the upper end $\hat{x}_b^u$ is given by

$$\hat{x}_b^u = -(1-\delta)\hat{x}_b + d_b^u(1-\delta)(1-\eta^{\frac{1}{\gamma_b^u}})$$

And for the lower end $\hat{x}_b^l$ the equation

$$\Big(1 + \frac{\hat{x}_b^l + (1-\delta)\hat{x}_b}{(1-\delta)d_b^l}\Big)^{\gamma_b^l} = \eta$$

provides

$$\hat{x}_b^l = -(1-\delta)\hat{x}_b - d_b^l(1-\delta)(1-\eta^{\frac{1}{\gamma_b^l}})$$

The η-cut of the number $D\hat{x}$ is obtained applying the interval calculus rules for the sum of the η-cuts of the numbers $\hat{x}_c$ and $-(1-\delta)\hat{x}_b$. Thus

$$D\hat{x}^u = D\hat{x} + d_c^u(1-\eta^{\frac{1}{\gamma_c^u}}) + d_b^l(1-\delta)(1-\eta^{\frac{1}{\gamma_b^l}})$$

$$D\hat{x}^l = D\hat{x} - d_c^l(1-\eta^{\frac{1}{\gamma_c^l}}) - d_b^u(1-\delta)(1-\eta^{\frac{1}{\gamma_b^u}})$$

For later use we find the value x^η – the distance between the right end of the η-cut and the right end of the support (i.e. 0-cut) equal to

$$d_{bc}^u = d_c^u + (1-\delta)d_b^l \tag{4.9}$$

Then

$$x^\eta = D\hat{x}+d_{bc}^u-D\hat{x}^u = D\hat{x}+d_c^u+(1-\delta)d_b^l-D\hat{x}-d_c^u(1-\eta^{\frac{1}{\gamma_c^u}})-d_b^l(1-\delta)(1-\eta^{\frac{1}{\gamma_b^l}}$$

or, after simple manipulations

$$x^\eta = d_c^u\eta^{\frac{1}{\gamma_c^u}} + d_b^l(1-\delta)\eta^{\frac{1}{\gamma_b^l}} \tag{4.10}$$

Let us now consider Fig. 4.9. We want to find the value x_α giving the area below the curve equal to A_α, which is the αth part of the whole area under the full membership function. By integrating the membership function we get

$$A_\alpha = \alpha\Big\{\frac{d_c^l}{1+\gamma_c^l} + \frac{d_c^u}{1+\gamma_c^u} + (1-\delta)[\frac{d_b^l}{1+\gamma_b^l} + \frac{d_c^u}{1+\gamma_b^u}]\Big\}$$

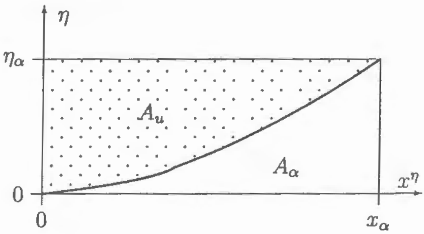

Figure 4.9: Calculation of x_α.

The area of the whole rectangle from the figure is $\eta_\alpha x_\alpha$. The are of the upper, shaded, part A_u can be found by integrating (4.10) as function $x^\eta(\eta)$. Thus

$$A_u = \int_0^{\eta_\alpha} \left[d_c^u \eta^{\frac{1}{\gamma_c^u}} + d_b^l(1-\delta)\eta^{\frac{1}{\gamma_b^l}}\right] d\eta$$

or, after integration,

$$A_u = \frac{d_c^u}{1+\gamma_c^u}\eta^{\frac{1}{\gamma_c^u}+1} + \frac{d_b^l(1-\delta)\gamma_b^l}{1+\gamma_b^l}\eta^{\frac{1}{\gamma_b^l}+1}$$

Finally we have

$$A_\alpha = \eta_\alpha x_\alpha - A_u$$

Inserting from (4.10) for $\eta = \eta_\alpha$ gives

$$A_\alpha = \eta_\alpha \left[d_c^u \eta_\alpha^{\frac{1}{\gamma_c^u}} + d_b^l(1-\delta)\eta_\alpha^{\frac{1}{\gamma_b^l}} - \frac{d_c^u \gamma_c^u}{1+\gamma_c^u}\eta_\alpha^{\frac{1}{\gamma_c^u}} - \frac{d_b^l(1-\delta)\gamma_b^l}{1+\gamma_b^l}\eta_\alpha^{\frac{1}{\gamma_b^l}}\right]$$

or after simplification

$$A_\alpha = \eta_\alpha \left[\frac{d_c^u}{1+\gamma_c^u}\eta_\alpha^{\frac{1}{\gamma_c^u}} + \frac{d_b^l(1-\delta)}{1+\gamma_b^l}\eta_\alpha^{\frac{1}{\gamma_b^l}}\right] \tag{4.11}$$

The area A under the membership function can be found by integration of the right brach, providing area A^u, and the left branch, providing area A^l. Thus

$$A^u = \int_0^1 d_c^u(1-\eta^{\frac{1}{\gamma_c^u}}) + d_b^l(1-\delta)(1-\eta^{\frac{1}{\gamma_b^l}})d\eta = \frac{d_c^u}{\gamma_c^u+1} + \frac{d_b^l(1-\delta)}{\gamma_b^l+1}$$

$$A^l = \int_0^1 d_c^l(1 - \eta^{\frac{1}{\gamma_c^l}}) + d_b^u(1-\delta)(1 - \eta^{\frac{1}{\gamma_b^u}})d\eta = \frac{d_c^l}{\gamma_c^l + 1} + \frac{d_b^u(1-\delta)}{\gamma_b^u + 1}$$

Therefore

$$A = \frac{d_c^u}{\gamma_c^u + 1} + \frac{d_b^l(1-\delta)}{\gamma_b^l + 1} + \frac{d_c^l}{\gamma_c^l + 1} + \frac{d_b^u(1-\delta)}{\gamma_b^u + 1}$$

and

$$A_\alpha = \alpha A = \frac{\alpha d_c^u}{\gamma_c^u + 1} + \frac{\alpha d_b^l(1-\delta)}{\gamma_b^l + 1} + \frac{\alpha d_c^l}{\gamma_c^l + 1} + \frac{\alpha d_b^u(1-\delta)}{\gamma_b^u + 1} \tag{4.12}$$

Inserting the above value A_α into (4.11), solving it for η_α and inserting the solution into (4.10) provides us with the value x_α, which cuts the right αth part of the area under the membership function of the fuzzy number $D\hat{x}$. Unfortunately, solving (4.11) in the general case is not possible analytically and must be solved numerically.

To find an approximate value of x_α we expand the function $A_\alpha(\eta_\alpha)$ given by equation (4.11) in the first order Taylor series around the value $\eta_0 = (2\alpha)^{\frac{\gamma_c^u}{\gamma_c^u+1}}$. This value of η_0 resembles the solution for the symmetric case. But it is worth to remeber that simplicity of the results obtained using this approach will depend very much on the ponit, around which the function is expanded. As

$$\frac{dA_\alpha(\eta_\alpha)}{d\eta_\alpha} = \frac{d_c^u}{\gamma_c^u}\eta_\alpha^{\frac{1}{\gamma_c^u}} + (1-\delta)\frac{d_b^l}{\gamma_b^l}\eta_\alpha^{\frac{1}{\gamma_b^l}}$$

then around the value $\eta_0 = (2\alpha)^{\frac{\gamma_c^u}{\gamma_c^u+1}}$ we have

$$\begin{aligned} A_\alpha(\eta_\alpha) \;&\approx 2\alpha\frac{d_c^u}{\gamma_c^u + 1} + (1-\delta)\frac{d_b^l}{\gamma_b^l + 1}(2\alpha)^{\frac{\gamma_c^u}{\gamma_b^l}}\frac{\gamma_b^l + 1}{\gamma_c^u + 1} + \\ &+ (2\alpha)^{\frac{1}{\gamma_c^u+1}}\Big[\frac{d_c^u}{\gamma_c^u} + (1-\delta)\frac{d_b^l}{\gamma_b^l}(2\alpha)^{\frac{\gamma_c^u}{\gamma_b^l}}\Big]\Delta\eta_\alpha \end{aligned}$$

Taking now into account (4.12), after some manipulations we get

$$\Delta\eta \approx \alpha\frac{\frac{d_c^l}{\gamma_c^l+1} - \frac{d_c^u}{\gamma_c^u+1} + (1-\delta)[\frac{d_b^u}{\gamma_b^u+1} + \frac{d_b^l}{\gamma_b^l+1}(1 - \frac{1}{2}(2\alpha)^{\frac{1+1/\gamma_b^l}{1+1/\gamma_c^u}})]}{(2\alpha)^{\frac{1}{\gamma_c^u+1}}[\frac{d_c^u}{\gamma_c^u} + (1-\delta)\frac{d_b^l}{\gamma_b^l}(2\alpha)^{\frac{\gamma_c^u}{\gamma_b^l}}]} = \alpha\beta$$

Thus, around the $\eta_0 = (2\alpha)^{\frac{\gamma_c^u}{\gamma_c^u+1}}$ we have

$$\eta_\alpha \approx (2\alpha)^{\frac{\gamma_c^u}{\gamma_c^u+1}} + \Delta\eta = 2\alpha[(2\alpha)^{-\frac{1}{\gamma_c^u+1}} + \frac{1}{2}\beta]$$

Inserting the above value to (4.10) and denoting by x_α the resulting value of x^η we finally get

$$x_\alpha \approx d_c^u[(2\alpha)^{\frac{\gamma_c^u}{\gamma_c^u+1}} + \alpha\beta]^{\frac{1}{\gamma_c^u}} + (1-\delta)d_b^l[(2\alpha)^{\frac{\gamma_b^l}{\gamma_b^l+1}} + \alpha\beta]^{\frac{1}{\gamma_b^l}} \quad (4.13)$$

For the symmetric case $\beta = 0$ and (4.13) reduces to the expression for the symmetric case, i.e. $x_\alpha = (2\alpha)^{\frac{1}{\gamma+1}} d_{bc}$.

Then, for a party to be compliant with risk α the following condition should be satisfied

$$D\hat{x} + d_{bc}^u \leq x_\alpha$$

where d_{bc}^u is given by (4.9). Thus

$$\hat{x}_c + d_{bc}^u - x_\alpha \leq (1-\delta)\hat{x}_b \quad (4.14)$$

For the approximate expression (4.13) the condition (4.14) is expressed as follows

$$\hat{x}_c + d_c^u\Big\{1 - [(2\alpha)^{\frac{\gamma_c^u}{\gamma_c^u+1}} + \alpha\beta]^{\frac{1}{\gamma_c^u}}\Big\} + (1-\delta)d_b^l\Big\{1 - [(2\alpha)^{\frac{\gamma_b^l}{\gamma_b^l+1}} + \alpha\beta]^{\frac{1}{\gamma_b^l}}\Big\} \leq (1-\delta)\hat{x}_b \quad (4.15)$$

Chapter 5

Conclusions

The paper deals with the problem of checking compliance of pollutant emission with a given limit in the case when the observed emission values are known with high uncertainty, which is the case of national inventories of emissions of the greenhouse gases. High uncertainty must influence trading in emission permits, which is frequently used to minimize the emission abatement cost [9].

Not only the inventory itself, but also its uncertainty is calculated with relatively low accuracy. This should be taken into account when deriving the compliance and emission trading rules. The idea proposed in this paper lies in grounding the derivations on the fuzzy set approach. A family of fuzzy numbers depending on a free parameter is introduced. This parameter can be chosen to appropriately shape the distribution of uncertainty. The approach provides the linear formulas, which can be used for designing a market for the efficient emission permits.

The results obtained are generalizations of the results derived for the interval type of uncertainty. It was, however, shown that the rules for the interval case can be still used instead of the generalized ones, provided the appropriately higher value of the risk of noncompliance is used.

Then a nonsymmetric membership functions are considered. In this case a closed analytical solution could not be found. But an approximate solution was considered and another generalized rule for compliance has been derived.

Appendix A: Fuzzy sets and fuzzy numbers

To introduce the notion of a fuzzy set let first us consider a classical set A from an universe U. It can be conveniently described by the characteristic function χ_A defined as

$$\chi_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}$$

which say that a point $u \in U$ belongs to the set, if $\chi_A(u) = 1$, or does not belong, if $\chi_A(u) = 0$.

In a fuzzy set the characteristic function χ_A is generalized to take any value from the interval $[0, 1]$. It is then called *a membership function* and is denoted μ_A. The value of a membership function $\mu_A(u)$ reflects the degree of acceptance of the point u to the set. Thus, *a fuzzy set* is characterized by the set A and the membership function μ_A. Then, an usual set is a special fuzzy set with the membership function being the characteristic function. A comparison of a membership function and a characteristic function of a set is shown in fig. 1.

A fuzzy set can be also fully characterized by a family of so called η-cuts[1] denoted by A_η, i. e. points of U, for which the value $\mu_A(u)$ assumes at least the value η, see fig. 1, where an example of a η-cut for $\eta = 0.5$ is depicted.

Two additional notions connected with a fuzzy set are worth to mention. One is *the support*, called supp A, which is the set of points u, for which the membership function is positive, i. e.:

$$\text{supp } A = \{u \in U : \mu_A(u) > 0\}$$

[1]Here we call as the η-cut of a fuzzy set A the notion usually called the α-cut, i.e. the set $A_\eta = \{x \in \text{supp } A | \mu_A(x) \geq \eta\}$, for $\eta \in (0, 1]$.

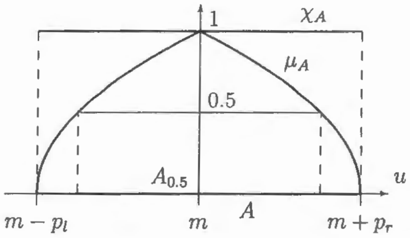

Figure 1: The characteristic function and a membership functions of the set A.

Another definition of the support may be formulated using η-cuts, as

$$\text{supp } A = \lim_{\eta \to 0} A_\eta$$

The second notion is *the core* of the fuzzy set, called core A, which is the set of points, for which the membership function is equal 1, i. e.:

$$\text{core } A = \{u \in U : \mu_A(u) = 1\}$$

Using the notion of the η-cuts we may also write

$$\text{core } A = A_1$$

A special case of a fuzzy set A is called *a fuzzy number*, if it satisfies three additional conditions:

1. core A consists of only one point.
2. The membership function does not increase starting from the core point towards both sides.
3. Every η-cut is a (connected) close interval.

The η-cuts for a fuzy number form a family of intervals. Each interval can be interpreted as our conviction in precision of knowledge of the core value. Values of the level η close to 1 mean that we are well convinced that the core value is precise. Small values of η, close to 0, mean that our conviction is small. See also [3] for more formal discussion of this subject. Calculations performed on fuzzy numbers allow us to process whole this knowledge in common.

Technically, two functions defined for nonnegative arguments may be introduced, L and R, [1], such that they have the unique value 1 at 0, $L(0) = R(0) = 1$, equal zero for arguments greater or equal 1, $L(u) = R(u) = 0$ for $u \geq 1$, and are not increasing. Then, given core $A = \{m\}$, the membership function of a fuzzy number may be constructed using the above functions as its left and right branches

$$\mu_A^l(u) = L\Big(\frac{m-u}{p_l}\Big) \quad \text{for } u \leq m \tag{1}$$

$$\mu_A^r(u) = R\Big(\frac{u-m}{p_r}\Big) \quad \text{for } u \geq m \tag{2}$$

where p_l and p_r are scale parameters, see Fig. 1. Let us denote the fuzzy number constructed this way as $A(m, p_l, p_r)_{LR}$.

Although operations on fuzzy sets or fuzzy numbers can be defined in a more general context, they are first restricted only to fuzzy numbers described in the above LR form. For two fuzzy numbers $A(m, p_l, p_r)_{LR}$ and $B(n, q_l, q_r)_{LR}$ the following operations are defined, see [2]:

1. Addition

$$A + B = (m + n, p_l + q_l, p_r + q_r)_{LR} \tag{3}$$

2. Multiplication by a positive real number c

$$cA = (cm, cp_l, cp_r)_{LR} \tag{4}$$

3. Multiplication by a negative real number c

$$cA = (cm, |c|p_r, |c|p_l)_{LR} \tag{5}$$

with interchange of the function L and R in (1) and (2)

$$\mu_{cA}^l(u) = R\Big(\frac{cm-u}{|c|p_r}\Big) \quad \text{for } u \leq cm$$

$$\mu_{cA}^r(u) = L\Big(\frac{u-cm}{|c|p_l}\Big) \quad \text{for } u \geq cm$$

If we further restrict attention to $L = R = M$ and $p_r = p_l = p$, then

$$\mu_{cA}(u) = M\Big(\frac{|u-cm|}{|c|p}\Big) \tag{6}$$

In the general case interval calculus for the η-cuts can be used to get the appropriate operation.

Appendix B: Determination of a membership function

The membership function can be determined freely. However, to be closer to reality, it can be set on the basis of some data. Here, a method of determination of the membership function parameters from the emission distribution obtained during Monte Carlo simulation is presented. We assume in this appendix that the data are shifted in such way that $\mu(0) = 1$.

Symmetric case

Let $\mu(x_i)$, $i = 1, 2, \ldots, N$, be the observed values of distribution. We want to fit to them the following distribution function

$$\mu^{\gamma}(x) = a\Big(1 - \frac{|x_i|}{d}\Big)^{\gamma}$$

where a is a scale coefficient. This is a highly nonlinear function and to make the problem "less" nonlinear we fit its logarithm using the sum of squares loss function

$$J = \sum_{i=1}^{N} \Big[\ln \mu(x_i) - \ln a - \gamma \ln \Big(1 - \frac{|x_i|}{d}\Big)\Big]^2$$

Denoting $\ln a = A$ and differentiating J to find the stationary points we get

$$\frac{\partial J}{\partial A} = -2 \sum_{i=1}^{N} \Big[\ln \mu(x_i) - A - \gamma \ln \Big(1 - \frac{|x_i|}{d}\Big)\Big] = 0$$

from where

$$A = \frac{1}{N} \sum_{i=1}^{N} \Big[\ln \mu(x_i) - \gamma \ln \Big(1 - \frac{|x_i|}{d}\Big)\Big] \qquad (7)$$

Further

$$\frac{\partial J}{\partial \gamma} = -2\sum_{i=1}^{N}\Big[\ln\mu(x_i) - \gamma\ln\Big(1-\frac{|x_i|}{d}\Big)\Big]\ln\Big(1-\frac{|x_i|}{d}\Big) = 0$$

which gives

$$\gamma = \frac{\sum_{i=1}^{N}\Big[\ln\mu(x_i) - A\Big]\ln\Big(1-\frac{|x_i|}{d}\Big)}{\sum_{i=1}^{N}\Big[\ln\Big(1-\frac{|x_i|}{d}\Big)\Big]^2} \tag{8}$$

Differentiation with respect to d provides a rather complicated formula in d

$$\frac{\partial J}{\partial d} = -2\Big[\ln\mu(x_i) - A - \gamma\ln\Big(1-\frac{|x_i|}{d}\Big)\Big]\frac{|x_i|}{1-\frac{|x_i|}{d}} = 0$$

At the same time, the value d can be easily approximately estimated from the emission distribution and should be practically only slightly adjusted. Thus, the following algorithm for estimation of the parameters A, γ and d is proposed:

1. Assume the starting values $d^{(0)}$ and $\gamma^{(0)}$.
2. Calculate $A^{(1)}$ according to (7).
3. Calculate improved value $\gamma^{(1)}$ according to (8).
4. Iterate calculations of $A^{(n)}$ and $\gamma^{(n)}$ for $n = 2, 3, \ldots$ until the stop condition is fulfilled.
5. Choose the value $d^{(k)}$ for $k = 1$ according to a one-dimensional optimization scheme for the function $J(d)$ and for the values of A and γ settled in iteration step 4.
6. Repeat steps 2 to 4 and then step 5 with incremented k until the stop condition for minimization of the function $J(d)$ is fulfilled.

The algorithm converged quickly for the two cases considered, for which the data were taken from the literature and were presented as histograms. The results are depicted in Figs. 4.4 and 2.

The approximate shapes of the function $J(d)$ for these two cases are shown in Figs. 3 and 4. These approximations were computed from only few points calculated during the execution of the estimation algorithm presented

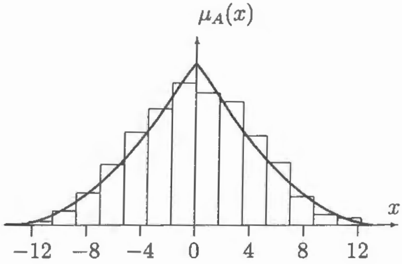

Figure 2: Fit of a membership function $\mu_A^\gamma(x)$ for $\gamma = 1.91$ and $d_A = 13.7$ to the histogram from [17], centered and normed.

above. Thus, it should be rather considered as an illustration of possible function $J(d)$ approximate shape. Nevertheless, the function seems to be quite convenient for optimization. An interesting further observation is that similar estimates of γ and d were obtained for both cases.

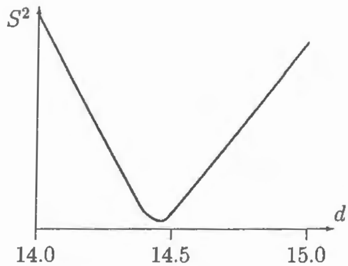

Figure 3: An approximate dependence of the sum of squares of errors on d for the data from [16].

Nonsymmetric case

We distinguish now among the obseved values of distributions those for negative arguments, $x_i < 0$, $\mu(x_i)$, $i = 1, 2, \ldots, N^l$, and those for the nonnegative ones, $x_j \geq 0$, $\mu(x_j)$, $j = 1, 2, \ldots, N^u$, $N^l + N^u = N$. The fitted function has

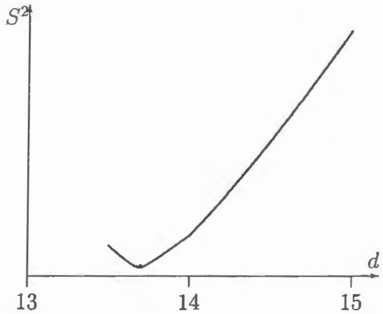

Figure 4: An approximate dependence of the sum of squares of errors on d for the data from [17].

the form

$$\mu^\gamma(x) = \begin{cases} a(1 - \frac{x}{d^u})^{\gamma^u} & \text{for} \quad 0 \le x \le d_c^u \\ a(1 + \frac{x}{d^l})^{\gamma^l} & \text{for} \quad -d^l \le x < 0 \end{cases}$$

Analogously as for the symmetric case we cosider the logarithms

$$J = \sum_{j=1}^{N^u} \left[\ln \mu^u(x_j) - \ln a - \gamma \ln \left(1 - \frac{x_j}{d^u}\right)\right]^2 + \sum_{i=1}^{N^l} \left[\ln \mu^l(x_i) - \ln a - \gamma \ln \left(1 + \frac{x_i}{d^l}\right)\right]^2$$

where μ^u and μ^l are the right and left branches of the function μ, respectively.

Denoted again $\ln a = A$ and differentiating J we get

$$A = \frac{1}{N}\Big\{ \sum_{j=1}^{N^u} [\ln \mu^u(x_j) - \gamma^u \ln(1 - \frac{x_j}{d^u})] + \sum_{i=1}^{N^l} [\ln \mu^l(x_i) - \gamma^l \ln(1 + \frac{x_i}{d^l})]\Big\}$$

$$\gamma^u = \frac{\sum_{j=1}^{N^u} [\ln \mu^u(x_j) - A] \ln(1 - \frac{x_j}{d^u})}{\sum_{j=1}^{N^u} [\ln(1 - \frac{x_j}{d^u})]^2} \qquad x_j \ge 0$$

$$\gamma^u = \frac{\sum_{i=1}^{N^l} [\ln \mu^l(x_i) - A] \ln(1 - \frac{x_i}{d^l})}{\sum_{i=1}^{N^l} [\ln(1 - \frac{x_i}{d^l})]^2} \qquad x_j < 0$$

To estimate the parameters A, γ^u, γ^l, d^u and d^l a simple generalization of the iteration procedure prezented for the symmetric case can be used.

Appendix C. Proofs of equations (4.1) - (4.3)

To prove these equations let us write the fuzzy numbers $\hat{x}_b^\gamma$ and $\hat{x}_c^\gamma$ using the LR notation used in expressions (3)-(5). Specifically, the notation with $L = R = M$ will be used below. Then, with

$$M(u) = (1-u)^\gamma$$

we have

$$\hat{x}_b^\gamma = (\hat{x}_b, d_b, d_b)_M \qquad \hat{x}_c^\gamma = (\hat{x}_c, d_c, d_c)_M$$

Thus, applying multiplication by a negative number rule (5) yields

$$-(1-\delta)\hat{x}_b^\gamma = \Big(-(1-\delta)\hat{x}_b, (1-\delta)d_b, (1-\delta)d_b\Big)_M$$

and for the interval calculus with the rule (3.2), after applying the addition operation (3) we get

$$D\hat{x}^\gamma = \hat{x}_c^\gamma - (1-\delta)\hat{x}_b^\gamma =$$

$$= \Big(\hat{x}_c - (1-\delta)\hat{x}_b, d_c + (1-\delta)d_b, d_c + (1-\delta)d_b\Big)_M = (D\hat{x}, d_{bc}, d_{bc})_M$$

Thus, from (6) the membership function is

$$\mu_{D\hat{x}^\gamma} = M\Big(\frac{|x - D\hat{x}^\gamma|}{d_{bc}}\Big) = \Big(1 - \frac{|x - D\hat{x}^\gamma|}{d_{bc}}\Big)^\gamma$$

and the support

$$\text{supp } D\hat{x}^\gamma = [D\hat{x}^\gamma - d_{bc}, D\hat{x}^\gamma + d_{bc}]$$

These are just the expressions (4.2) and (4.3).

It seems obvious from the above reasoning that its generalization for the operator (2.2) requires changing of the addition operator (3) to

$$A +_\zeta B = \Big(m + n, (1 - \zeta)(p_l + p_r), (1 - \zeta)(p_r + q_r)\Big)_M \qquad (9)$$

leaving (4) and (5) intact. Then, repeating the reasoning the equations (4.1) - (4.3) are obtained again.

Let us notice, however, that change of the addition operator to (9) brings a disdvantage. As far as (3) is commutative and associative, that is it holds

$$A + B = B + A \quad \text{and} \quad A + B + C = (A + B) + C = A + (B + C)$$

Then the operator (9) is only commutative, because

$$(A +_\zeta B) +_\zeta C \neq A +_\zeta (B +_\zeta C)$$

Thus, practically, the operator (9) can be used only for pairs of numbers. But this is actually exactly what is needed in the application considered in this paper.

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