NOTE ON THE TRANSFORMATION OF AN HEINEAN SERIES.

By Prof. *L. J. Rogers.*

THE properties of the series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2+\ldots$$

in which the $(r+1)^{\text{th}}$ term is

$$\frac{(1-a)(1-aq)\ldots(1-aq^{r-1})(1-b)\ldots(1-bq^{r-1})}{(1-q)\ldots(1-q^r)(1-c)\ldots(1-cq^{r-1})}x^r$$

have been investigated by Heine in his *Kugelfunctionen*, Vol. I., Chap. 2, under the functional form $\phi[a, b, c, q, x]$.

Moreover, in *Crelle.*, Vol. XXXII., he establishes a chain-fractional form for the quotient of

$$\frac{1-b}{1-c}\phi[a, bq, cq, q, x]\div\phi[a, b, c, q, x],$$

from the easily proved identity

$$\phi[a, bq, cq, q, x]-\phi[a, b, c, q, x]$$
$$=\frac{(1-a)(b-c)}{(1-c)(1-cq)}\phi[aq, bq, cq^2, q, x].$$

This form is

$$\frac{1-b}{1-c-}\;\frac{(1-a)(b-c)x}{1-cq-}\;\frac{(1-bq)(a-cq)x}{1-cq^2-}$$
$$\times\frac{(1-aq)(b-cq)qx}{1-cq^3-}\;\frac{(1-bq^2)(a-cq^2)qx}{1-cq^4-}\;\ldots\ldots\ldots(1),$$

the $2r^{\text{th}}$ link being

$$\frac{(1-aq^{r-1})(b-cq^{r-1})q^{r-1}x}{1-cq^{2r-1}-},$$

and the $(2r+1)^{\text{th}}$ being

$$\frac{(1-bq^r)(a-cq^r)q^{r-1}x}{1-cq^{2r}-}.$$

Now, if $b=1$ and $ax=cq$, we get

$$\phi\left[\frac{cq}{x},\ q,\ cq,\ q,\ x\right]$$

$$=\frac{1}{1-}\ \frac{x-cq}{1-cq-}\ \frac{(1-q)(1-x)cq}{1-cq^2-}\ \frac{(x-cq^2)(1-cq)q}{1-cq^3-}$$

$$=\frac{1}{1-}\ \frac{x-cq}{1-cq-}\ \frac{(1-q)(1-x)cq}{1-cq^2-(1-cq)\left(1-\frac{1}{\phi_1}\right)}\ \text{say} \ \ldots\ldots(2),$$

where $$\phi_1\equiv\frac{1}{1-}\ \frac{(x-cq)q}{1-cq^3-}\ \frac{(1-q^2)(1-xq)cq^2}{1-cq^4-\ldots}.$$

Reducing (2), we see that

$$\phi=\frac{1-cq}{1-q}+cq\frac{(x-cq)(1-q)}{(1-cq)(1-x)}\phi_1 \ \ldots\ldots\ldots\ldots(3).$$

Similarly if

$$\phi_2\equiv\frac{1}{1-}\ \frac{(x-cq^2)q^2}{1-cq^3-}\ \frac{(1-q^3)(1-xq^2)}{1-cq^5-\ldots},$$

then $$\phi_1=\frac{1-cq^3}{1-xq}+cq^3\frac{(x-cq^2)(1-q^2)}{(1-cq^2)(1-xq)}\phi_2.$$

Proceeding in this way we get a series for

$$\phi\left[\frac{cq}{x},\ q,\ cq,\ q,\ x\right]/(1-c),$$

viz.

$$\frac{1-cq}{(1-c)(1-x)}+cq\frac{(x-cq)(1-q)}{(1-c)(1-cq)(1-x)(1-xq)}(1-cq^3)$$

$$+c^2q^4\frac{(x-cq)(x-cq^2)(1-q)(1-q^2)}{(1-c)(1-cq)(1-cq^2)(1-x)(1-xq)(1-xq^2)}(1-cq^5)$$

$$+\ldots\ldots\ldots\ldots\ldots(4),$$

in which the $(r+1)^{\text{th}}$ term is

$$c^rq^{r^2}\frac{(x-cq)(x-cq^2)\ldots(x-cq^r)(1-q)(1-q^2)\ldots(1-q^r)}{(1-c)(1-cq)\ldots(1-cq^r)(1-x)(1-xq)\ldots(1-xq^r)}(1-cq^{2r+1}).$$

A few well-known identities may be derived from this transformation, and as we get very rapidly converging series, the results are not without interest.

Putting $c=1$ after dividing by $1-c$, we have

$$\prod_{n=0}^{\infty}\left(\frac{1-q^{n+1}}{1-xq^{n}}\right)=\frac{1-q}{1-x}+q\,\frac{x-q}{(1-x)(1-xq)}(1-q^{3})$$
$$+q^{4}\frac{(x-q)(x-q^{2})}{(1-x)(1-xq)(1-xq^{2})}(1-q^{5})+\dots.$$

From this equation we may derive the following well-known relations.

Let $x=0$, then

$$\prod_{n=0}^{\infty}(1-q^{n+1})=1-q-q^{2}(1-q^{3})+q^{7}(1-q^{5})-q^{15}(1-q^{7})+\dots.$$

Let $x=q^{\frac{1}{2}}$. Then, changing q into q^{2}, we get

$$\prod_{n=0}^{\infty}\left(\frac{1-q^{2n+2}}{1-q^{2n+1}}\right)=1+q+q^{3}(1+q^{3})+q^{10}(1+q^{5})+\dots.$$

Let $x=-1$, then

$$\prod_{n=0}^{\infty}\left(\frac{1-q^{n+1}}{1+q^{n+1}}\right)=1-q-q(1-q^{3})+q^{4}(1-q^{5})-q^{9}(1-q^{7})+\dots$$
$$=1-2q+2q^{4}-2q^{9}+\dots.$$

Again in (4) if $x=q$, then we have

$$\frac{1}{1-q}+\frac{c}{1-q^{2}}+\frac{c^{2}}{1-q^{3}}+\dots$$
$$=\frac{1-cq}{(1-c)(1-q)}+\frac{cq^{2}(1-cq^{3})}{(1-cq)(1-q^{2})}+\frac{c^{2}q^{6}(1-cq^{5})}{(1-cq^{2})(1-q^{3})}+\dots,$$

or better, writing cq for c and multiplying by c,

$$\frac{cq}{1-q}+\frac{c^{2}q^{2}}{1-q^{2}}+\frac{c^{3}q^{3}}{1-q^{3}}+\dots=\frac{cq}{1-q}+\frac{c^{2}q^{4}}{1-q^{2}}+\frac{c^{3}q^{9}}{1-q^{3}}+\dots$$
$$+\frac{c^{2}q^{2}}{1-cq}+\frac{c^{3}q^{6}}{1-cq^{2}}+\frac{c^{4}q^{12}}{1-cq^{3}}+\dots,$$

as may easily be verified by equating coefficients of c^{r}.

When $c \equiv 1$, we get Clausen's identity,

$$\Sigma \frac{q^r}{1-q^r} = q\frac{1+q}{1-q} + q^4\frac{1+q^2}{1-q^2} + \ldots,$$

and when $c = -1$,

$$\frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} - \ldots = q\frac{1+q^2}{1-q^2} - q^4\frac{1+q^4}{1-q^4} + q^9\frac{1+q^6}{1-q^6} - \ldots.$$

Leeds, *June*, 1893

PROOF OF A THEOREM IN THE THEORY OF NUMBERS.

By H. W. Segar.

§ 1. On page 59 of vol. XXII. of the *Messenger* the fact that the product of the differences of any r unequal numbers is divisible by $r-1!\, r-2! \ldots 3!\, 2!\, 1!$ was incidentally discovered, and it may be worth while to give an independent proof of this property. Let the unequal numbers be denoted by $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$; then the product of their differences is $\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)$, and this we shall shew is divisible by $r-1!\, r-2! \ldots 3!\, 2!\, 1!$ or $r-1!!$, say; which is the product of the differences of $1, 2, 3, \ldots, r$.

Let a be one of the prime factors of $r-1!!$; then we shall first shew that there are at least as many of the differences of $\alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_r$, divisible by a as there are differences of $1, 2, 3, \ldots, r$ so divisible.

The investigation will be simplified if we restrict ourselves to the most unfavourable case. This will be when the number of letters $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$ having the various remainders $0, 1, 2, 3, \ldots a-1$, after having been divided by a, are as nearly equal as can be; that is, when the number of letters having any one of the remainders does not differ by more than unity from the number of letters having any other remainder. For suppose that we have in this case p groups of letters having x letters each and q groups having $x+1$ letters each, using the word 'group' to denote all the letters having the same remainder. The differences which are divisible by a are obtained by subtracting from one another the numbers which have the same remainders when divided by a. Hence each