Moreover, substituting for p, q, r, s, &c., their values, we have finally for the required involution

$$\begin{aligned}
&[\beta\beta' x^2 + (\alpha\alpha' + \beta + \beta')\, y^2 + z^2 - (\alpha + \alpha')\, yz - (\alpha\beta' + \alpha'\beta)\, xy] \\
&\qquad \times [\beta''\beta''' x^2 + (\alpha''\alpha''' + \beta'' + \beta''')\, y^2 + z^2 \\
&\qquad\qquad - (\alpha'' + \alpha''')\, yz - (\alpha''\beta''' + \alpha'''\beta'')\, xy] \\
&- [\beta\beta'' x^2 + (\alpha\alpha'' + \beta + \beta'')\, y^2 + z^2 - (\alpha + \alpha'')\, yz - (\alpha\beta'' + \alpha''\beta)\, xy] \\
&\qquad \times [\beta'\beta''' x^2 + (\alpha'\alpha''' + \beta' + \beta''')\, y^2 + z^2 \\
&\qquad\qquad - (\alpha' + \alpha''')\, yz - (\alpha'\beta''' + \alpha'''\beta')\, xy], \\
&- (y^2 - zx) \times
\end{aligned}$$

$$\left\{\begin{aligned}
&y^2\,[(\alpha\beta''' - \alpha'''\beta)(\alpha' - \alpha'') + (\alpha'\beta'' - \alpha''\beta')(\alpha - \alpha''') - (\beta - \beta''')(\beta' - \beta'')] \\
&+ yz\,[(\alpha - \alpha''')(\beta' - \beta'') + (\alpha' - \alpha'')(\beta - \beta''')] \\
&- zx\,[(\beta - \beta''')(\beta' - \beta'')] \\
&- xy\,[(\alpha\beta''' - \alpha'''\beta)(\beta' - \beta'') + (\alpha'\beta'' - \alpha''\beta')(\beta - \beta''')]
\end{aligned}\right\} = 0.$$

It will be recollected that this is the solution for the case $A = 1234$, $F = 5678$; $B = 1256$, $F' = 3478$, which is that to which the present paper has reference.

ON THE DEVELOPMENT OF $(1+n^2x)^{\frac{m}{n}}$.

By Professor *Cayley*.

It is a known theorem that, if $\frac{m}{n}$ be any fraction in its least terms, the coefficients of the development of $(1+n^2x)^{\frac{m}{n}}$ are all of them integers, or, what is the same thing, that

$$\frac{m \,.\, m-n \ldots m-(r-1)\,n}{1 \,.\, 2 \;\ldots\; r}\, n^r$$

is an integer. The greater part, but not the whole, of this result comes out very simply from Mr. Segar's very elegant theorem, *Messenger*, August, 1892, p. 59, "the product of the differences of any r unequal numbers is divisible by $(r-1)\,!\,!$" or, as it may be stated, if $\alpha, \beta, \gamma, \ldots$ are any r unequal numbers, then $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots)$ is divisible by $\zeta^{\frac{1}{2}}(0, 1, 2 \ldots r-1)$.

In fact, writing $r+1$ for r and considering the numbers $m+n$, n, $2n$, $3n$, ... $(r-1)n$; then neglecting signs

$$\begin{aligned}\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots) \text{ is} = \;& m\,.\,m-n \ldots m-(r-1)\,n,\\ & \times 1n\,.\,2n \ldots (r-1)\,n,\\ & \times 1n\,.\,2n \ldots (r-2)\,n,\\ & \vdots\\ & \times 1n\,.\,2n,\\ & \times 1n,\end{aligned}$$

which is

$$= m\,.\,m-n \ldots m-(r-1)\,n \times n^{\frac{1}{2}r\,.\,r-1} \times \zeta^{\frac{1}{2}}(0, 1, 2, \ldots r-1),$$

and similarly

$$\zeta^{\frac{1}{2}}(0,\ 1,\ 2,\ \ldots\ r) = 1\,.\,2\,.\,3 \ldots r \times \zeta^{\frac{1}{2}}(0, 1, 2, \ldots r-1);$$

so that, omitting the common factor $\zeta^{\frac{1}{2}}(0, 1, 2, \ldots r-1)$, we have

$$m\,.\,m-n\,.\ldots m-(r-1)\,n\,.\,n^{\frac{1}{2}r\,.\,r-1} \text{ divisible by } 1\,.\,2\,.\,3 \ldots r.$$

It thus appears that the fraction

$$\frac{m\,.\,m-n \ldots m-(r-1)\,n}{1\ .\ 2\ \ldots\ \ r}$$

when reduced to its least terms will contain in the denominator only products of power of the prime factors of n; and it remains to show that multiplying this by n^r it will become integral, or what is the same thing that

$$\frac{n^r}{1\,.\,2\,.\ldots r}$$

in its least terms will not contain in the denominator any prime factor of n.

Considering in succession the prime numbers 2, 3, 5, ..., first the number 2, we see that in the product $1\,.\,2\,.\,3\,.\ldots r$, the number of terms divisible by 2 is $=\left(\frac{r}{2}\right)$, the number of terms divisible by 4 is $=\left(\frac{r}{4}\right)$, that by 8 is $=\left(\frac{r}{8}\right)$, and so on, where $\left(\frac{r}{2}\right)$ denotes the integer part of $\frac{r}{2}$, and so in other cases. Hence the product contains the factor 2, with the exponent $\left(\frac{r}{2}\right)+\left(\frac{r}{4}\right)+\left(\frac{r}{8}\right)+\ldots$, which exponent is less than

$$\frac{r}{2}+\frac{r}{4}+\frac{r}{8}+\ldots\ ad.\ inf.$$

is less than r, say it is less than (r). Similarly for the number 3, the product contains the factor 3 with the exponent

$$\left(\frac{r}{3}\right)+\left(\frac{r}{9}\right)+\left(\frac{r}{27}\right)+\ldots,$$

which exponent is less than

$$\frac{r}{3}+\frac{r}{9}+\frac{r}{27}+\ldots \textit{ ad. inf.}$$

is less than $\frac{1}{2}r$, say it is at most $=(\frac{1}{2}r)$; and so it contains the factor 5 with an exponent which is less than $\frac{1}{4}r$, say it is at most $=(\frac{1}{4}r)$, and generally the prime factor p with an exponent which is less than $\frac{1}{p-1}r$: say it is at most $=\left(\frac{1}{p-1}r\right)$.

This is

$$1.2.3\ldots r=\frac{1}{K}2^{(r)}\,3^{(\frac{1}{2}r)}\,5^{(\frac{1}{4}r)}\ldots,$$

where K is a whole number. Hence if $n=2^{\alpha}3^{\beta}5^{\gamma}\ldots$, we have

$$\frac{n^r}{1.2.3\ldots r}=K\,2^{r\alpha-(r)}.\,3^{r\beta-(\frac{1}{2}r)}.\,5^{r\gamma-(\frac{1}{4}r)}\ldots,$$

and here for every prime number 2, 3, 5, ... which is a factor of n, that is for which the corresponding exponent α, β, γ, ... is not $=0$, the exponents $r\alpha-(r)$, $r\beta-(\frac{1}{2}r)$, $r\gamma-(\frac{1}{4}r)$, ... are all of them positive; and thus the fraction in its least terms does not contain in the denominator any prime factor of n; which is the theorem which was to be proved.

Mr. Segar's theorem may without loss of generality be stated as follows: if β, γ, ... are any $r-1$ unequal positive integers (which for convenience may be taken in order of increasing magnitude), then $\zeta^{\frac{1}{2}}(0,\beta,\gamma,\ldots)$ is divisible by $\zeta^{\frac{1}{2}}(0,1,2,\ldots r-1)$. A proof, in principle the same as his, is as follows:

We have the determinant

$$\begin{vmatrix} 1, & a^{\beta}, & a^{\gamma}, \ldots \\ b & & \\ c & & \\ \vdots & & \end{vmatrix} \text{ divisible by } \begin{vmatrix} 1, & a, & a^{2}, \ldots \\ b & & \\ c & & \\ \vdots & & \end{vmatrix},$$

viz. the quotient is a rational and integral function of $a, b, c, \ldots$ with coefficients which are positive integers; hence putting $a=b=c, \ldots=1$, the quotient will be a positive integer number. Considering the numerator determinant, and for $a, b, c, \ldots$

writing therein $1+a$, $1+b$, $1+c$, ... respectively where $a, b, c, \ldots$ are ultimately to be put each $=0$, the value is

$$=\begin{vmatrix} 1, & 1+\beta_1 a+\beta_2 a^2 \ldots, & 1+\gamma_1 a+\gamma_2 a^2+\ldots, \ldots \\ & b & \\ & c & \\ & \vdots & \end{vmatrix},$$

where $\beta_1, \beta_2, \ldots$ denote the binomial coefficients

$$\frac{\beta}{1}, \frac{\beta.\beta-1}{1.2}, \text{\&c.}:$$

attending only to the lowest powers of $a, b, c, \ldots$ which enter into the formula, this is

$$=\begin{vmatrix} 1, & & \\ 1, & \beta_1, & \beta_2 \\ 1, & \gamma_2, & \gamma_3 \\ \vdots & & \end{vmatrix} \begin{vmatrix} 1, & a, & a^2, \ldots \\ 1, & b, & b^2, \\ 1, & c, & c^2, \\ \vdots & & \end{vmatrix},$$

or what is the same thing it is

$$=M\begin{vmatrix} 1, & & \ldots \\ 1, & \beta, & \beta^2, \\ 1, & \gamma, & \gamma^2, \\ \vdots & & \end{vmatrix} \div \begin{vmatrix} 1, & a, & a^2 \\ 1, & b, & b^2 \\ 1, & c, & c^2 \end{vmatrix}, =M\zeta^{\frac{1}{2}}(0,\beta,\gamma,\ldots)\begin{vmatrix} 1, & a, & a^2, \ldots \\ 1, & b, & b^2, \\ 1, & c, & c^2, \\ \vdots & & \end{vmatrix},$$

where M is a mere number: it will be recollected that in this form, $a, b, c, \ldots$ are not the original $a, b, c, \ldots$. Putting herein $\beta, \gamma, \ldots = 1, 2, \ldots$, the denominator determinant is

$$=M\zeta^{\frac{1}{2}}(0,1,2,\ldots)\begin{vmatrix} 1, & a, & a^2 \\ 1, & b, & b^2 \\ 1, & c, & c^2 \\ \vdots & & \end{vmatrix},$$

and hence the quotient, which as already seen is an integer number, is equal to $\zeta^{\frac{1}{2}}(0, \beta, \gamma, \ldots) \div \zeta^{\frac{1}{2}}(0, 1, 2, \ldots)$, the theorem in question.

The original theorem as to the form of $(1+n^2x)^{\frac{m}{n}}$ is a particular case of Eisenstein's very general theorem that in the development of any algebraical function of x, it is always possible by substituting for x a proper multiple of x, to make all the coefficients integers. It may be remarked that this would not be so if we had only

$$m.m-n\ldots m-(r-1)\,n.n^{\frac{1}{2}r.r-1}$$

divisible by $1.2\ldots.r$; for then, writing Nx for x, the form of the coefficient would have been

$$\frac{KN^r}{n^r.n^{\frac{1}{2}r.r-1}}, = \frac{KN^r}{n^{\frac{1}{2}r.r+1}},$$

and there would be no value (however great) of N by which the denominator factor $n^{\frac{1}{2}r.r+1}$ could be got rid of.

NOTE ON LINEAR SUBSTITUTIONS.

By *W. Burnside.*

THE points z_1, z_2 which are unchanged by a linear substitution

$$w = \frac{\alpha z + \beta}{\gamma z + \delta} \; (\alpha\delta - \beta\gamma = 1) \;\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

are the roots of

$$\gamma z^2 + (\delta - \alpha)\, z - \beta = 0,$$

and are therefore given by

$$\gamma\,(z_1 \text{ or } z_2) = \tfrac{1}{2}(\alpha - \delta) \pm \tfrac{1}{2}\sqrt{\{(\alpha+\delta)^2 - 4\}}.$$

When $\alpha + \delta$ is not $= 2$, equation (i) may be written in the form

$$\frac{w - z_1}{w - z_2} = K\frac{z - z_1}{z - z_2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and K, the multiplier, is given by

$$K + \frac{1}{K} = (\alpha + \delta)^2 - 2.$$

When $\alpha + \delta = 2$, so that the fixed points z_1, z_2 coincide, the form (ii) becomes illusory, and we have instead

$$\frac{1}{w - z_1} = \frac{1}{z - z_1} + c \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

When the substitution takes this latter form it is called parabolic: when it is of the form (ii) it is called hyperbolic, elliptic or loxodromic, according as K is real, imaginary with modulus unity, or imaginary with modulus other than unity.

It is the object of this note to determine the conditions under which two substitutions, neither of which is loxodromic, when performed successively on a variable leads to a substitution which also is not loxodromic.

The first of the two substitutions may, without loss of generality, be taken with its fixed points at 0 and ∞; or if parabolic with its fixed point at infinity and its additive constant real.

Let this be either

$$w = n^2 z \quad \text{(iv)}$$

or

$$w = z + m \quad \text{(iv)}'.$$

The result of performing first (iv) and then (i) is

$$w = \frac{\alpha n z + \frac{\beta}{n}}{\gamma n z + \frac{\delta}{n}}$$

and the multiplier of this substitution is given by

$$K + \frac{1}{K} = \left(\alpha n + \frac{\delta}{n}\right)^2 - 2,$$

and if K is either to be real or to have modulus unity

$$\alpha n + \frac{\delta}{n} \text{ must be real.}$$

Also if (i) is not loxodromic $\alpha + \delta$ is real.

Suppose then that

$$\alpha = a + ci, \; \delta = b - ci.$$

Then (a) when n is real

$$c\left(n - \frac{1}{n}\right) = 0;$$

(b) when $n = e^{i\theta}$

$$(a + ci)\, e^{i\theta} + (b - ci)\, e^{i\theta} \text{ is real,}$$

or

$$(a - b) \sin\theta = 0.$$

Hence in the first case $c = 0$ or α and δ are both real, and in the second case $a = b$ or α and δ are conjugate imaginaries. The result of performing first (iv)$'$ and then (i) is

$$w = \frac{\alpha z + \alpha m + \beta}{\gamma z + \gamma m + \delta},$$

and if in this case (c) the substitution is not loxodromic $\alpha + \delta + \gamma m$ must be real, and therefore γ must be real.

All possible cases are thus exhausted. Now in case (a) when (i) is hyperbolic the fixed points of (i) lie on a line through the origin, since γz_1 and γz_2 are both real. Hence in this case the two given substitutions being both hyperbolic their four fixed points lie on a line, and if the hyperbolic substitution (iv) be replaced by one with both its fixed points at finite distance the corresponding condition would be that the four fixed points should lie on a circle. In case (a) when (i) is elliptic the line joining its fixed points is bisected at right-angles by a line through the origin. In this case the substitutions are respectively elliptic and hyperbolic, and the corresponding general condition is that the fixed points of the elliptic substitution should be inverse points in the respect to some circle through the fixed points of the hyperbolic substitution. Case (a) when (i) is parabolic may be regarded as the limit of either of the above cases as the fixed points approach each other indefinitely. In the first of these cases the circle through the four fixed points, and in the second the circle through the two fixed points of the hyperbolic substitution with respect to which the other two fixed points are inverses, are unchanged circles for both the given substitutions; and hence in each of these cases the condition that the resulting substitutions should not be loxodromic is that there should be one circle (at least) which is unchanged by each of the given substitutions. A similarly detailed consideration of cases (b) and (c) leads to the same condition. Hence in every case it is a necessary condition that there should be one circle unchanged by both the given substitutions. From the nature of a loxodromic substitution which leaves no circle unchanged, it is obvious that this is a sufficient condition.

It follows almost immediately that the group of substitutions arising from any given set of fundamental substitutions will or will not contain loxodromic substitutions according as there is not or is at least one circle which is unchanged by the given set; and that if such a group contains no loxodromic substitution there must be a circle which is unchanged by the set of fundamental substitutions.

CORRECTION TO THE NOTE ON THE NINE SCHOOLGIRLS PROBLEM, (p. 159).

THE first paragraph of the Note should run:

'This is a parallel question to the well-known one of fifteen schoolgirls extended to the supposition of their walking for *one* week, three and three together, so that in any the same *day* no two, and at the end of the *week* no three, taking four walks a day, shall have walked more than once together.'

END OF VOL XXII.

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