we have $\phi(x) = e^{Ax^2+B} F(x^2)$, where $F(x^2)$ is a simple uniform function of x^2 of class zero. The example given by Laguerre may be derived from this by putting $-nx$ for x^2, and then making n infinite. By differentiating the original integral polynomial with respect to x^2, and then proceeding as above, we arrive at the same result for the function

$$1 - \frac{q^{1.2r+1}x^2}{1.n+r+1} + \frac{q^{2.2r+2}x^4}{1.2.n+r+1.n+r+2} - \frac{q^{3.2r+3}x^6}{1.2.3.n+r+1.n+r+2.n+r+3} + \dots$$

as we got for $\phi(x)$.

ON A CASE OF THE INVOLUTION $AF+BG+GH=0$, WHERE A, B, C, F, G, H ARE TERNARY QUADRICS.

By Prof. *Cayley*.

We have here the six conics

$$A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0;$$

the curves $AF=0$ and $BG=0$ are quartics intersecting in 16 points, and if 8 of these lie in a conic $H=0$, then the remaining 8 will be in a conic $C=0$. I take the first set of eight points to be 1, 2, 3, 4, 5, 6, 7, 8; the quartics $AF=0$ and $BG=0$ each pass through these eight points; and I assume for the moment

$$A=1234,\ F=5678;\ B=1234,\ G=5678,$$

viz. that $A=0$ is a conic through the points 1, 2, 3, 4, and similarly for F, G, B. Here $H=0$ is a conic through the points 1, 2, 3, 4, 5, 6, 7, 8, or attending only to the last four points it is a conic through 5, 6, 7, 8; we have therefore a linear relation between F, G, H, and supposing the implicit constant factors to be properly determined, this may be taken to be $F+G+H=0$; the identity $AF+BG+CH=0$ thus becomes $F(A-C)+G(B-C)=0$. We have thus F a numerical multiple of $B-C$, and by a proper determination of the implicit factor we may make this relation to be $F=B-C$; the last equation then gives $G=C-A$, and from the equation $F+G+H=0$, we have $H=A-B$; the six functions thus are

$$\begin{array}{lll} A,\ B-C & \text{or if we please} & A-D,\ B-C, \\ B,\ C-A & & B-D,\ C-A, \\ C,\ A-B & & C-D,\ A-B, \end{array}$$

where D is an arbitrary quadric function. The solution

$$(A-D)(B-C)+(B-D)(C-A)+(C-D)(A-B)=0$$

of the involution is an obvious and trivial one.

But the case which I proceed to consider is

$$A=1234,\ F=5678\ ;\ B=1256,\ G=3478\ ;$$

here $AF=0$, and $BG=0$, meet as before in the points 1, 2, 3, 4, 5, 6, 7, 8, and in eight other points, say that

$A=0,\ B=0$	meet in	1, 2	and in two other points	α, β,
$A=0,\ G=0$	,,	3, 4	,, ,,	γ, δ,
$F=0,\ B=0$	,,	5, 6	,, ,,	ε, ζ,
$F=0,\ G=0$	,,	7, 8	,, ,,	η, θ;

then the 8 points $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$ will lie in a conic $C=0$.

I take $y^2-zx=0$ for the conic $H=0$; for any point in this conic we have $x:y:z=1:\theta:\theta^2$, and we may take $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8$ for the parameters of the points 1, 2, 3, 4, 5, 6, 7, 8 respectively.

Write $(a, b, c, f, g, h \between x, y, z)^2=0$ for the conic $A_1=1234=0$; therefore we have

$$a+b\theta^2+c\theta^4+f\theta^3+g\theta^2+h\theta=\theta-\theta_1.\theta-\theta_2.\theta-\theta_3.\theta-\theta_4;$$

or if

$$\begin{aligned}
p_{1234}&=\theta_1+\theta_2+\theta_3+\theta_4,\\
q_{1234}&=\theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4,\\
r_{1234}&=\theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4,\\
s_{1234}&=\theta_1\theta_2\theta_3\theta_4,
\end{aligned}$$

then $c=1, f=-p_{1234}, b+g=q_{1234}, h=-r_{1234}, a=s_{1234}$;

or writing $g=-\lambda$, we have

$$s_{1234}x^2+q_{1234}y^2+z^2-p_{1234}yz-r_{1234}xy+\lambda(y^2-zx)=0$$

for the equation of the conic in question. We may without loss of generality put $\lambda=0$; and then if in general

$$\Omega=sx^2+qy^2+z^2-pyz-rxy,$$

we have $A=\Omega_{1234}=0$ for the conic $A=0$. And thus the equations of the four conics are

$$A=\Omega_{1234}=0,\ F=\Omega_{5678}=0\ ;\ B=\Omega_{1256}=0,\ C=\Omega_{3478}=0,$$

or, as for shortness I write them,

$$A=\Omega=0,\ F=\Omega'=0\ ;\ B=\Omega''=0,\ C=\Omega''',$$

viz. in Ω the suffixes are 1, 2, 3, 4, in Ω' they are 5, 6, 7, 8, in Ω'' they are 1256, and in Ω''' they are 3, 4, 7, 8.

I find that the implicit constant factors of AF and BG are 1, – 1, and consequently that the form of the identity is

$$\Omega\Omega' - \Omega''\Omega''' + (y^2 - zx)\,C = 0,$$

where C is a quadric function to be determined; or, what is the same thing, we have

$$\begin{aligned}&(sx^2 + qy^2 + z^2 - pyz - rxy)(s'x^2 + q'y^2 + z^2 - p'yz - r'xy),\\ &-(s''x^2 + q''y^2 + z^2 - p''yz - r''xy)(s'''x^2 + q'''y^2 + z^2 - p'''yz - r'''xy),\\ &+(y^2 - zx)\,C = 0.\end{aligned}$$

Writing for shortness

$$\begin{aligned}\theta_1 + \theta_2 &= \alpha\ \ , & \theta_1\theta_2 &= \beta\ \ ,\\ \theta_3 + \theta_4 &= \alpha'\ , & \theta_3\theta_4 &= \beta'\ ,\\ \theta_5 + \theta_6 &= \alpha''\,, & \theta_5\theta_6 &= \beta''\,,\\ \theta_7 + \theta_8 &= \alpha''', & \theta_7\theta_8 &= \beta''',\end{aligned}$$

we have

$$\left.\begin{aligned}p &= \alpha + \alpha'\\ q &= \alpha\alpha' + \beta + \beta'\\ r &= \alpha\beta' + \alpha'\beta\\ s &= \beta\beta'\end{aligned}\ \right|\ \begin{aligned}p' &= \alpha'' + \alpha'''\\ q' &= \alpha''\alpha''' + \beta'' + \beta'''\\ r' &= \alpha''\beta''' + \alpha'''\beta''\\ s' &= \beta''\beta'''\end{aligned}\ \Bigg|$$

$$\Bigg|\ \begin{aligned}p'' &= \alpha + \alpha''\\ q'' &= \alpha\alpha'' + \beta + \beta''\\ r'' &= \alpha\beta'' + \alpha''\beta\\ s'' &= \beta\beta''\end{aligned}\ \Bigg|\ \begin{aligned}p''' &= \alpha' + \alpha'''\\ q''' &= \alpha'\alpha''' + \alpha'\beta''' + \alpha'''\beta'\\ r''' &= \alpha'\beta''' + \alpha'''\beta'\\ s''' &= \beta'\beta'''\end{aligned}$$

In the last mentioned equation, the first and second lines together are a quartic function of (x, y, z), say the value is

$$\begin{aligned}=\ &Ax^4 + By^4 + Cz^4,\\ &+ Fy^3z + Gz^3x + Hx^3y,\\ &+ Iyz^3 + Jzx^3 + Kxy^3,\\ &+ Lx^2yz + Mxy^2z + Nxyz^2,\\ &+ Py^2z^2 + Qz^2x^2 + Rx^2y^2,\end{aligned}$$

where after all reductions

$$
\begin{aligned}
A &= ss' - s''s''' && = 0,\\
B &= qq' - q''q''' && = (\alpha\beta''' - \alpha'''\beta)(\alpha' - \alpha'')\\
&\quad + (\alpha'\beta'' - \alpha''\beta')(\alpha - \alpha''') - (\beta' - \beta'')(\beta - \beta'''),\\
C &= \quad 1 - 1 && = 0,\\
F &= -pq' - p'q + p''q''' + p'''q'' && = (\alpha - \alpha''')(\beta' - \beta'')\\
&&& \quad + (\alpha' - \alpha'')(\beta - \beta'''),\\
G &= \quad 0 - 0 && = 0,\\
H &= -rs' - r's + r''s''' + r'''s'' && = 0,\\
I &= -p - p' + p'' + p''' && = 0,\\
J &= \quad 0 - 0 && = 0,\\
K &= -qr' - q'r + q''r''' + q'''r'' && = (\alpha\beta''' - \alpha'''\beta)(\beta'' - \beta')\\
&&& \quad + (\alpha'\beta'' - \alpha''\beta')(\beta''' - \beta),\\
L &= -ps' - p's + p''s''' + p'''s'' && = (\alpha\beta''' - \alpha'''\beta)(\beta' - \beta'')\\
&&& \quad + (\alpha'\beta'' - \alpha''\beta')(\beta - \beta'''),\\
M &= \quad pr' + p'r - p''r''' - p'''r'' && = (\alpha\beta''' - \alpha'''\beta)(\alpha'' - \alpha')\\
&&& \quad + (\alpha'\beta'' - \alpha''\beta')(\alpha''' - \alpha),\\
N &= -r - r' + r'' + r''' && = (\alpha - \alpha''')(\beta'' - \beta')\\
&&& \quad + (\alpha' - \alpha'')(\beta''' - \beta),\\
P &= \quad pp' + q + q' - p''p''' - q'' - q''' && = 0,\\
Q &= \quad s + s' - s'' - s''' && = (\beta' - \beta'')(\beta - \beta'''),\\
R &= rr' + qs' + q's - r''r''' - q''s''' - q'''s'' = 0:
\end{aligned}
$$

values which satisfy

$$F + N = 0,$$

$$K + L = 0,$$

$$B + M + Q = 0.$$

The quartic function is thus seen to be

$$= (y^2 - zx)(By^2 + Fyz - Qzx + Kxy = 0,$$

viz. we have $By^2 + Fyz - Qzx + Kxy = 0$ for the equation of the conic $C = 0$.

Moreover, substituting for p, q, r, s, &c., their values, we have finally for the required involution

$$[\beta\beta'x^2+(\alpha\alpha'+\beta+\beta')y^2+z^2-(\alpha+\alpha')yz-(\alpha\beta'+\alpha'\beta)xy]$$
$$\times[\beta''\beta'''x^2+(\alpha''\alpha'''+\beta''+\beta''')y^2+z^2-(\alpha''+\alpha''')yz-(\alpha''\beta'''+\alpha'''\beta'')xy]$$
$$-[\beta\beta''x^2+(\alpha\alpha''+\beta+\beta'')y^2+z^2-(\alpha+\alpha'')yz-(\alpha\beta''+\alpha''\beta)xy]$$
$$\times[\beta'\beta'''x^2+(\alpha'\alpha'''+\beta'+\beta''')y^2+z^2-(\alpha'+\alpha''')yz-(\alpha'\beta'''+\alpha'''\beta')xy],$$
$$-(y^2-zx)\times\left\{\begin{array}{l} y^2[(\alpha\beta'''-\alpha'''\beta)(\alpha'-\alpha'')+(\alpha'\beta''-\alpha''\beta')(\alpha-\alpha''')-(\beta-\beta''')(\beta'-\beta'')] \\ +yz[(\alpha-\alpha''')(\beta'-\beta'')+(\alpha'-\alpha'')(\beta-\beta''')] \\ -zx[(\beta-\beta''')(\beta'-\beta'')] \\ -xy[(\alpha\beta'''-\alpha'''\beta)(\beta'-\beta'')+(\alpha'\beta''-\alpha''\beta')(\beta-\beta''')] \end{array}\right\}=0.$$

It will be recollected that this is the solution for the case $A=1234$, $F=5678$; $B=1256$, $F'=3478$, which is that to which the present paper has reference.

ON THE DEVELOPMENT OF $(1+n^2x)^{\frac{m}{n}}$.

By Professor *Cayley*.

It is a known theorem that, if $\frac{m}{n}$ be any fraction in its least terms, the coefficients of the development of $(1+n^2x)^{\frac{m}{n}}$ are all of them integers, or, what is the same thing, that

$$\frac{m\,.\,m-n\,\ldots\,m-(r-1)\,n}{1\,.\,2\;\ldots\;r}\,n^r$$

is an integer. The greater part, but not the whole, of this result comes out very simply from Mr. Segar's very elegant theorem, *Messenger*, August, 1892, p. 59, "the product of the differences of any r unequal numbers is divisible by $(r-1)!\,!$" or, as it may be stated, if $\alpha, \beta, \gamma, \ldots$ are any r unequal numbers, then $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots)$ is divisible by $\zeta^{\frac{1}{2}}(0, 1, 2 \ldots r-1)$.