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COMPLIANCE AND EMISSION TRADING RULES FOR ASYMETRIC EMISSION UNCERTAINTY ESTIMATES

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Compliance and emission trading rules for asymmetric emission uncertainty estimates

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Abstract. Greenhouse gases emission inventories are computed with rather low precision. Moreover, their uncertainty distributions may be asymmetric. This should be accounted for in the compliance and trading rules. In this paper we model the uncertainty of inventories as intervals or using fuzzy numbers. The latter allows us to shape better the uncertainty distributions. Obtained compliance and emission trading rules generalize those for the symmetric uncertainty distributions, which were considered in the earlier papers. However, unlike in the symmetric distribution, it is necessary in the asymmetric fuzzy case to apply approximations due to nonlinearities in the formulas. The final conclusion is that the interval uncertainty rules can be applied, but with much higher substitutional noncompliance risk, which is a parameter of the rules.

Keywords: greenhouse gases emission inventories, uncertainty, interval calculus, fuzzy sets, compliance, emission permit trading.

1. Introduction

Emission of greenhouse gases is a basic element of the climate change models, see e. g. (Stern, 2007) where results are presented in probabilistic terms. However, greenhouse gases inventories estimates are not calculated exactly. Possible error magnitudes depend on types of gases considered, activities, and countries, ranging from few to over 100 percent. Moreover, distributions of errors for different gases as well as for national inventories may be asymmetric (Ramirez at al., 2006; Winiwarter and Muik, 2007). The methods of checking compliance and particularly establishing rules for emission trading proposed up to now for the uncertain inventories (Jonas at al., 2007; Jonas and Nilsson, 2007; Nahorski et al., 2007; Nahorski and Horabik, 2008a) concern only the symmetric distributions and mostly the interval uncertainty models.

In (Nahorski et al., 2007) the compliance and trading rules were considered for the interval uncertainties of emissions. In order to have high enough likelihood of fulfilling the compliance, lower limit of reductions were required (undershooting), and an appropriate recalculation

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of the traded emissions to be performed. However, interval uncertainty approach provides too conservative reduction of limits and recalculation of traded emissions. Although the stochastic case may be useful for the determination of new compliance rule, see also (Gillenwater et al., 2007), only a complicated formula for recalculation of the traded emissions has been provided (Nahorski et al., 2007), practically useless, because it is valid only for uncorrelated inventories. In this paper a fuzzy uncertainty is considered. The fuzzy set calculus basically inherits the rules from the interval calculus, and this way provides simpler calculations than that for the stochastic variables. At the same time the fuzzy variables may be shaped to have distributions more concentrated around observed values than for the intervals, where the information on distribution is lost. And therefore it can better approximate the real distributions. This paper also deals with the asymmetric cases. This way it aims at improving precision of assessment of satisfying the given emission limits or reductions, in the sense of guarantying (with a prescribed small risk) fulfillment of this limit or reduction, including in it emission trading among parties and other possible flexible mechanisms included in the Kyoto Protocol. Improved precision, as compared with the interval case, means lower costs of compliance and more reliable estimates of inventories for the climate change models.

We derive in this paper a new formula for recalculation of the trading quantities for the fuzzy and symmetric distributions, which is generalization of that for the interval approach. To obtain an analogous formula for the asymmetric fuzzy case an approximation is required. The one proposed in this paper is a generalization of both those for the symmetric fuzzy case and for the asymmetric interval approach.

Summing up, we derive here new rules for checking compliance and for emission trading, for asymmetric fuzzy distributions. They are generalizations of the rules presented in (Nahorski et al., 2007; Nahorski and Horabik, 2008a) for symmetric distributions and interval uncertainty and reduce to them as special instances when appropriate parameters are taken. Comparison of the rules obtained for the fuzzy approach with those for the interval one shows that the latter can be used equivalently, but with a much bigger substitutional parameter than originally designed for the noncompliance risk.

In section 2 we formulate the problem and introduce some basic notation. Then, in section 3, we deal with the asymmetric interval uncertainty and we derive conditions for checking compliance and formulas for so called efficient emissions, which can be directly traded, without taking into account the emission uncertainty. In section 4 a family of fuzzy numbers is introduced. They are used to model the full inventory uncertainty and form the basis for derivations of generalized compliance and emission trading rules. These rules are compared with the interval approach rules and their equivalence in applications considered in the paper, for appropriately chosen parameters, is shown. Section 5 concludes.

2. Problem formulation

Two systems for reducing greenhouse gases emissions have been applied. One, called "cap and trade", like for example in the European Trade System, where the limits on emission from chosen activities are distributed between member countries in the first stage and then, finally, between companies within the European Union. The problem here is to check, if L, the given emission limit for the company, expressed as emission permit, has not been exceeded, i.e. if

$$x \leq L \tag{1}$$

where x is the real, unknown emission of a party in a considered year. Unfortunately, x is not known exactly, as only its available estimate of the emission $\hat{x}$ can be calculated. The estimate of the total emission by a party is calculated by inventory of emissions from every contributing activity, including absorption by sinks. They are, however, highly unsure, see (Winiwarter, 2004; Monni at al., 2007). Moreover, uncertainties of inventories $\hat{x}$ differ between different activities both in the range and distributions.

Another system is used in the Kyoto Protocol, which requires from each participating country to reduce a prespecified percent of its basic year emission within the given period (from 1990 to 2008-12 for most countries), although some countries are granted a possibility of stabilizing the emission at the basic year level or even of a limited increase of its emission. Three so called flexible mechanisms are connected with the Kyoto Protocol. These are: Joint Implementation, Clean Development Mechanism and Permit Trading. All of them are related with some forms of buying the emission saved by other parties. In all these cases a problem is to check, if the declared reduction has been actually achieved.

With emission reduction, the compliance checking is slightly more complicated than in the "cap and trade" system, because also the referred limit is unsure. This leads to the problem of comparison of two uncertain values. This problem will be, however, transformed here to the form similar to (1), that is to comparison of unsure value with the exactly known limit. Let us denote by δ the fraction of the party emission to be reduced. The value of δ may be negative, for parties, which were alloted limitation of the emission increase. Denoting by x_b the basic emission and by x_c the emission to be checked, the following inequality should be satisfied

$$x_c - (1 - \delta)x_b \leq 0 \tag{2}$$

This inequality has the same form as (1), with the inspected variable $x_c - (1-\delta)x_b$ and the limit $L = 0$. And similarly as earlier, neither x_c nor x_b are known precisely enough. Thus, only the difference of estimates can be calculated

$$\hat{x}_c - (1 - \delta)\hat{x}_b \tag{3}$$

where both $\hat{x}_c$ and $\hat{x}_b$ are known inaccurately. In the Kyoto Protocol context, x_b is the emission in the basic year and x_c the emission in the compliance period. We are not, however, interested here in reference and compliance times, but only in the values to be compared.

Moreover, the emission estimate of a Party may be modified by selling or buying uncertain emissions, which adds to the final uncertainty of the left hand side. These problems are discussed in the sequel using two models of uncertainty: interval and fuzzy.

3. Interval uncertainty

Material in this section is a generalization of the results for the symmetric intervals given in (Nahorski et al., 2007). The idea of the methods is the same, but the results differ due to changed assumption, although they reduce to the previous ones when the symmetric intervals are considered in the equations. The derivations of this section are fundamental for the rest of the material and therefore are presented in a rather complete way, even if they are rather straight generalizations of those for the symmetric intervals.

3.1. Compliance

Let us denote the lower spread of the uncertainty interval by d^l and the upper by d^u. Then, the real basic emission x_b and the real checked emission x_c are situated in the intervals

$$x_b \in [\hat{x}_b - d_b^l, \hat{x}_b + d_b^u], \qquad x_c \in [\hat{x}_c - d_c^l, \hat{x}_c + d_c^u]$$

Known limit

We start with the simpler case of the limit L known exactly. To be fully sure that a party (typically a company) fulfills the limit, its emission inventory should satisfy the following condition, see fig. 1 (a).

$$\hat{x}_c + d_c^u \leq L \tag{4}$$

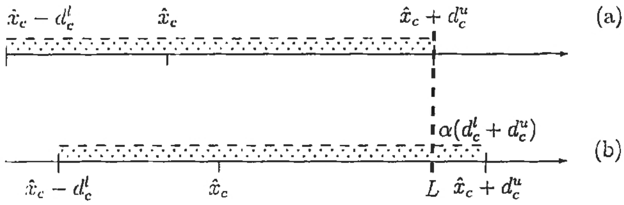

Figure 1. Full compliance (a) and the compliance with risk α (b) in the interval uncertainty approach for the known limit case.

As the bounds can be quite large, a weaker condition will be used, see (Nahorski et al., 2007). A party is compliant with the risk α if its emission inventory satisfies the condition

$$\hat{x}_c + d_c^u \leq L + \alpha(d_c^l + d_c^u) \tag{5}$$

The risk is here understood as a likelihood that the party may not fulfill the agreed obligation as to the emission limit or reduction due to uncertainty of emission inventory.

Condition (5) means that it is allowed for the party that the αth part of its emission estimate (inventory) uncertainty interval lies above the limit L, see fig. 1 (b). After some algebraic manipulations the condition (5) can be also written in the following form

$$\hat{x}_c + [1 - (1 + \frac{d_c^l}{d_c^u})\alpha] d_c^u \leq L \tag{6}$$

The above condition shows that a part of the upper spread of the uncertainty interval is added to the emission estimate before checking compliance. This can be also interpreted that some unreported emission, due to uncertainty, is included in the condition to reduce the risk of noncompliance.

For the symmetric interval $d_c^l = d_c^u = d_c$ the condition (6) takes the form

$$\hat{x}_c + (1 - 2\alpha)d_c \leq L$$

which has been derived in (Nahorski and Horabik, 2008a).

Emission reduction

A more difficult case of checking reduction of the emission, when both the checked and the basic emission are unsure, will be transformed to the problem of known limit by considering the difference of the checked and reduced emissions, as mentioned earlier. Using the interval calculus rules, we get

$$x_c - (1-\delta)x_b \in [D\hat{x} - d_{bc}^l, D\hat{x} + d_{bc}^u]$$

where

$$D\hat{x} = \hat{x}_c - (1-\delta)\hat{x}_b \tag{7}$$

and the lower and upper spreads are

$$d_{bc}^l = d_c^l + (1-\delta)d_b^u, \qquad d_{bc}^u = d_c^u + (1-\delta)d_b^l \tag{8}$$

However, the inventories $\hat{x}_b$ and $\hat{x}_c$ are dependent and the values of d_{bc}^l and d_{bc}^u are usually much smaller than those resulting from the above expression. In (Nahorski et al., 2007) it was proposed to take it into account by modification of the formulas (8) to

$$d_{bc}^l = (1-\zeta)(d_c^l + (1-\delta)d_b^u) \tag{9}$$

$$d_{bc}^u = (1-\zeta)(d_c^u + (1-\delta)d_b^l) \tag{10}$$

where $0 \leq \zeta \leq 1$ is an appropriate chosen dependency coefficient. This will be also assumed in this paper[1].

Now, to be fully credible, that is to be sure that (2) is satisfied, the party should prove

$$D\hat{x} + d_{bc}^u \leq 0 \tag{11}$$

This nonequality condition is analogous to (4), with the upper limit $L = 0$.

When a party is compliant with risk α, then the part of its distribution that lies above zero is not bigger than α, see fig. 2 for the geometrical interpretation. That is, it holds $D\hat{x} + d_{bc}^u \leq 2\alpha d_{bc}^u$. After simple algebraic manipulations this gives the condition

$$\hat{x}_c + [1 - (1 + \frac{d_{bc}^l}{d_{bc}^u})\alpha]d_{bc}^u \leq (1-\delta)\hat{x}_b \tag{12}$$

[1] Modification of the addition operator has a disadvantage. As far as the usual addition is commutative and associative, i. e. for the intervals A, B and C it holds $A + B = B + A$ and $A + B + C = (A + B) + C = A + (B + C)$, then the modified operator with operations (9) and (10), denoted below as $+_\zeta$, is only commutative and not associative, because then $(A +_\zeta B) +_\zeta C \neq A +_\zeta (B +_\zeta C)$. Thus, practically, the operator $+_\zeta$ can be only used for pairs of numbers. But this is actually exactly what is needed in the application considered in this paper.

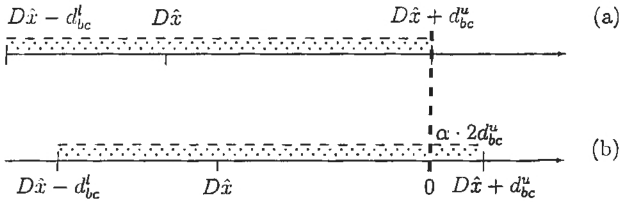

Figure 2. Full compliance (a) and the compliance with risk α (b) in the interval uncertainty approach for the emission reduction case.

This condition is analogous to (6). Thus, to prove the compliance with risk α the party has to fulfill its obligation with the inventory emission estimate increased by the value $[1 - (1 + \frac{d_{bc}^l}{d_{bc}^u})\alpha]d_{bc}^u$, dependent on its uncertainty.

3.2. Emission trading

The above compliance proving policy can be used to modify rules of emission trading. The main idea presented in earlier papers (Nahorski et al., 2007; Nahorski and Horabik, 2008a) consists in transferring the emissions seller uncertainty to the emissions buyer together with the traded quota of emission and then including it in the buyer's emission balance. Here it is adapted to the asymmetric distributions.

Let us denote by $R_c^{uS} = d_c^{uS}/\hat{x}_c^S$ and $R_c^{lS} = d_c^{lS}/\hat{x}_c^S$ the respective relative upper and lower spreads of uncertainty intervals of the seller and by $\hat{E}^S$ the traded amount of estimated emission. This emission is associated with lower and upper spreads of the uncertainty intervals $\hat{E}^S R_c^{lS}$ or $\hat{E}^S R_c^{uS}$, respectively.

Known limit

First, let us consider the simpler case of known limit L. Before the trade the buyer has to satisfy the condition (6), which is reformulated to

$$\hat{x}_c^B + d_c^{uB} - (d_c^{lB} + d_c^{uB})\alpha \le L^B$$

After buying $\hat{E}^S$ units of emissions from the seller and including the corresponding uncertainty in the formula, the new condition looks like

$$\hat{x}_c^B - \hat{E}^S + d_c^{uB} + \hat{E}^S R_c^{uS} - (d_c^{uB} + \hat{E}^S R_c^{uS} + d_c^{lB} + \hat{E}^S R_c^{lS})\alpha \le L^B$$

The above conditions differ in the following value called the effective emission (Nahorski et al., 2007)

$$E_{eff} = \hat{E}^S - \hat{E}^S R_c^{uS} + \hat{E}^S(R_c^{uS} + R_c^{lS})\alpha$$

which can be transformed to the form

$$E_{eff} = \hat{E}^S\Big\{1 - [1 - (1 + \frac{d_c^{lS}}{d_c^{uS}})\alpha]R_c^{uS}\Big\} \qquad (13)$$

The effective emission is smaller than the estimated emission. The bigger the relative upper spread of the uncertainty interval of the seller is, the smaller is the effective emission. But it also depends on the ratio d_c^{uS}/d_c^{lS}, and obviously on α.

Emission reduction

When emission reduction is required, before the trade the buying party checks the following condition

$$\hat{x}_c^B + d_{bc}^{uB} - (d_{bc}^{uB} + d_{bc}^{lB})\alpha \leq \ (1 - \delta^B)\hat{x}_b^B$$

After the transaction the condition changes into

$$\hat{x}_c^B - \hat{E}^S + d_{bc}^{uB} + \hat{E}^S R_c^{uS} - (d_{bc}^{uB} + \hat{E}^S R_c^{uS} + d_{bc}^{lB} + \hat{E}^S R_c^{lS})\alpha \leq (1 - \delta^B)\hat{x}_b^B$$

Due to partial cancellation of the subtracted estimated emission and its uncertainty in the buyer's emission balance the effective emission is

$$E_{eff} = \hat{E}^S\Big\{1 - [1 - (1 + \frac{d_c^{lS}}{d_c^{uS}})\alpha]R_c^{uS}\Big\} \qquad (14)$$

This is exactly the same formula as (13). The bigger the seller's upper spread of uncertainty interval is, the less purchased unit is accounted for the buyer. Expressions (13) and (14) reduce emissions estimated with an arbitrary precision to globally comparable values, which can be directly subtracted from country's estimated emission. This way it is possible to construct a market for the effective emissions, see (Nahorski et al., 2007; Nahorski and Horabik, 2007) for details.

4. Fuzzy uncertainty

Interval uncertainty approach does not use any information on the distribution of inventory errors. Thus, it ends with too conservative results. Modeling the uncertainty using stochastic approach causes problems related with nonlinearities of the underlying algebra. Instead, we propose to use the fuzzy approach in modeling uncertainty distribution. It allows for good approximation of the distribution while keeping simple algebra of the interval calculus. A short information on fuzzy sets and some related notions is given in the Appendix.

In this paper the fuzzy numbers (see Appendix for a definition) are used to model imperfect knowledge of the uncertainty. A fuzzy number is a straight generalization of an ordinary number, whose value is unsure. This is the situation, which we spot in the greenhouse gas inventories.

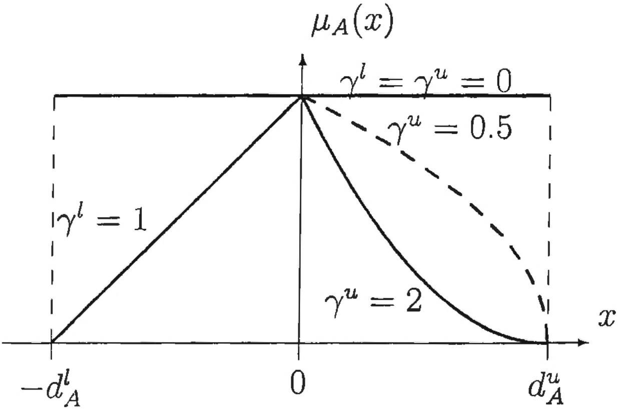

Figure 3. Membership functions for different values of γ.

Usually, the main problem with the fuzzy set approach is to determine the membership function. Here, we introduce analytical membership functions dependent on parameters. To estimate the parameters, the function can be fitted to the distribution obtained from Monte Carlo simulations, as shown in the sequel. In lack of experimental distributions, the parameter can be fixed to fit the experimenter expectation[2].

The most popular membership functions are the triangular or trapezoidal ones. These functions are, however, rather inconvenient for our

[2] It is perhaps worth to mention at this point that we treat the fuzzy approach only as an approximation of distribution and algebraic rules for the variables and not for introduction of the possibility function, see e. g. (Bandemer, 2006), that is another possible approach to the problem.

purpose because of their bad approximations of the distribution tails, which are very important in the applications described here.

Consider a family of fuzzy numbers $A^\gamma = \{(x, \mu_A^\gamma(x)) | x \in \text{supp } A^\gamma\}$ indexed by a vector parameter $\gamma = [\gamma_1^u, \gamma_2^l] \in C^+ \times C^+$, with the support supp $A^\gamma = [-d_A^l, d_A^u]$. The proposed membership function has the form, see fig. 3.

$$\mu_A^\gamma(x) = \begin{cases} a(1 - \frac{x}{d_A^u})^{\gamma^u} & \text{for } 0 \leq x \leq d_A^u \\ a(1 + \frac{x}{d_A^l})^{\gamma^l} & \text{for } d_A^l \leq x < 0 \end{cases} \qquad \gamma^l, \gamma^u \neq 0 \tag{15}$$

where a is a normalizing factor used for fitting the membership function to empirical distributions. In the theoretical considerations it can be assumed that the membership function has been normalized and therefore $a = 1$ is taken in the sequel. Let us note that taking $\gamma^l = \gamma^u = 0$ we get the even distribution, see fig. 3, and actually reduce the considerations to the interval case.

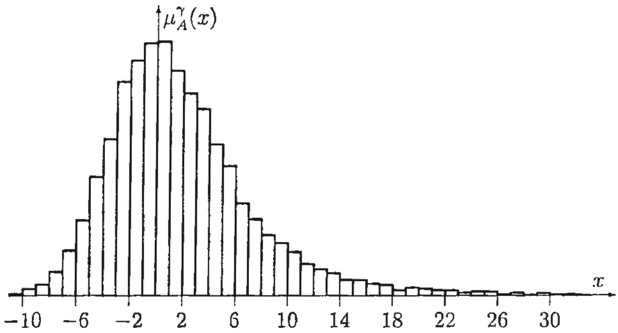

Figure 4. An estimate of a membership function $\mu_A^\gamma(x)$ calculated using the Monte Carlo method.

Fig. 4 presents an estimate of an asymmetric distribution obtained using the Monte Carlo method and presented in (Winiwarter and Muik, 2007).

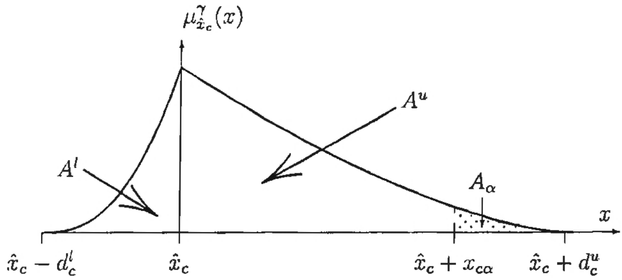

Figure 5. Definition of areas under asymmetric fuzzy number membership function.

4.1. Compliance

It is assumed that the uncertainty of the estimate $\hat{x}_b$ is described by the membership function

$$\mu^{\gamma}_{\hat{x}_b}(x) = \begin{cases} (1 - \frac{x-\hat{x}_b}{d^u_b})^{\gamma^u_b} & \text{for } \hat{x}_b \le x \le \hat{x}_b + d^u_b \\ (1 + \frac{x-\hat{x}_b}{d^l_b})^{\gamma^l_b} & \text{for } \hat{x}_b - d^l_b \le x < \hat{x}_b \end{cases}$$

and of the estimate $\hat{x}_c$ by

$$\mu^{\gamma}_{\hat{x}_c}(x) = \begin{cases} (1 - \frac{x-\hat{x}_c}{d^u_c})^{\gamma^u_c} & \text{for } \hat{x}_c \le x \le \hat{x}_c + d^u_c \\ (1 + \frac{x-\hat{x}_c}{d^l_c})^{\gamma^l_c} & \text{for } \hat{x}_c - d^l_c \le x < \hat{x}_c \end{cases} \tag{16}$$

Known limit

We start with the exactly known limit case. First, we calculate by integration the whole area A under the membership function. It is the sum of two areas, see fig. 5

$$A = A^l + A^u$$

$$A^l = \int_{\hat{x}_c - d^l_c}^{\hat{x}_c} (1 + \frac{x - \hat{x}_c}{d^l_c})^{\gamma^l_c} dx = \frac{d^l_c}{1 + \gamma^l_c}$$

$$A^u = \int_{\hat{x}_c}^{\hat{x}_c + d^u_c} (1 - \frac{x - \hat{x}_c}{d^u_c})^{\gamma^u_c} dx = \frac{d^u_c}{1 + \gamma^u_c}$$

We want now to find the distance $x_{c\alpha}$ between $\hat{x}_c$ and $\hat{x}_c + x_{c\alpha}$, where the latter is the value cutting off the most right αth part of the area under the membership function, see fig. 5. This area, denoted A_α, for $0 \le \alpha \le A^u/(A^l + A^u)$, where A^l is the area under the left branch of the membership function and A^u under the right branch is

$$A_\alpha = \int_{\hat{x}_c + x_{c\alpha}}^{\hat{x}_c + d_c^u} (1 - \frac{x - \hat{x}_c}{d_c^u})^{\gamma_c^u} dx = \frac{d_c^u}{1 + \gamma_c^u} (1 - \frac{x_{c\alpha}}{d_c^u})^{1+\gamma_c^u}$$

Now, it must hold

$$A_\alpha = \alpha A$$

which after some algebraic manipulations gives

$$x_{c\alpha} = \left\{1 - [(1 + \frac{d_c^l}{d_c^u} \frac{1 + \gamma_c^u}{1 + \gamma_c^l})\alpha]^{\frac{1}{1+\gamma_c^u}}\right\} d_c^u$$

Finally, the compliance checking condition is

$$\hat{x}_c + \left\{1 - [(1 + \frac{d_c^l}{d_c^u} \frac{1 + \gamma_c^u}{1 + \gamma_c^l})\alpha]^{\frac{1}{1+\gamma_c^u}}\right\} d_c^u \le L \tag{17}$$

For the interval uncertainty case $\gamma_c^u = \gamma_c^l = 0$. Then the above condition is the same as (6), and the symmetric case $d_c^l = d_c^u = d_c$ and $\gamma_c^l = \gamma_c^u = \gamma_c$, and the above condition takes the form

$$\hat{x}_c + [1 - (2\alpha)^{\frac{1}{1+\gamma_c}}] d_c \le L$$

This formula has been derived in (Nahorski and Horabik, 2008a).

For the symmetric case only the range $0 \le \alpha \le 0.5$ is practically worth to be considered, as for $\alpha = 0.5$ the above condition takes the form $\hat{x}_c \le L$, and for $\alpha > 0.5$ we would let for exccesion of the limit, i.e. for $\hat{x} > L$. For the asymmetric case the range $0 \le \alpha \le A^u/(A^l + A^u)$ should be considered. Thus, the limiting α can take values greater or smaller than 0.5. For the interval uncertainty the range will be $0 \le \alpha \le d^u/(d^l + d^u)$.

Let us then notice that for the asymmetric distribution, like in fig. 4, the likelihood of noncompliance when $\hat{x}_c$ is only compared with the limit L is greater than 0.5, in the sense that for the random occurance of the inventory $\hat{x}_c$, but compatible with the distribution, the limit will be more frequently exceeded then not achieved.

Emission reduction

For the emission reduction case, to find the membership function of the fuzzy number $D\hat{x} = \hat{x}_c - (1 - \delta)\hat{x}_b$ as a linear combination of the fuzzy numbers $\hat{x}_b$ and $\hat{x}_c$, the η-cuts will be used, see Appendix for

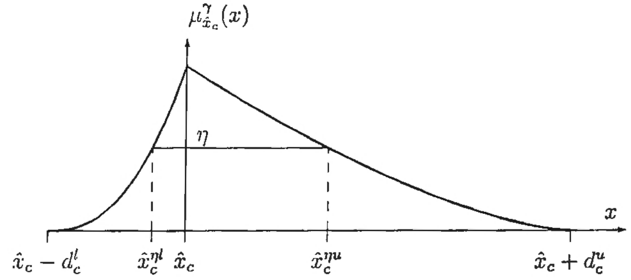

Figure 6. Asymmetric fuzzy number and definitions of related parameters.

explanation of this notion. For the number $\hat{x}_c$ the upper $\hat{x}_c^{\eta u}$ and the lower $\hat{x}_c^{\eta l}$ ends of the η-cut are as follows, see Fig. 6. For $\hat{x}_c^{\eta u}$ we have

$$\left(1 - \frac{\hat{x}_c^{\eta u} - \hat{x}_c}{d_c^u}\right)^{\gamma_c^u} = \eta$$

Then, assuming $\gamma_c^u \neq 0$,

$$\hat{x}_c^{\eta u} = \hat{x}_c + d_c^u(1 - \eta^{\frac{1}{\gamma_c^u}})$$

In the same way, for $\hat{x}_c^{\eta l}$, assuming $\gamma_c^l \neq 0$,

$$\left(1 + \frac{\hat{x}_c^{\eta l} - \hat{x}_c}{d_c^l}\right)^{\gamma_c^l} = \eta$$

and

$$\hat{x}_c^{\eta l} = \hat{x}_c - d_c^l(1 - \eta^{\frac{1}{\gamma_c^l}})$$

For $\gamma_c^u = 0$ or $\gamma_c^l = 0$ we have $\eta = 1$. For this case the expression like $\eta^{\frac{1}{\gamma_c^u}}$ is not formally defined. Thus, we additionally define

$$\eta^{\frac{1}{\gamma_c^u}} = 0, \qquad \text{for} \quad \gamma_c^u = 0$$

$$\eta^{\frac{1}{\gamma_c^l}} = 0 \qquad \text{for} \quad \gamma_c^l = 0$$

The fuzzy number $\hat{x}_b$ can be treated analogously. But we consider the number $-(1-\delta)\hat{x}_b$. Taking analogous asumptions and additional definitions as above, we now look for $\hat{x}_b^{\eta u}$ satisfying

$$\left(1 - \frac{\hat{x}_b^{\eta u} + (1-\delta)\hat{x}_b}{(1-\delta)d_b^u}\right)^{\gamma_b^u} = \eta$$

from where the upper end $\hat{x}_b^{\eta u}$ of the η-cut is given by

$$\hat{x}_b^{\eta u} = -(1-\delta)\hat{x}_b + d_b^u(1-\delta)(1-\eta^{\frac{1}{\gamma_b^u}})$$

For the lower end $\hat{x}_b^{\eta l}$ of the η-cut the equation

$$\Big(1 + \frac{\hat{x}_b^{\eta l} + (1-\delta)\hat{x}_b}{(1-\delta)d_b^l}\Big)^{\gamma_b^l} = \eta$$

provides

$$\hat{x}_b^{\eta l} = -(1-\delta)\hat{x}_b - d_b^l(1-\delta)(1-\eta^{\frac{1}{\gamma_b^l}})$$

Finally, the η-cut of the number $D\hat{x}$ is obtained applying the modified interval calculus rules (9) and (10) for the sum of the η-cuts of the numbers $\hat{x}_c$ and $-(1-\delta)\hat{x}_b$. Thus

$$D\hat{x}^{\eta u} = D\hat{x} + (1-\zeta)[d_c^u(1-\eta^{\frac{1}{\gamma_c^u}}) + d_b^l(1-\delta)(1-\eta^{\frac{1}{\gamma_b^l}})] \qquad (18)$$

$$D\hat{x}^{\eta l} = D\hat{x} - (1-\zeta)[d_c^l(1-\eta^{\frac{1}{\gamma_c^l}}) + d_b^u(1-\delta)(1-\eta^{\frac{1}{\gamma_b^u}})] \qquad (19)$$

The above equations show dependences of $D\hat{x}^{\eta l}$ and $D\hat{x}^{\eta u}$ on η, that is they are the reverse functions of the two branches of the membership function $\mu_{D\hat{x}}^{\gamma}(x)$, see fig. 6.

Let us now transform (18) to

$$1 - \frac{D\hat{x}^{\eta u} - D\hat{x}}{d_{bc}^u} = \frac{d_c^u\eta^{1/\gamma_c^u} + d_b^l(1-\delta)\eta^{1/\gamma_b^l}}{d_c^u + d_b^l(1-\delta)} \qquad (20)$$

where d_{bc}^u is given by (10), and define γ_{bc}^u to satisfy the equation

$$\frac{d_c^u\eta^{1/\gamma_c^u} + d_b^l(1-\delta)\eta^{1/\gamma_b^l}}{d_c^u + d_b^l(1-\delta)} = \eta^{\frac{1}{\gamma_{bc}^u}}$$

From the above

$$\gamma_{bc}^u = \frac{1}{\log_\eta \frac{d_c^u\eta^{1/\gamma_c^u} + d_b^l(1-\delta)\eta^{1/\gamma_b^l}}{d_c^u + d_b^l(1-\delta)}} = \frac{\log\eta}{\log \frac{d_c^u\eta^{1/\gamma_c^u} + d_b^l(1-\delta)\eta^{1/\gamma_b^l}}{d_c^u + d_b^l(1-\delta)}} \qquad (21)$$

In the spirit of earlier additional definitions we also define

$$\gamma_{bc}^u = 0 \qquad \text{for} \quad \gamma_c^u = 0 \quad \text{or} \quad \gamma_b^l = 0$$

Now it is possible to write the right branch of the membership function as

$$\mu_{\hat{x}_{bc}}^{\gamma u}(x) = (1 - \frac{x - D\hat{x}}{d_{bc}^u})^{\gamma_{bc}^u} \qquad D\hat{x} \le x \le D\hat{x} + d_{bc}^u \qquad (22)$$

Likewise we get

$$\mu^{\gamma^l}_{\hat{x}_{bc}}(x) = (1 + \frac{x - D\hat{x}}{d^l_{bc}})^{\gamma^l_{bc}} \qquad D\hat{x} - d^l_{bc} \leq x \leq D\hat{x} \tag{23}$$

where d^l_{bc} is given by (9), and

$$\gamma^l_{bc} = \frac{1}{\log_\eta \frac{d^l_c \eta^{1/\gamma^l_c} + d^u_b(1-\delta)\eta^{1/\gamma^u_b}}{d^l_c + d^u_b(1-\delta)}} = \frac{\log \eta}{\log \frac{d^l_c \eta^{1/\gamma^l_c} + d^u_b(1-\delta)\eta^{1/\gamma^u_b}}{d^l_c + d^u_b(1-\delta)}} \tag{24}$$

with

$$\gamma^l_{bc} = 0 \qquad \text{for} \quad \gamma^l_c = 0 \quad \text{or} \quad \gamma^u_b = 0$$

Now, the most right αth part of the area under the membership function (22) is

$$A_\alpha = \int_{D\hat{x}+x_\alpha}^{D\hat{x}+d^u_{bc}} (1 - \frac{x - D\hat{x}}{d^u_{bc}})^{\gamma^u_{bc}} dx = \frac{d^u_{bc}}{1+\gamma^u_{bc}} (1 - \frac{x_\alpha}{d^u_{bc}})^{1+\gamma^u_{bc}}$$

and the area under the entire membership function (22) - (23) is

$$A = \int_{D\hat{x}-d^l_{bc}}^{D\hat{x}} (1 + \frac{x - D\hat{x}}{d^l_{bc}})^{\gamma^l_{bc}} dx + \int_{D\hat{x}}^{D\hat{x}+d^u_{bc}} (1 - \frac{x - D\hat{x}}{d^u_{bc}})^{\gamma^u_{bc}} dx =$$

$$= \frac{d^l_{bc}}{1+\gamma^l_{bc}} + \frac{d^u_{bc}}{1+\gamma^u_{bc}} \tag{25}$$

Because $A_\alpha = \alpha A$, then its solution for x_α, denoted $x_{bc\alpha}$, has the form

$$x_{bc\alpha} = \left\{1 - \left[(1 + \frac{d^l_{bc}}{d^u_{bc}} \frac{1+\gamma^u_{bc}}{1+\gamma^l_{bc}})\alpha\right]^{\frac{1}{1+\gamma^u_{bc}}}\right\} d^u_{bc} \tag{26}$$

and finally the compliance condition is

$$\hat{x}_c + \left\{1 - \left[(1 + \frac{d^l_{bc}}{d^u_{bc}} \frac{1+\gamma^u_{bc}}{1+\gamma^l_{bc}})\alpha\right]^{\frac{1}{1+\gamma^u_{bc}}}\right\} d^u_{bc} \leq (1-\delta)\hat{x}_b \tag{27}$$

This condition is analogous to (17). For the interval case $\gamma^l_{bc} = \gamma^u_{bc} = 0$ and (27) reduces to (12). For the symmetric distribution $d^l_{bc} = d^u_{bc} = d_{bc}$ and $\gamma^l_{bc} = \gamma^u_{bc} = \gamma_{bc}$ and it reduces to

$$\hat{x}_c + [1 - (2\alpha)^{\frac{1}{1+\gamma_{bc}}}] d_{bc} \leq (1-\delta)\hat{x}_b \tag{28}$$

The condition (28) has been derived in (Nahorski and Horabik, 2007).

4.2. Emission trading

The formula for the efficient emission can be quite easily obtained for the symmetric distribution (28) using derivations similar to interval case. Before the trade the buying party has to satisfy the condition

$$\hat{x}_c^B + [1 - (2\alpha)^{\frac{1}{1+\gamma_{bc}^B}}]d_{bc}^B \leq (1 - \delta^B)\hat{x}_b^B$$

and after buying $\hat{E}^S$ emission units from the seller it becomes

$$\hat{x}_c^B - \hat{E}^S + [1 - (2\alpha)^{\frac{1}{1+\gamma_{bc}^B}}](d_{bc}^B + \hat{E}^S R_c^S) \leq (1 - \delta^B)\hat{x}_b^B$$

Then the efficient emission is

$$E_{eff} = \hat{E}^S\left\{1 - [1 - (2\alpha)^{\frac{1}{1+\gamma_{bc}^B}}]R_c^S\right\} \quad (29)$$

However, the problem becomes more difficult for the asymmetric distributions, as then the uncertainty distribution bounds d_{bc}^l and d_{bc}^u enter nonlinearly in the compliance condition (27). This is why linearization is now used to obtain the result. The exact derivation is presented in (Nahorski and Horabik, 2008b). This way the following expression for the efficient emission is obtained

$$E_{eff} = \hat{E}^S\left\{1 - \{1 - [(1 + \frac{d_c^{lS}}{d_c^{uS}})\alpha]^{\frac{1}{1+\gamma_{bc}^{uB}}}\}R_c^{uS}\right\} \quad (30)$$

It generalizes expressions for simpler cases. In particular, for the known limit case the following substitution should be made: $\gamma_{bc}^{uB} \to \gamma_c^{uB}$. For the symmetric distributions the substitutions are: $d_c^{lS} \to d^S$, $d_c^{uS} \to d^S$, $\gamma_{bc}^{uB} \to \gamma_{bc}^B$, which provide (29). For the interval uncertainty: $\gamma_{bc}^{uB} \to 0$, which gives (14).

In comparison with the formula (14) for the interval uncertainty, the formulas (29) and (30) depend on parameters γ_{bc}^B or γ_{bc}^{uB} of the emission buyer uncertainty distributions. This would complicate considerably the market, as the traded quota depends in such case both on the seller and the buyer uncertainty distributions. This problem will not be discussed in this paper.

4.3. Equivalence of approaches

Let us notice that actually the fuzzy approach formulas (17) and (27) can be considered equivalent to the interval approach ones (6) and (12), provided appropriate values of α are chosen for both cases. Equivalence of the compliance checking formulas causes also equivalence of the emission trading counterparts. Denoting the interval case by the subscript $_I$ and the fuzzy case by $_F$, and equaling the effective emitions $E_{eff,F} = E_{eff,I}$, after simple algebraic manipulations we arrive at the following condition

$$(1 + \frac{d_c^{lS}}{d_c^{uS}})\alpha_F = [(1 + \frac{d_c^{lS}}{d_c^{uS}})\alpha_I]^{1+\gamma_{bc}^{uB}}$$

If the cases $\alpha_F = 0$ (no noncompliance risk) and $\gamma_{bc}^{uB} = 0$ (interval uncertainty) are excluded, then for $0 < \alpha_I, \alpha_F \leq \frac{d_c^{uS}}{d_c^{uS}+d_c^{lS}}$ and $\gamma_c^{uS} > 0$ we have

$$\alpha_I > \alpha_F$$

Dependence of α_I on α_F and γ_{bc}^{uB} is shown in Table I. The results show that α_I rises quickly when γ_{bc}^{uB} rises. In cases considered in our calculations estimates of γ_{bc}^{uB} close to or much higher than 1.5 were obtained. Then, practically it seems that $\alpha_I \geq 0.3$ should be taken even for small values of α_F.

An interpretation of these results is quite straightforward. Within the considered family of distributions ignorance of the uncertainty distribution in the interval case requires bigger reduction. To obtain the same effective emitions as for the fuzzy uncertainties, a bigger substitutional noncompliance risk should be adopted in the interval approach. Thus, for α_I, at least the values 0.3 or higher should be taken to compensate for ignorance of the exact knowledge of the uncertainty interval distribution, even if a small noncompliance risk is actually meant.

5. An example

In the example the data from the Monto Carlo simulation presented in (Ramirez at al., 2006) are used. Uncertainty distributions of three gases emissions, carbon dioxide CO_2, methane CH_4 and fluorine F, are considered. The uncertainty distributions were chosen to illustrate the proposed rules of trade. They are depicted in figs. 7, 8 and 9 together with fits of the distribution functions (15). It is assumed that each emission is related to different companies, called CO, CH and F, respectively. Table II contains parameters of the distributions.

We do not consider the compliance, and only the trade. Let us then suppose that the three companies mentioned: CO, CH and F, want to trade with each other. The uncertainty of emission in the company CO is small, less than 4%, while in the rest it is around 38%. On the other hand, the shape of the uncertainty distributions of CO and CH are similar, with values of γ of the order 2 – 2.5 for the lower and 4 –

Table I. Dependence of α_I on α_F and γ_{bc}^{uB}.

$\alpha_F \downarrow \gamma_{bc}^{uB} \rightarrow$	0.1	0.5	1	1.5	2	2.5
			$d_c^l/d_c^u = 0.2$			
0.05	0.06	0.13	0.20	0.27	0.33	0.37
0.10	0.12	0.20	0.29	0.36	0.41	0.45
0.15	0.18	0.27	0.35	0.42	0.47	0.51
0.20	0.23	0.32	0.41	0.47	0.52	0.55
0.25	0.28	0.37	0.46	0.51	0.56	0.59
			$d_c^l/d_c^u = 0.5$			
0.05	0.06	0.12	0.18	0.24	0.28	0.32
0.10	0.12	0.19	0.26	0.31	0.35	0.39
0.15	0.17	0.25	0.32	0.37	0.41	0.44
0.20	0.22	0.30	0.37	0.41	0.45	0.47
0.25	0.27	0.35	0.41	0.45	0.48	0.50
			$d_c^l/d_c^u = 1$			
0.05	0.06	0.11	0.16	0.20	0.23	0.26
0.10	0.12	0.17	0.22	0.26	0.29	0.32
0.15	0.17	0.22	0.27	0.31	0.33	0.35
0.20	0.22	0.27	0.32	0.35	0.37	0.38
0.25	0.27	0.32	0.35	0.38	0.40	0.41

Table II. Parameters of the distributions.

Distribution	d^l [Tg]	γ^l	γ^u	d^u [Tg]
CO_2	4.8	2.6	4.5	6.9
CH_4	4.3	2.1	3.9	6.7
F	2.0	1.4	1.4	3.1

4.5 for the upper branch, while the shape of F is close to triangular, with γ equal 1. In Table III the values of E_{eff} are depicted for three assumed trades, when each company at turn is the seller while other are buyers. Two values of the original noncompliance risk $\alpha = 0.05$ or 0.1 were assumed and substitutional values of α_I are given in the right side of the table. Most of them are of the order of 0.4. For CO, with

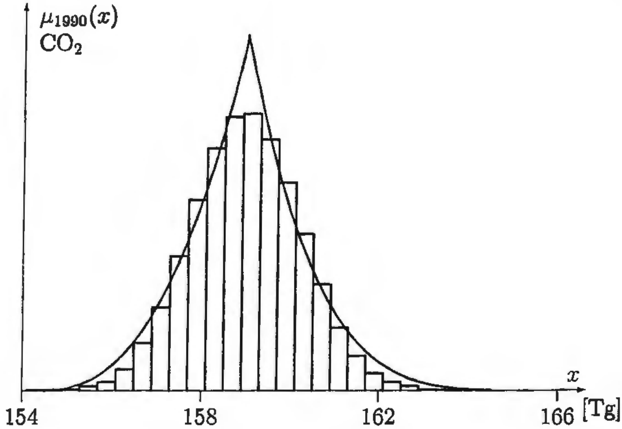

Figure 7. Fit of a membership function $\mu_A^\gamma(x)$ to the histogram for emission of CO_2.

small uncertainty, the values E^1_{eff} are only sligtly smaller than 1. The values E^2_{eff} and E^3_{eff} are much smaller, around 0.8 – 0.9.

Let us notice that for the fuzzy distribution there is no unique substitutional risk parameter α_I related with the seller, because it also depends on who is the buyer. This is what causes problems in the trade as compared to the interval case. A solution to avoid it might be that a common value 0.4 or a smaller one, like 0.35, is taken for α_I to organize the market with a substitutional interval uncertainty. This way the market schedulled in (Nahorski et al., 2007) can be applied. A market with substitutional risk parameters α_I dependent on the buyer is, however, an interesting question. It will be considered elsewhere.

6. Conclusions

The paper deals with the problem of checking compliance of pollutant emission with a given limit in the case when the observed emission values are highly uncertain with asymmetric uncertainty distributions. High uncertainty should be also considered in trading in emission permits, which is frequently used to minimize the emission abatement cost, and this is also done in the paper. Asymmetric uncertainty is evidenced

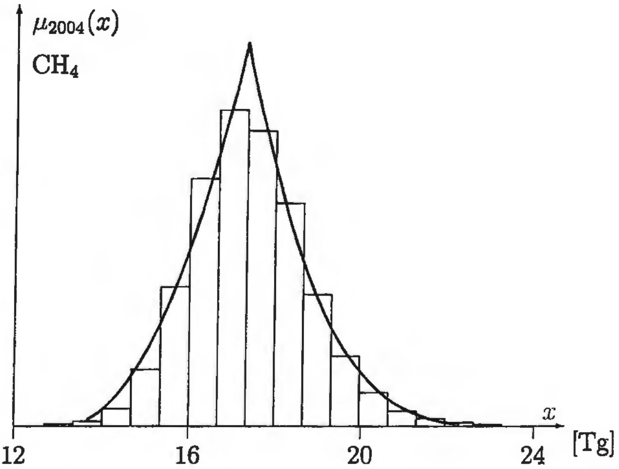

Figure 8. Fit of a membership function $\mu_A^7(x)$ to the histogram for emission of CH_4.

by recent investigations, and particularly by Monte Carlo simulations of uncertainty distributions.

An interesting case[3] of an asymmetric distribution of uncertainty is connected with the risk in valuing forest carbon offsets caused by accidental losses, e.g. due to wildfires (Hurteau et al., 2009). The uncertainty there has a specific one-sided distribution. This case entered already the implementation stage in the U. S. forest carbon storage project (Mignone at al., 2009). However, the solutions applied there take into account that the related uncertainty is eventually resolved in the future, as the damages are known after they have happened. This is in opposition to the case discussed in this paper, where uncertainties are inherent part of data considered in all stages of decision making.

A market with the effective emission permits has been outlined in earlier papers (Nahorski et al., 2007; Nahorski and Horabik, 2008a) for the symmetric case. That construction is valid also in the asymmetric case discussed in this paper.

[3] This direction of research has been brought to our attention by one of undisclosed reviewers.

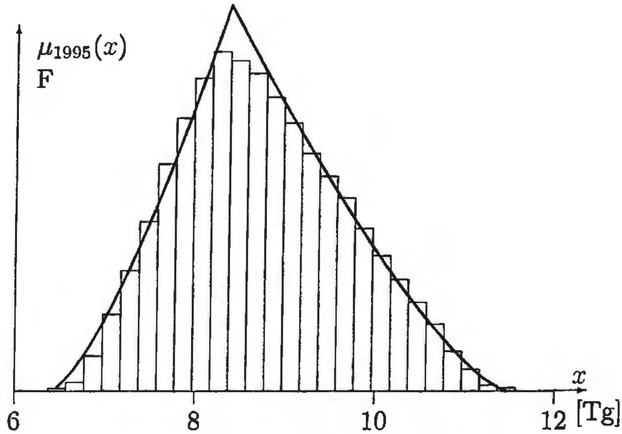

Figure 9. Fit of a membership function $\mu_A^7(x)$ to the histogram for emission of F.

The idea proposed in this paper lies in grounding the derivations in the fuzzy set approach. A family of fuzzy numbers depending on free parameters is introduced. These parameters can be chosen to appropriately shape the distribution of uncertainty. The approach provides the closed form formulas, which can be used for designing a market for the efficient emission permits. However, for the most general case of an asymmetric membership functions a closed analytical solution could not be found. An approximate solution was considered for this case and a generalized rule for compliance has been derived.

Application of the fuzzy numbers and consideration of asymmetric distributions enabled us to much more precisely determine the required level of reduced inventories to get high likelihood of fulfilling the given limit or reduction. Moreover, the better accuracy in determining the level give rise to better scaling the amounts of emission emitted by parties for using them in trading, which has a measurable financial meaning. Approximating distribution by a function dependent on parameters allowed us to derive the analytical expressions for reduction of emissions and for scaling the traded emissions. The distribution parameters have been acquired by fitting the distribution functions to the data from the Monte Carlo simulations.

Table III. Efficient emissions in the trade and substitutional values of α_I for interval uncertainty.

Emission	R_c^u	E_{eff}^1	E_{eff}^2	E_{eff}^3	α_I^1	α_I^2	α_I^3
			$\alpha = 0.05$				
CO	0.043	seller	0.86	0.86	-	0.21	0.36
CH	0.385	0.98	seller	0.85	0.39	-	0.36
F	0.371	0.97	0.75	seller	0.39	0.37	-
			$\alpha = 0.1$				
CO_2	0.043	seller	0.89	0.90	-	0.28	0.41
CH_4	0.385	0.99	seller	0.88	0.44	-	0.42
F	0.371	0.98	0.79	seller	0.44	0.42	-

The results obtained are generalizations of the results derived for the interval and symmetric uncertainty models. However, it was shown that the rules for the interval case can be used instead of the generalized ones, provided the appropriately higher value of the risk of noncompliance is substituted in the interval case.

Although the fits of the functions presented in this paper to the data are quite good, except perhaps in the central part of the uncertainty interval, a question of a possible better fit to the data has been risen by one of the undisclosed reviewers. As this is certainly possible with a more flexible class of functions, a possibility of obtaining close analytical solution may be a challenging problem. It will be a subject of further investigations.

Appendix: Fuzzy sets and fuzzy numbers

To introduce the notion of a fuzzy set let us first consider a classical set A from an universe U. It can be conveniently described by the characteristic function χ_A defined as

$$\chi_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}$$

which says that a point $u \in U$ belongs to the set, if $\chi_A(u) = 1$, or does not belong, if $\chi_A(u) = 0$.

In a fuzzy set the characteristic function χ_A is generalized to take any value from the interval $[0, 1]$. It is then called *a membership function* and is denoted μ_A. The value of a membership function $\mu_A(u)$ reflects the degree of acceptance of the point u to the set. Thus, ***a fuzzy set*** is characterized by the set A and the membership function μ_A. Then, an usual set is a special fuzzy set with the membership function being the characteristic function. A comparison of a membership function and a characteristic function of a set is shown in fig. 10.

A fuzzy set can be also fully characterized by a family of so called η-cuts[4] denoted by A_η, i. e. points of U, for which the value $\mu_A(u)$ assumes at least the value η, see fig. 10, where an example of a η-cut for $\eta = 0.5$ is depicted.

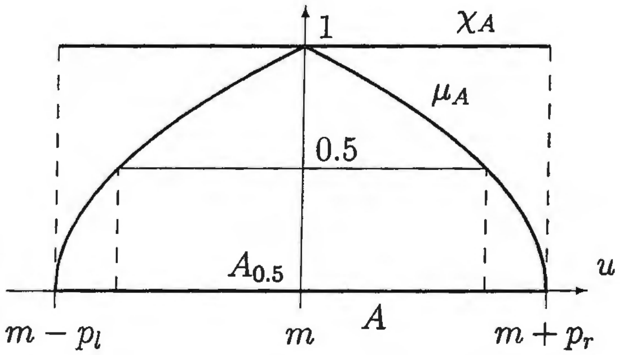

Figure 10. The characteristic function and a membership functions of the set A.

Two additional notions connected with a fuzzy set are worth to mention. One is ***the support***, called supp A, which is the set of points u, for which the membership function is positive, i. e.:

$$\text{supp } A = \{u \in U : \mu_A(u) > 0\}$$

Another definition of the support may be formulated using η-cuts, as

$$\text{supp } A = \lim_{\eta \to 0} A_\eta$$

The second notion is ***the core*** of the fuzzy set, called core A, which is the set of points, for which the membership function is equal 1, i. e.:

$$\text{core } A = \{u \in U : \mu_A(u) = 1\}$$

[4] Here we call as the η-cut of a fuzzy set A the notion usually called the α-cut, i.e. the set $A_\eta = \{x \in \text{supp } A | \mu_A(x) \geq \eta\}$, for $\eta \in (0, 1]$.

Using the notion of the η-cuts we may also write

$$\text{core } A = A_1$$

A fuzzy set A is called *a fuzzy number*, if it satisfies three additional conditions:

1. core A consists of only one point.
2. The membership function does not increase starting from the core point toward both sides.
3. Every η-cut is a (connected) close interval.

A weaker definition of a fuzzy number is often used, with the first condition replaced by

1' There exists a point belonging to the core A.

But in this paper we use the stronger former definition.

The η-cuts for a fuzy number form a family of intervals. Each interval can be interpreted as our conviction in precision of knowledge of the core value. Values of the level η close to 1 mean that we are well convinced that the core value is precise. Small values of η, close to 0, mean that our conviction is small. See also (Dubois and Prade, 2005) for more formal discussion of this subject. Calculations performed on fuzzy numbers allow us to process whole this knowledge in common.

Technically, two functions defined for nonnegative arguments may be introduced, L and R, (Bandemer, 2006), such that they have the unique value 1 at 0, $L(0) = R(0) = 1$, equal zero for arguments greater or equal 1, $L(u) = R(u) = 0$ for $u \geq 1$, and are not increasing. Then, given core $A = \{m\}$, the membership function of a fuzzy number may be constructed using the above functions as its left and right branches

$$\mu^l_A(u) = L\Big(\frac{m-u}{p_l}\Big) \quad \text{for } u \leq m \tag{31}$$

$$\mu^r_A(u) = R\Big(\frac{u-m}{p_r}\Big) \quad \text{for } u \geq m \tag{32}$$

where p_l and p_r are scale parameters, see Fig. 10. Let us denote the fuzzy number constructed this way as $A(m, p_l, p_r)_{LR}$.

Although operations on fuzzy sets or fuzzy numbers can be defined in a more general context, they are first restricted only to fuzzy numbers described in the above LR form. For two fuzzy numbers $A(m, p_l, p_r)_{LR}$ and $B(n, q_l, q_r)_{LR}$ the following operations are defined, see (Dubois and Prade, 1978):

1. Addition

$$A + B = (m + n, p_l + q_l, p_r + q_r)_{LR} \tag{33}$$

2. Multiplication by a positive real number c

$$cA = (cm, cp_l, cp_r)_{LR} \tag{34}$$

3. Multiplication by a negative real number c

$$cA = (cm, |c|p_r, |c|p_l)_{RL} \tag{35}$$

with interchange of the function L and R in (31) and (32)

$$\mu^l_{cA}(u) = R\Big(\frac{cm - u}{|c|p_r}\Big) \quad \text{for } u \leq cm$$

$$\mu^r_{cA}(u) = L\Big(\frac{u - cm}{|c|p_l}\Big) \quad \text{for } u \geq cm$$

In the general case interval calculus for the η-cuts can be used to get the appropriate operation.

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